University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2020
Assignment 4

Due 12:30pm Monday, November 2, 2020

Thanks for following the submission guidelines on hw 01. Please ask me by email if you have any trouble.

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1. **Brain-warmer.** If I tell you that $\langle \chi_R, \chi_R \rangle \equiv \frac{1}{|G|} \sum_{g \in G} \chi_R^\star(g) \chi_R(g) = 2$ for some representation R of G, do you know whether R is irreducible? Explain.

2. **Brain-warmer.** Given two representations A, B of a group G (with carrier spaces U and V respectively) and an intertwiner between them

$$\Lambda : U \to V, \quad \Lambda A(g) = B(g) \Lambda \forall g \in G$$

show that $\ker \Lambda \subset U$ and $\text{Im}\, \Lambda \subset V$ are invariant subspaces.

3. **Character table for the quaternions.** Figure out the character table for the quaternion group Q_8 (on page 11 of the lecture notes) by whatever means necessary (don't look it up).

4. **Irreps and conjugacy classes.**

Consider the object

$$S_\alpha^a \equiv \frac{1}{n_\alpha} \sum_{g \in C_\alpha} D^a(g),$$

a linear operator on V_a, the carrier space for an irrep of G. C_α is a conjugacy class of G.

(a) Show that S_α^a commutes with all the $D^a(g)$.

(b) Use Schur's lemma to conclude that $S_\alpha^a = \lambda_\alpha^a \mathbb{1}_a$ and find λ_α^a in terms of familiar objects.

(c) [Bonus problem] Conclude that $\mathbf{C}_\alpha \mathbf{P}^a = \lambda_\alpha^a \mathbf{P}^a$ where $\mathbf{P}^a = \frac{d_a}{|G|} \sum_{g \in G} (\chi^a(g))^\star \mathbf{g}$ is the projector in the group algebra associated with the irrep a and $\mathbf{C}_\alpha \equiv \frac{1}{n_\alpha} \sum_{g \in C_\alpha} \mathbf{g}$ are elements of the center of the group algebra.

(d) [This part is a bonus problem because I am not sure if it is possible.] Show directly from the definitions that $\mathbf{C}_\alpha \mathbf{P}^a = \lambda_\alpha^a \mathbf{P}^a$ for some eigenvalues λ_α^a.

5. **Diffusion on the vertices of the cube.**

(a) The group $\mathsf{O} = S_4$ of rotational symmetries of the cube acts on functions on vertices of the cube. Decompose this representation into irreps of S_4.

(b) Assign a temperature to each vertex of the cube T_i. Suppose the temperature evolves according to $T_i(t+1) = \frac{1}{3} \sum_{\text{neighbors, } j \text{ of } i} T_j(t)$. Find the rate at which the temperature function approaches its final distribution. (Do it without explicitly diagonalizing any matrices.)

(c) [bonus problem] Construct explicit projectors onto the eigenmodes using the character table of S_4.

6. **Diffusion on the edges of the tetrahedron.** [bonus problem]

(a) Show that the group of rotations which are symmetries of a regular tetrahedron is isomorphic to A_4.

(b) Construct the character table for A_4.

(c) A_4 acts on the edges of the tetrahedron in a 6-dimensional representation. Decompose this into irreps.

(d) Suppose we have a tight-binding model on the edges of the tetrahedron, $H = -t \sum_{\langle e_1 e_2 \rangle} |e_1\rangle\langle e_2| + h.c.$ where two edges are regarded as neighbors if they both lie in the boundary of the same face. Find the spectrum.

7. **Projection operators.** [bonus problem] Write a program to make the pictures of the normal modes of the 'triatomic molecule'. Write it in such a way that it is easy to redo it for the generalization to n particles arranged in a regular n-sided polygon.

8. **Using the algebra of classes to construct the character table for S_3.** [bonus problem] I'm postponing this problem until next week, since we didn't get to it in lecture.

(a) Write explicit expressions for the $\mathbf{C}_\alpha$ operators for the case $G = S_3$, and find the structure constants $c^\gamma_{\alpha\beta}$ of the algebra of classes.

(b) Construct the matrices $(\mathbf{C}_\alpha)^\gamma{}_\beta$, check that they commute, and find their eigenvalues, λ^a_α

(c) Check that you reproduce the character table for S_3 using the fact that the eigenvalues should be $\lambda^a_\alpha = \chi^a_\alpha / \chi^a_e$.