University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 239/139 Fall 2019
Assignment 8

Due 12:30pm Wednesday, November 27, 2019

1. **Brain-warmer.** Let $Z = \sum_j |j\rangle \langle j| \omega^j$ be the clock operator on a d-dimensional Hilbert space, $\omega \equiv e^{2\pi \mathbf{i}/d}$. Show that $K_j = Z^j/\sqrt{d}$ are Kraus operators for the diagonal-part channel.

2. **Equivalent Kraus representations.** Show that $\{\mathcal{K}\} \simeq \{\tilde{\mathcal{K}}\}$ produce the same channel iff $\mathcal{K}_k = \sum_l u_{kl} \tilde{\mathcal{K}}_l$ where u_{kl} is a unitary matrix in the kl indices.

   Open-ended bonus problem: find invariants of $\{\mathcal{K}\}$ which label equivalence classes under the above equivalence relation.

3. **The entropy exchange.** Many of the quantum channels we considered in previous homeworks have the property that they increase the von Neumann entropy of their victim. An exception is the amplitude damping channel, which makes the state more pure. An implementation of a second law of thermodynamics would require us to identify a total entropy which probably increases.

   We showed that any channel can be regarded as unitary evolution on a larger space (followed by partial trace), and unitary evolution doesn't change the entropy (of the whole system) at all. Even without introducing the environment explicitly, the Kraus operators give us a way to keep track of the entropy:

   Suppose a CPTP map $\mathcal{E} : \mathrm{End}(A) \to \mathrm{End}(B)$ has Kraus representation $\{K_i\}_{i=1..r}$. For any density matrix on A, let

   $$\varsigma_{ij} \equiv \mathrm{tr} K_i \boldsymbol{\rho} K_j^\dagger .$$

   (Recall that the index on the Kraus operators comes from a basis on the environment.)

   (a) Show that ς_{ij} is positive and has unit trace as a matrix in the ij space.

   (b) The von Neumann entropy of ς_{ij}, $S(\varsigma) = -\mathrm{tr}\varsigma \log \varsigma$ is called the *entropy exchange*. Show that $S(\varsigma)$ is preserved by the equivalence relation between Kraus representations of $\mathcal{E}$ in problem 2.

   (c) Show (by Stinespring dilation) that σ is the reduced density matrix of the *environment* after the action of the channel, and therefore that the entropy exchange is equal to the entropy of the environment after the action of the channel.

4. **Ordering versus entanglement.** Here we give a simple illustration that there is competition between ordering in a quantum state (non-vanishing local order parameter) and entanglement between the degrees of freedom.

   Consider just a single spin in a many-body system; call its reduced density matrix ρ.

   Then its contribution to the magnetization is $s = \text{tr}\rho\sigma^z$, and its entanglement with anyone else (either neighbors or some conduction electrons or something else more long-ranged) can be quantified by $S_{vN}(\rho) = -\text{tr}\rho\log\rho$.

   Show that the entropy is bounded above by a function of s. Find the function of s. Verify the inequality numerically in some random states. Make pictures.

   Hint: Consider the probability distribution obtained from the diagonal elements of ρ.

In lecture we showed several results beginning with monotonicity of the relative entropy as the starting point. Here we will show, following Ruskai, that SSA is just as good a starting point.

5. **SSA implies concavity of the conditional entropy.**

   (a) Show that SSA can be rewritten as

   $$D(\boldsymbol{\rho}_{12}||\boldsymbol{\rho}_2) \leq D(\boldsymbol{\rho}_{123}||\boldsymbol{\rho}_{23}) \tag{1}$$

   where $\boldsymbol{\rho}_2$ means $\mathbb{1}_1 \otimes \boldsymbol{\rho}_2$ etc. (Note that in this expression the arguments are not density matrices and positivity of the BHS is not guaranteed.)

   (b) Consider a bipartite state $\boldsymbol{\rho}_{12}$. Show that

   $$D(\boldsymbol{\rho}_{12}||\mathbb{1}/d_1 \otimes \boldsymbol{\rho}_2) = -S(12) + S(2) + \log d_1 = -S(1|2) + \log d_1 \ .$$

   (c) Apply SSA in the form (1) to state

   $$\boldsymbol{\rho}_{123} = \sum_i p_i \boldsymbol{\rho}_{12}^i \otimes |i\rangle\langle i|_3 \, .$$

   Conclude the statement in the title of this problem.

6. **SSA implies monotonicity of the relative entropy.**

   (a) Show that for $F(A)$ convex and homogeneous $F(xA) = xF(A)$,

   $$\lim_{x\to 0} \frac{F(A + xB) - F(A)}{x} \leq F(B). \tag{2}$$

(b) Recall from problem (5) that SSA implies concavity of $S(2|1) \equiv S(\boldsymbol{\rho}_{12}) - S(\boldsymbol{\rho}_1)$.

(c) Combine the first two parts of this problem, setting

$$A \equiv \boldsymbol{\sigma}_{12}, B \equiv \boldsymbol{\rho}_{12}$$

in (2) to show monotonicity of the relative entropy under partial trace.

7. **SSA implies joint convexity of relative entropy.**

(a) **Monotonicity of the relative entropy implies joint convexity.** Apply monotonicity of the relative entropy to the following block-diagonal bipartite states

$$\boldsymbol{\rho}_{AB} = \sum_i p_i \boldsymbol{\rho}_A^i \otimes |i\rangle \langle i|_B \,, \quad \boldsymbol{\sigma}_{AB} = \sum_i p_i \boldsymbol{\sigma}_A^i \otimes |i\rangle \langle i|_B \quad .$$

Conclude the boldface statement.

(b) Conclude from the previous part (7a) and (6) that SSA implies joint convexity of the relative entropy.

(c) Check that there are no loops in the above chains of reasoning.