University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2020
Assignment 3

Due 12:30pm Monday, October 26, 2020

Thanks for following the submission guidelines on hw 01. Please ask me by email if you have any trouble.

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1. **Brain-warmer.** Consider the action of a group G on itself by left multiplication. What kinds of terms are allowed in the cycle indicator for this group action? Compute the cycle indicators for the action by left multiplication of Q_8, of D_4 and of $\mathbb{Z}_n$ (do the case of n prime first, and do general n as a bonus problem). What do you get if you set all the $z_i = 1$?

2. **Brain-warmer.** Given a group homomorphism $\phi : G \to K$, show that its kernel $H \equiv \ker \phi \subset G$ is always a subgroup. Further, show that it is a *normal* subgroup.

3. **Representation on cosets.** Given a group G and a subgroup H, we can construct a representation of G by its action on the cosets $x \in G/H$. The action is

$$\begin{array}{ccc} G/H & \to & G/H \\ x = \{g_1, g_2 \cdots\} & \mapsto & \{gg_1, gg_2 \cdots\} \end{array}.$$

   To see more explicitly what this does, choose a representative x_i of each distinct coset. Then $G = \cup_i x_i H$, and for each $g \in G$, gx_iH is again a coset and therefore equal to one of the x_iH. So left multiplication by g permutes the cosets. [1]

   Consider the case where $H = \langle(123)\rangle$ is the subgroup of S_3 generated by the order-3 element (123). First decompose S_3 into cosets by H. Write out the matrices in a basis. What representation of S_3 do you get by the construction above? (If it is reducible, decompose it into irreps, for example by computing its character.)

   What about the case where $H' = \langle(12)\rangle$? In each case, decompose it into irreps.

   This construction plays an important role in the idea of induced representations. We will learn to call it the representation of G induced by the trivial representation of H.

[1] A word of clarification here: an action of a group G on a set (recall that for a set $X = \{x_1, ..x_k\}$ of order k, this is a homomorphism from $G \to S_k$) is also a representation in the following sense: define the carrier space $V = \text{span}\{|x_i\rangle, i = 1..k\}$, and define the linear operators to act on the basis vectors by $D(g)|x\rangle = |gx\rangle$. This is a special kind of representation called a permutation representation, since the linear operators take basis vectors to basis vectors, rather than making linear combinations – the matrix elements in this basis are a single 1 in each row and column and zeros in all the other entries.

4. **The class of inverses.** Given a conjugacy class c, define the class $\bar{c}$ to consist of the inverses of each of the elements in c. Convince yourself that this is well-defined. Show that for a unitary representation R,

$$\chi_R(\bar{c}) = \chi_R(c)^\star.$$

5. **Character exercise.** Recall the definition of the regular representation of a finite group G:

$$\mathcal{H}_\mathsf{G} \equiv \text{span}\{|g\rangle\,,\ g \in \mathsf{G}\}.$$

   This space also carries a representation of $\mathsf{G} \times \mathsf{G}$ by

$$\Gamma(m,n)\,|h\rangle = \left|mhn^{-1}\right\rangle\,,\ (m,n) \in \mathsf{G} \times \mathsf{G}.$$

   $\mathcal{H}_\mathsf{G}$ is also reducible as a representation of $\mathsf{G} \times \mathsf{G}$. Show that

$$\mathcal{H}_\mathsf{G} = \oplus_{R,\text{irreps of }\mathsf{G}} R \otimes \bar{R}$$

   where $\bar{R}$ is the conjugate representation to R, with $\Gamma_{\bar{R}}(g) = \Gamma_R(g)^\star$.

   Hint: Find the characters of $\mathcal{H}_\mathsf{G}$ as a representation of $\mathsf{G} \times \mathsf{G}$, and compute $\left\langle \chi_{\mathcal{H}_\mathsf{G}} | \chi_{R_i \otimes R_j} \right\rangle$, where R_i are all the irreps of G.

6. Using the character table, make unitary representation matrices for the 2-dimensional representation of $D_3 = S_3$ in which $D(123)$ and $D(132)$ are diagonal. Note that $1 + \omega + \omega^2 = 0$ where $\omega \equiv e^{2\pi \mathbf{i}/3}$.