Groupaction of G on X is a group homomorphism

from $G \longrightarrow S_n$ $\quad n = |X|$.

$g \longmapsto (x \mapsto gx)$

- orbit of $x \equiv \underline{Gx} = \{ gx \mid g \in G \}$
- stabilizer subgroup of $x \in X \equiv \underline{G_x} \equiv \{ g \in G \mid gx = x \}$
- Fixed-point set of $g \in G \equiv X^g \equiv \{ x \in X \mid gx = x \}$

last time: $\underline{Gx} \cong G/G_x$ = set of cosets of G_x

$$\Rightarrow |G| = |G_x||Gx| \quad (\forall x \in X)$$

Not-Burnside's Lemma:

$$\#\text{ of orbits} \equiv |X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

Pf: ① $X = \dot{\bigcup}_{O \in G/X} \bigcup_{x \in O} x$ $\qquad X =$ 

$$|X/G| = \sum_{O \in X/G} 1 = \sum_{O \in X/G} \underbrace{\sum_{x \in O}}_{O = Gx} \frac{1}{|O|} = \sum_{x \in X} \frac{1}{|Gx|}$$

② Use $\frac{1}{|Gx|} = \frac{|G_x|}{|G|}$

$$\Rightarrow |X/G| = \sum_{x \in X} \frac{|G_x|}{|G|}$$

③ $\sum_{x \in X} |G_x| = |\{(g,x) \in G \times X \mid gx = x\}| = \sum_{g \in G} |X^g|$ ∎

- $\underline{\text{eg}}$: $X = G$ by conjugation. $Gx = C_x$
  $G_x = Z_x \quad X^g = Z_g$

  $\#$ of conjugacy classes $= \frac{1}{|G|} \sum_{g \in G} |X^g| = \frac{1}{|G|} \sum_{g \in G} |Z_g|$
  ("class formula")

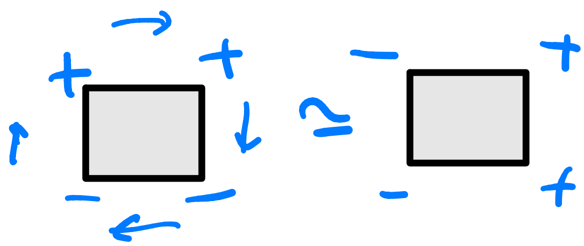
- choose k of n things w/o order.
  $X = S_n$ $\quad (\underbrace{\pi_1\ \pi_2\ \pi_3 \ldots \pi_k}_{\text{choose 1st } k.}\ \underbrace{\ldots\ \pi_n}_{n-k})$

  Action of $\underline{G = S_k \times S_{n-k}}$ regard as equivalence
  $\equiv$ $\underline{\text{gauging}}$

$\#$ of orbits $= \frac{1}{|S_k \times S_{n-k}|} \sum_{(\sigma,\pi) \in S_k \times S_{n-k}} |X^{(\sigma,\pi)}| = \frac{1}{k!(n-k)!}(n! + 0) = \binom{n}{k}$

eg: $X = \{\text{spin}_{\pm} \text{ configs on } \square\}$

$G = \{\text{rots. of } \square\} = \mathbb{Z}_4$



$$\#\text{of orbits} = \frac{1}{4}\Big(\underset{2^4}{|X^0|} + \underset{\substack{++\ --\\++\ --}}{|X^{\pi/2}|} + \underset{\substack{+-\ -+\\-+\ +-\\++\ --\\++\ --}}{|X^{\pi}|} + \underset{\text{same as }\pi/2}{|X^{3\pi/2}|} \Big)$$

$$= \frac{1}{4}\big(\underline{16+2+4+2} \big) = 6$$

$\begin{matrix}++\\++\end{matrix}$	1 orbit w/	$4+\ 0-$	$\sim p^4 m^0$
$\begin{matrix}++\\+-\end{matrix}$	1 " "	$3+\ 1-$	$p^3 m^1$
$\begin{matrix}++\\--\end{matrix}\ \ \begin{matrix}+-\\-+\end{matrix}$	2 " "	$2+\ 2-$	$p^2 m^2$
$\begin{matrix}+-\\--\end{matrix}$	1	$1+\ 3-$	$p\, m^3$
$\begin{matrix}--\\--\end{matrix}$	1	$0+\ 4-$	$p^0 m^4$
	6		

$p = e^h, \ m = e^{-h}$

$$Z = \sum_{\text{ORBITS}} e^{-\beta \underbrace{H(\text{ORBIT})}_{\text{gauge inv.}}} = 1p^4m^0 + 1p^3m + 2p^2m^2 + 1pm^3 + 1p^0m^4.$$

Weighted not-Burnside Lemma:

Given an action of G on X, a G-inv't weight function $W(x) = W(Gx)$:

$$Z \equiv \sum_{O \in X/G} W(O) = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X^g} W(x)$$

$$\xrightarrow{W(x)=1} \frac{1}{|G|} \sum_{g \in G} |X^g| .$$

Pf: $X_W = \{ x \in X \text{ s.t. } W(x) = W \}$ carries an action of G.

(add together $\sum_W X_W$, use not-Burnside for each.) ∎

Check: $$Z = \frac{1}{4}\left(\sum_{x \in X^0} W(x) + \sum_{x \in X^{\pi/2}} W(x) + \sum_{x \in X^{\pi}} W(x) + \sum_{x \in X^{3\pi/2}} W(x) \right)$$

$$= \frac{1}{4}\left((p+m)^4 + p^4 + m^4 + (p^2+m^2)^2 + p^4 + m^4 \right)$$

// Expand $= Z_{\text{above}}$ ✓

$$Z_{\text{ungauged}} = \sum_{\{s_i = \pm 1\}} e^{-\beta H(s)}$$

$$= p^4 m^0 + 4 p^3 m + \ldots$$

$$\neq Z = Z_{\text{gauged}\ \mathbb{Z}_4} = \sum_{\substack{\text{ORBITS, } O \\ \text{of } \mathbb{Z}_4 \\ \text{on } \{s\}}} e^{-\beta H(O)}$$

$$= p^4 m^0 + 1\, p^3 m + \ldots$$

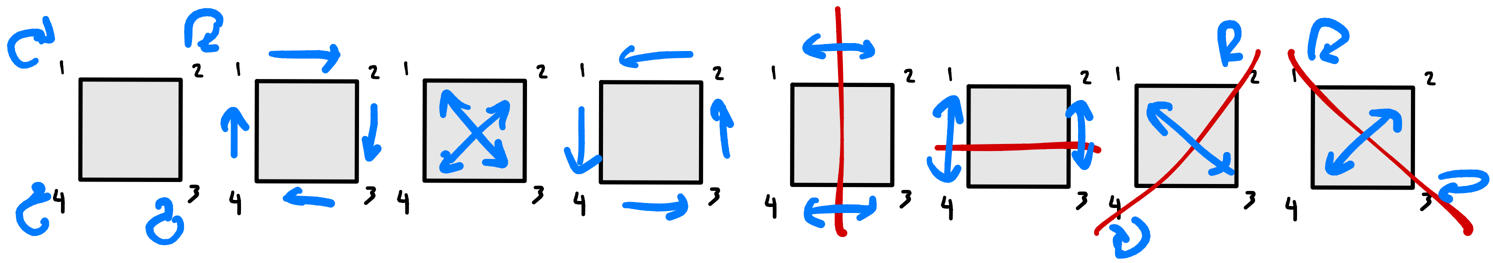
Polya Enumeration Thm:

Ass. w/ each $g \in G$ is $\sigma \in S_{n=|X|}$.

let $z(\sigma) \equiv z_1^{c_1} z_2^{c_2} \ldots$ $\quad c_j$ = # of cycles of length j in σ.

z_i = formal vars.

$$Z(G, X) \equiv \frac{1}{|G|} \sum_{\sigma \in G} z(\sigma)$$ "cycle index"



$\sigma =$ id, $\pi/2$, π, $3\pi/2$, d_y, d_x, d_{xy}, d_{yx}

$S_n \ni$ $(1)(2)(3)(4)$, (1234), $(13)(24)$, (4321), $(12)(34)$, $(14)(23)$, $(13)(2)(4)$, $(24)(1)(3)$

$Z(\sigma)$: z_1^4, z_4, z_2^2, z_4, z_2^2, z_2^2, $z_2 z_1^2$, $z_2 z_1^2$

$$Z(D_4, \square) = \frac{1}{8}\left(z_1^4 + 2 z_1^2 z_2 + 3 z_2^2 + 2 z_4 \right)$$

$$\xrightarrow[z_i = 1]{} 1 \qquad n = \sum_j j c_j$$

Given action of G on X then G acts on colorings of X ($\overset{k-}{=}$ spin configs on X) $\equiv Y$

$$Y \ni y = \{(x, s(x)) \mid x \in X\} \qquad s(x) \in \{1..k\}$$

G acts on Y by: $\bar{\sigma}(y) = \bar{\sigma}(\{(x, s(x))\})$
$= \{(\sigma(x), s(x))\}$.

$$X = \overset{\uparrow}{\underset{1}{\bullet}} \ \overset{\downarrow}{\underset{2}{\bullet}} \xrightarrow{(12)} \overset{\downarrow}{\underset{1}{\bullet}} \ \overset{\uparrow}{\underset{2}{\bullet}}$$

Polya Enumeration Thm:

$$\sum_{\text{orbits } O \in Y/G} W(O) = Z(G, X)\Big|_{z_i = b_1^i + b_2^i + \dots b_k^i}$$

$$W(y) = b_1^{\#\text{ of } x \text{ w/ } s(x)=1}\, b_2^{\#\text{ of } x \text{ w/ } s(x)=2} \dots$$

Pf: wtd not-Burnside $\Rightarrow$ LHS is

$$\sum_{O \in Y/G} W(O) = \frac{1}{|G|} \sum_{\sigma \in G} \sum_{y \in Y^{\bar\sigma}} W(y)$$

$y \in Y^{\bar\sigma}$ $\quad$ y is fixed by $\bar\sigma$ $\quad$ ie

$$y = \{(x, s(x))\} \overset{\downarrow}{=} \{(\sigma(x), s(x))\}$$

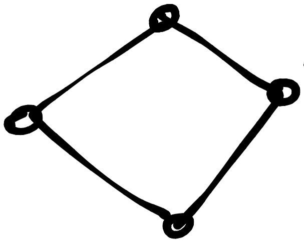
$$y = \{(x, s(x)) \mid x \in X\} = \{(\sigma(x), s(\sigma(x))) \mid x \in X\}$$

$$\iff \underline{s(x) = s(\sigma(x))} \quad \forall x \in X$$

$\rightarrow$ all pts in a given cycle have same color.

$$\sum_{y \in Y^{\bar\sigma}} W(y) = \underbrace{(b_1 + b_2 + \dots b_k)}_{\text{ways of coloring a 1-cycle}}{}^{c_1} (b_1^2 + b_2^2 + \dots b_k^2)^{c_2} \cdots (b_1^n + \dots)^{c_n}$$

$$= Z(\sigma)\Big|_{z_i = b_1^i + b_2^i + \dots b_k^i} \quad \blacksquare$$



spin 1 identical boson.

$$H = -\sum_i B\, S_i^z$$

$S_i^z : \{-1, 0, +1\}$
R G B

$$Z = Z(D_4, \square)\Big|_{z_i \to R^i + G^i + B^i}$$

$$\left(\begin{array}{l} R = e^{-\beta h} \\ G = e^{0} \\ B = e^{+\beta h} \end{array}\right)$$

$$= \frac{1}{8}\left(z_1^{\,4} + 2 z_1^2 z_2 + 3 z_2^{\,2} + 2 z_4\right)\Big|_{z_i \to R^i + G^i + B^i}$$

$z_i \equiv z[i]$

```
/. {z[i_] :> R^i + G^i + B^i}
```

$\text{Coefficient}(\downarrow,\ RBG^2) = 2$

= # of necklaces w/ 1 Red, 1 B, 2 Green.

2. Representations of Groups

Def:

A rep. R of G associates a linear op.

$D_R(g) : V_R \to V_R$ to each element of G.

s.t.
- $D_R(e) = \mathbb{1}$
- $\underline{D_R(g_1) D_R(g_2) = D_R(g_1 g_2)}$.

(ie D_R is a group homomorphism $G \to \underline{GL(d_R, \mathbb{C})}$)

$d_R = \dim V_R \equiv$ dim. of rep.

$V_R =$ "carrier space".

V_R is a vector space over $\mathbb{C}$.

Motivations: ① $V = \mathcal{H}$.

special role for $\underline{\text{unitary}}$ reps : $D(g)^\dagger D(g) = \mathbb{1} \quad \forall g$.

If $D(g_1) D(g_2) = D(g_1 g_2) e^{i\phi(g_1, g_2)}$

$\equiv$ **projective representation**.

② learn about G by doing linear algebra

$$V = \text{span}\{|i\rangle\}_{i=1\sim n} \qquad \mathbb{1} = \sum_i |i\rangle\langle i|$$

let $D = D(g)$.

$$\langle i|D|j\rangle \equiv D_{ij}$$

$$(D(g_1g_2))_{ij} = (D(g_1)\,D(g_2))_{ij} = \sum_k (D(g_1))_{ik}\,(D(g_2))_{kj}$$

$\mathbb{1} = \sum_k |k\rangle\langle k|$

matrix mult.

examples : • trivial rep. $\underline{1}$ $\qquad D(g) = 1 \quad \forall g \in G$

dim $= 1$.

• A 1d rep of $\mathbb{Z}_n$ $= \langle g \mid g^n = e\rangle$:

$$D(e) = 1$$

$$D(g) = \omega = e^{2\pi i/n}$$

$$D(g^2) = \omega^2$$

$\vdots$

If $n > 2$: $D'(g) = \omega^2$ is another.

$\vdots$