University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2020
Assignment 10

Due 12:30pm Monday, December 14, 2020

Thanks for following the submission guidelines on hw 01. Please ask me by email if you have any trouble.

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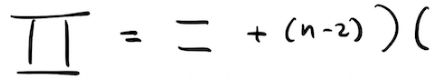
1. **Brain-warmer.** Verify the identity



(with $n = 3$) in $\mathsf{SO}(3)$ using birdtracks.

2. **Ramond-Ramond sectors.** [Bonus problem]

   (a) The *Ramond* sector of the superstring worldsheet contains a Hilbert space on which 8 majorana operators γ_i act. The $\mathsf{SO}(8)$ acting on the index i in the fundamental is part of the spacetime symmetry. Physical states of the superstring are those which have definite eigenvalue of $\gamma_F = C\prod_{i=1}^{8}\gamma_i$ (where C is chosen so that $\gamma_F^\dagger = \gamma_F$ and $\gamma_F^2 = 1$).

   The Ramond-Ramond sector of the closed superstring Hilbert space is the tensor product two Ramond sectors (one from right-moving modes and one from the left-moving modes on the closed string worldsheet). In type IIB, both copies have the same eigenvalue of γ_F and in type IIA, the two copies have opposite γ_F eigenvalue.

   How do the physical states of each type of closed superstring transform under $\mathsf{SO}(8)$?

   Removing the string theory jargon, the question is: how do $\mathbf{8}_+ \otimes \mathbf{8}_+$ and $\mathbf{8}_+ \otimes \mathbf{8}_-$ decompose into irreps of $\mathsf{SO}(8)$?

   One way to do it is to consider the transformation law for objects of the form $\langle s_1 s_2 s_3 s_4|\, \gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_k}\, |s_1 s_2 s_3 s_4\rangle$.

   (b) [Super-bonus problem – requires some field theory] Consider the action

$$S[X^i, \psi^i] = \int d^2x((\partial X)^2 + \psi \partial\!\!\!/ \psi),$$

   where $i = 1..n$. Take space to be periodic $x \equiv x + L$ and take periodic (Ramond) boundary conditions on the fermions fields (and the scalars). Show that its groundstates form a tensor product of two spinor representations of $\mathsf{SO}(n)$.

Here is a hint: we can decompose the fields $X(x,t)$ and $\psi(x,t)$ into left-movers and right-movers:

$$X(x,t) = X_L(x-t) + X_R(x+t), \quad \psi(x,t) = \psi_L(x-t) + \psi_R(x+t).$$

If we impose periodic boundary conditions on both ψ_L and ψ_R we get the RR sector described above.

If we impose periodic boundary conditions on ψ_L and antiperiodic boundary conditions on ψ_R we get the R-NS sector. Show that these states transform as a spinor representation of $\mathsf{SO}(n)$ (and, consistent with the spin-statistics theorem, they are fermions).

3. **Schwinger bosons.**

   What happens if in our construction of spinor reps, we replace the fermions $\{c_a, c_b^\dagger\} = \delta_{ab}$ with bosons $[b_a, b_b^\dagger] = \delta_{ab}$?

   (a) Consider what representations this produces of the $\mathsf{SU}(n)$ subalgebra

   $$H_a = b_a^\dagger b_a - \frac{1}{2}, \quad E_{ab} = b_a^\dagger b_b, a \neq b.$$

   Hint: consider states of fixed particle number $\sum_a H_a = k$.

   First check that these operators actually do satisfy the $\mathsf{SU}(n)$ algebra.

   I recommend starting with the case of $n = 2$.

   (b) Can you make representations in this way of the full $\mathsf{SO}(n)$ algebra which includes

   $$E'_{ab} = b_a^\dagger b_b^\dagger, a \neq b.$$

   What about $b_a^\dagger b_b^\dagger$ with $a = b$?

4. **Swap operator on qudits.** Let $\mathcal{H}_n = \text{span}\{|i\rangle, i = 1..n\}$ be an n-dimensional Hilbert space. In terms of the generators of $\mathsf{SU}(n)$ in the fundamental representation $(T_A)_{ij}$, write a formula for the *swap* operator S on $\mathcal{H}_n \otimes \mathcal{H}_n$. The swap operator is defined by its action on the basis states:

   $$\mathsf{S}\,|ij\rangle = |ji\rangle\,.$$

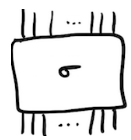
   Check that your answer makes sense for $n = 2$.

5. [Bonus problem] Compute the second Casimir for the adjoint rep of $\mathsf{SU}(n)$ using birdtracks.

   [Cultural remark: Notice that this quantity will appear in the vacuum polarization amplitude for the gluon, and hence the running of the QCD gauge coupling.]

6. **Projector onto the antisymmetric tensor rep.**

   For a positive integer k, consider the object (in the Brauer algebra)

$$P_{A^k} \equiv \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma$$  .

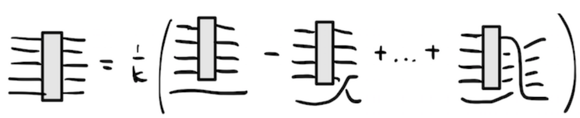
   (a) Show that it is a projector.

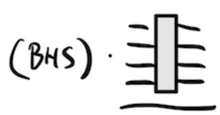
   (b) Show that its trace is

$$\mathrm{tr} P_{A^k} = \binom{n}{k} \equiv \frac{n(n-1)\cdots(n-k+1)}{k!}. \tag{1}$$

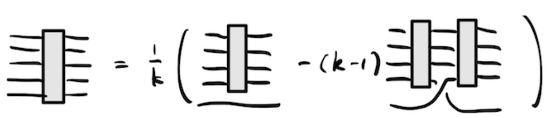
   Hint: find a recursion relation between $\mathrm{tr} P_{A^k}$ and $\mathrm{tr} P_{A^{k-1}}$.

   Elaboration of hint: First show that



   Then by doing this: 

 show that



   (c) Show that this vanishes if n is any integer less than k.

   (d) [Bonus problem] Show that the expression (1) can be used as a generating function for the number of Young diagrams with n boxes and k columns.

7. Schur-Weyl duality test. [Bonus problem]

   (a) Compute the character of the representation of S_3 on $\mathbf{n}^{\otimes 3}$ where $\mathbf{n}$ is the fundamental representation of $\mathsf{SU}(n)$.

   (b) Decompose this representation into irreps of S_3. Check that the dimensions match the expected dimensions of the associated irreps of $\mathsf{SU}(n)$.

8. Checking the reality of spinors. [Bonus problem]

   (a) Compute the characters of the spinor representations of $\mathsf{SO}(2n+1)$ and $\mathsf{SO}(2n)$ evaluated on an element of the Cartan subgroup $e^{\mathbf{i}\theta_a H_a}$.

   (b) Verify the reality properties of the spinor reps from the behavior of the characters, by computing their Frobenius-Schur indicators or otherwise. In particular: what property of their character shows that the spinor irreps of $\mathsf{SO}(2n)$ with n odd are complex?

   To do the integrals over the group, use the *Weyl integration formula*: for a class function $F(g)$,

$$\int_G dg F(g) = \int_T d\theta F(\theta)\Delta(\theta)\bar{\Delta}(\theta)$$

   where T is the Cartan subgroup with coordinates $\theta_a, a = 1..r$, and

$$\Delta(\theta) \equiv \prod_{\text{positive roots},\alpha} \left(e^{\frac{\mathbf{i}\theta\cdot\alpha}{2}} - e^{-\frac{\mathbf{i}\theta\cdot\alpha}{2}} \right).$$

9. [Bonus problem]

   (a) Show (using birdtracks or otherwise) that the character of the adjoint representation of $\mathsf{SU}(n)$ is

$$\chi_{\mathbf{adj}}(U) = \text{tr}U\text{tr}U^\dagger - 1.$$

   (b) Conclude that the number of singlets in $\mathbf{adj}^{\otimes k}$ can be written as

$$\int_{\mathsf{SU}(n)} d\mu(U) \left(\text{tr}U\text{tr}U^\dagger - 1\right)^k$$

   where $d\mu(U)$ is the Haar measure, with $\int_{\mathsf{SU}(n)} d\mu(U) = 1$.

   (c) Using the large-n identity

$$\int_{\mathsf{SU}(n)} d\mu(U) \prod_\ell \left(\text{tr}U^\ell\right)^{m_\ell} \left(\text{tr}U^{\dagger\ell}\right)^{\bar{m}_\ell} \overset{n\gg1}{=} \prod_\ell \delta_{m_\ell \bar{m}_\ell} \ell^{m_\ell} (m_\ell)!\ , \qquad (2)$$

   conclude that the number of singlets in $\mathbf{adj}^{\otimes k}$ is $\leq k!$.

   (d) Understand what the identity (10) is true. Actually, at least for the case where $m_\ell = \delta_{\ell,1}$, this is possible using only information from the lecture notes! (No large-n assumption is required for this case.) Hint: use Schur-Weyl duality.