1. **Two faces of strong subadditivity.** Show that for any state $\rho_{ABC}$ on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$,

$$S(A) + S(B) \leq S(AC) + S(BC)$$

and that this statement is equivalent to strong subadditivity in the form

$$S(A'C') + S(A'B') \geq S(A') + S(A'B'C') \quad \forall A'B'C'.$$

Hint: use the Araki-Lieb purification trick.

2. **Consequences of SSA for mutual information.** Prove that SSA implies

$$I(A : B) + I(A : C) \leq 2S(A).$$

Is the analogous inequality for Shannon entropies true?

Find an example of a state where $I(A : B) > S(A)$.

3. **Measurement is coarse-graining.**

Let $\rho, \sigma$ be two states on $\mathcal{H}$, and let $\{M_x\}$ be a POVM. Define the classical probability distributions $p_x, q_x$ from the outcomes of a measurement of $\{M_x\}$ on the states $\rho, \sigma$ respectively (that is, $p_x = \text{tr}\rho M_x$ etc). Show that

$$\hat{D}(\rho || \sigma) \geq D(p || q).$$

4. **Scramble.**

For this problem $\mathcal{H}_A$ has dimension $d$.

(a) **Warmup.** The set of linear operators $\text{End}(\mathcal{H}_A)$ is itself a Hilbert space with the Hilbert-Schmidt inner product $\langle A, B \rangle = \text{tr}A^\dagger B$. Find an orthogonal basis $\{U_a\}$ for this space (over $\mathbb{C}$) whose elements are themselves unitary operators, $\text{tr}U_a^\dagger U_b = d\delta_{ab}$. 

[Hint: consider the algebra generated by the unitaries $X, Z$ on the qdit teleportation problem on the previous problem set.]

Bonus: For the case of $|A| = 2^k$ find such a basis whose elements square to one.
(b) Consider a maximally entangled state \(|\Phi\rangle \equiv \frac{1}{\sqrt{d}} \sum_i |ii\rangle \in H_A \otimes H_A\). Show that the \(d^2\) maximally entangled states

\[ |\Phi_a\rangle \equiv U_a \otimes \mathbb{1} |\Phi\rangle \]

form an orthonormal basis of \(H_A \otimes H_A\).

(c) Check your answers to the previous two parts for the case of qbits \(d = 2\). Make a basis of product states from linear combinations of maximally entangled states.

(d) For an arbitrary operator \(A \in \text{End}(A)\) find \(\{p_a, U_a\}\) with \(p_a\) probabilities and \(U_a\) unitary such that the associated channel *scrambles* \(A\) in the sense that

\[ \sum_a p_a U_a A U_a^\dagger = \frac{\text{tr} A}{d} \mathbb{1}. \]

(e) Use the previous result and the concavity of the entropy to show that the uniform state \(u = \mathbb{1}/d\) has the maximum von Neumann entropy on \(A\).

(f) Bonus problem: for the case where \(A\) is Hermitian, find a set of only \(d\) unitaries which scramble \(A\).

5. **Random quantum expanders.** [somewhat open-ended and numerical]

Consider the family of quantum channels of the form

\[ \rho \mapsto \mathcal{E}_\chi(\rho) = \sum_{i=1}^\chi p_i U_i \rho U_i^\dagger \]

with \(\{U_i\}\) a collection of unitaries. Such a channel is called a *quantum expander*. Show that such a channel is unital.

Sample \(\chi\) random unitaries from the Haar measure on \(U(d)\) e.g. in Mathematica\(^1\). (You can take \(p_i = 1/\chi\) for definiteness if you wish.)

Sample a random initial density matrix\(^2\).

Consider the rate at which repeated action of the channel \(\mathcal{E}_\chi\), \(\rho_n = \mathcal{E}^n(\rho)\) mixes the initial state \(\rho\) as a function of \(\chi\) (and \(d\)). We can use the von Neumann entropy as a measure of this mixing. Make some plots and some estimates.

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\(^1\) Haar measure means the measure which is invariant under the group action. I did this by choosing a \(d \times d\) complex matrix \(X\) with entries chosen from the gaussian distribution (which is indeed invariant under \(U(d)\)) and then taking \(Y = X + X^\dagger\) to make it hermitian, and then using the matrix \(U\) which diagonalizes \(Y\).

\(^2\) I did this by choosing a complex matrix \(X\) with entries chosen from the gaussian distribution, and then taking \(Y = X + X^\dagger\) to make it hermitian and then taking \(Z = Y^2\) to make it positive and then taking \(\rho = Z/\text{tr}Z\) to make it a density matrix. What distribution did I use?
If $n$ is very large, how many terms do I actually need to include in the sum in
\[ \mathcal{E}^n(\rho) = \sum_{i_1 \cdots i_n} p_{i_n} \cdots p_{i_1} U_{i_1} \cdots U_{i_n} \rho U_{i_n}^\dagger \cdots U_{i_1}^\dagger \ ? \]

Consider the eigenstates (eigenoperators) of the (super)operator $\mathcal{E}_\chi$. Can you show that any state orthogonal (in the Hilbert-Schmidt norm) to $\mathds{1}$ has a an eigenvalue less than 1?