A group action is a group hom. $G \to S_n$

" " representation " " " $G \to GL(n, \mathbb{C})$

A unitary " " " " $G \to U(n)$.

examples: • trivial rep $D_{\underline{1}}(g) = 1$.

• regular rep of G acts on

$$\mathcal{H}_G \equiv \text{span}_{\text{ON}}\{|g\rangle, g \in G\}$$

$$D(g_1)|g_2\rangle \equiv |\underbrace{g_1 g_2}_{\text{group product}}\rangle$$

is a rep: $D(e)|g_2\rangle = |eg_2\rangle = |g_2\rangle$

$D(g_1)D(g_2)|g_3\rangle = |g_1 g_2 g_3\rangle \overset{!}{=} D(g_1 g_2)|g_3\rangle$ ✓

$$\dim R_{\text{reg}} = |G|$$

eg: $G = \mathbb{Z}_3 = \langle g | g^3 = e \rangle$ $\quad D(e) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$$D(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D(g^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

New reps from old: Given $R_{1,2}$ reps of G

direct sum: $R_1 \oplus R_2$. carrier space is $V_1 \oplus V_2$

$$D_{R_1 \oplus R_2}(g) = \begin{pmatrix} D_{R_1}(g) & 0 \\ \underbrace{0}_{\dim R_2 \times \dim R_1} & D_{R_2}(g) \end{pmatrix}$$

$0 \leftarrow \dim R_1 \times \dim R_2$

$\dim(R_1 \oplus R_2) = \dim R_1 + \dim R_2$

direct product: $R_1 \otimes R_2$ on $V_1 \otimes V_2 = \mathrm{span}\{|i,\alpha\rangle\}$

$V_1 = \mathrm{span}\{|i\rangle\}$

$V_2 = \mathrm{span}\{|\alpha\rangle\}$

$$\langle i\alpha | D_{R_1 \otimes R_2}(g) | j\beta\rangle = \left(D_{R_1 \otimes R_2}(g)\right)_{i\alpha, j\beta}$$

$$\equiv \left(D_{R_1}(g) \otimes D_{R_2}(g)\right)_{i\alpha, j\beta} = \left(D_{R_1}(g)\right)_{ij} \left(D_{R_2}(g)\right)_{\alpha\beta}$$

$\dim R_1 \otimes R_2 = \dim R_1 \times \dim R_2$.

Reps of S_n: let $\text{sign}(\pi) = (-1)^{\pi}$

$$\equiv (-1)^{k_2+k_4+k_6+\dots} \leftarrow \text{\# of even-length cycles of } \pi$$

$k_j \equiv$ # of j-cycles of π

$(12)(12) = e$ $\qquad$ $(12)(23) = (123)$

even-length cycles annihilate in pairs

$\text{sign}(\pi_1)\,\text{sign}(\pi_2) = \text{sign}(\pi_1 \cdot \pi_2)$

$\Rightarrow$ sign is a 1d rep of S_n.

$$\{e\} \to A_n \xrightarrow{i} S_n \xrightarrow{\text{sign}} \mathbb{Z}_2 \xrightarrow{j} \{e\}$$

"exact seq."

$\text{Im}(\text{sign}) = \ker(j)$

$A_n \equiv \ker(\text{sign}) = \text{Im}(i)$

alternating subgroup.

$\ker(j) \equiv \{$ elements g of G w/ $j(g) = e \}$

$j: G \to H$

is a normal subgroup $\Rightarrow$ $S_n / A_n = \mathbb{Z}_2$.

Defining or fundamental rep of S_n : $\boxed{S_n < U(n).}$

$$\mathcal{H} = \underset{ON}{\text{span}} \{ |j\rangle \ \ j = 1..n \}$$

$$D(\pi)|j\rangle = |\pi_j\rangle$$

eg $n=3$ $\quad D(123)|1\rangle = |2\rangle \quad D(123)|2\rangle = |3\rangle$

$D(123)|3\rangle = |1\rangle$

$D(12)|1\rangle = |2\rangle \quad D(12)|2\rangle = |1\rangle$

$D(12)|3\rangle = |3\rangle$

$$D(e)_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}_{ij} \qquad D(123)_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{ij}$$

$$D(12)_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ij} \quad \ldots$$

"permutation matrices"

Equivalence of reps: changing basis of V preserves reps

$D' \sim D$ iff $D'(g) = S^{-1} D(g) S$ $(\det S \neq 0)$

$(D' = D)$

(same S $\forall g$!!)

Reducibility: A rep is <u>reducible</u> if it has an <u>invariant subspace</u>

$\equiv W \subset V$ s.t. $D(g)|w\rangle \in W$

$\forall w \in W, g \in G$.

Let P_W be the projector onto W.

$= \sum_{w\in W} |w\rangle\langle w|$ $\qquad P_W^2 = P_W$.

W is an inv't subspace $\iff \underline{P_W} D(g) P_W = \underbrace{D(g) P_W}_{\in W}$ $\forall g \in G$

$\Longrightarrow D_W(g) \equiv P_W D(g) P_W$

form a rep. on carrier sp W.

<u>eg</u>: $R_{reg.}$ $\qquad |u\rangle \equiv \underline{\sum_{g\in G} |g\rangle} \xrightarrow[\text{sudoku}]{D(h)} |u\rangle$

$P_W = |u\rangle\langle u|$ is a 1d inv't subspace.

[<u>eg</u>: $G = \mathbb{Z}_3$ $\qquad u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $P_W = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$]

$R_{reg.} = \underline{1} \oplus R_{|G|-1}$

trivial

A rep is completely reducible or decomposable if

$$S^{-1} D(g)\, S = \begin{pmatrix} D_1(g) & 0 & 0 \\ 0 & D_2(g) & 0 \\ 0 & 0 & D_3(g) \\ & & & \ddots \end{pmatrix}$$

$$D \sim D_1 \oplus D_2 \oplus D_3 \oplus \dots$$

eg: reg. rep of $\mathbb{Z}_3$. all $D(g)$ commute

$$D'(e) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad D'(g) = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix}$$

$$D'(g^2) = \begin{pmatrix} 1 & & \\ & \omega^2 & \\ & & \omega \end{pmatrix} \quad \omega = e^{2\pi i/3}.$$

$$D(g) = \underline{S}\, D'(g)\, S^{-1} \qquad D' = \underline{1} \oplus \underline{1}_1 \oplus \underline{1}_2$$

Why? Indecomposable rep: $\begin{pmatrix} D_1(g) & B \\ 0 & D_2(g) \\ & & \ddots \end{pmatrix}$

eg: $D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ is a rep of $\mathbb{Z}$ under $+$.

$$D(x)D(y) = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix} = D(x+y).$$

is reducible: $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad D(x)P = P$

$\Rightarrow PD(x)P = P$
$= D(x)P$

$\Rightarrow$ Im(P) is inv't subspace.

Im($\mathbb{1}$-P) is NOT

$D(x)(\mathbb{1}-P) \neq \mathbb{1} - P$

$= \begin{pmatrix} 0 & x \\ 0 & 1 \end{pmatrix}$

This can happen for $\mathbb{Z}$ because it is NON-COMPACT.

Thm A: Any unitary rep'n R is decomposable.

Pf: If R is reducible $\exists P = P^\dagger$

$\left(P D(g) P = D(g) P \right)^\dagger \quad \forall g \in G$

$P \underbrace{D(g)^\dagger}_{} P = P D(g)^\dagger \qquad \forall g \in G$

unitary: $\quad = D(g)^{-1} = D(g^{-1})$

$P D(h) P = P D(h) \qquad \forall\, h = g^{-1} \in G.$

$\Leftrightarrow (\mathbb{1}-P) D(g) (\mathbb{1}-P) = D(g)(\mathbb{1}-P)$ $\quad \Rightarrow \mathbb{1}-P$ is an invariant projector.

$\cancel{D} - PD - \cancel{DP} + PDP = \cancel{D} - \cancel{DP}$

Thm B: For a compact group, every rep is equivalent to a unitary rep!

Pf: let $S \equiv \sum_{g \in G} D(g)^\dagger D(g)$

- $S^\dagger = S$. $\Rightarrow S = \sum_a a |a\rangle\langle a|$
- $\underline{S \geq 0}$ $\rightsquigarrow\Rightarrow \sqrt{S} = \sum_a \sqrt{a} |a\rangle\langle a| = (\sqrt{S})^\dagger$.

claim: $S > 0$ (so $\sqrt{S}$ is invertible).

pf: if not, $\exists v$ s.t. $Sv = 0$.

$$0 = v^\dagger S v$$
$$= \sum_{g \in G} v^\dagger D^\dagger(g) D(g) v$$
$$= \sum_{g \in G} \| D(g) v \|^2 \Longleftrightarrow D(g)v = 0 \;\; \forall g.$$

But $D(e) = \mathbb{1}$ contradiction

claim: $D'(g) \equiv \sqrt{S}\, D(g) \sqrt{S}^{-1}$ are unitary

$$D'(g)^\dagger D'(g) = \sqrt{S}^{-1} D(g)^\dagger \underbrace{\sqrt{S}\sqrt{S}}_{S} D(g) \sqrt{S}^{-1}$$

$$= \sqrt{S}^{-1} D(g)^\dagger \sum_h D(h)^\dagger D(h) D(g) \sqrt{S}^{-1}$$

$$= \sqrt{S}^{-1} \sum_h D(g)^\dagger D(h)^\dagger D(h) D(g) \sqrt{S}^{-1}$$

$$= \sqrt{S}^{-1} \sum_h \underline{D(hg)}^\dagger D(hg) \sqrt{S}^{-1}$$

$h' = hg$

$$= \sqrt{S}^{-1} S \sqrt{S}^{-1} = \sqrt{S}^{-1} \sqrt{S} \sqrt{S} \sqrt{S}^{-1}$$

$$= \mathbb{1}. \quad \blacksquare$$

(for infinite but compact groups,

$$\sum_{g \in G} D^\dagger(g) D(g) \longrightarrow \int_g D^\dagger(g) D(g).$$
)

If a rep is <u>NOT reducible</u>, it's an <u>irrep</u>.

<u>Schur's lemma</u> : given $A_\alpha : U \to U$ (e.g. $\alpha \in G$)

$B_\alpha : V \to V$

irreducible.

and an <u>intertwiner</u> $\Lambda : U \to V$

$$\underline{\Lambda A_\alpha = B_\alpha \Lambda} \quad \forall \alpha . \quad (*)$$

then either a) $\Lambda = 0$

or b) Λ is a bijection, $\dim U = \dim V$

and $A_\alpha = \Lambda^{-1} B_\alpha \Lambda$.

<u>Pf</u>: $\ker(\Lambda) \subset U$ $\quad \ker(\Lambda) \equiv \{ |u\rangle \in U \text{ s.t. } \Lambda|u\rangle = 0 \}$

is an invariant subspace.

If $|u\rangle \in \ker(\Lambda)$ $\quad \Lambda|u\rangle = 0$

$$\Lambda \underbrace{A_\alpha |u\rangle} = B_\alpha \underbrace{\Lambda |u\rangle} = 0 .$$

$$\Rightarrow A_\alpha |u\rangle \in \ker(\Lambda) .$$

$\mathrm{Im}(\Lambda) \subset V$ $\quad \mathrm{Im}\,\Lambda \equiv \{ |v\rangle = \Lambda|u\rangle, |u\rangle \in U \}$

is, too.

A_α irreducible $\Rightarrow$ only invt subspaces are U or $\underline{\underline{0}}$.
B_α " " " V or 0

$\Rightarrow$ either $\ker \Lambda = U$, $\mathrm{Im}\,\Lambda$, ie $\Lambda = 0$.
OR $\ker \Lambda = 0$, $\mathrm{Im}\,\Lambda = V$ ie $A_\alpha = \Lambda^{-1} B_\alpha \Lambda$. ∎

<u>Corollo.</u> If $\{A_\alpha\}$ act irreducibly on V
$(V=U)$ and $\underline{\underline{\Lambda A_\alpha = A_\alpha \Lambda}}$ $\forall \alpha$

Then $\Lambda = \lambda \mathbb{1}_V$.

<u>Pf:</u> $(\Lambda - x\mathbb{1})A_\alpha = A_\alpha(\Lambda - x\mathbb{1})$

$\det(\Lambda - x\mathbb{1}) = P_{\dim V}(x)$ has a root at $x = \lambda$

$\Rightarrow \Lambda - \lambda\mathbb{1}$ is not invertible. $\xRightarrow[\text{Schur}]{} \underline{\Lambda - \lambda\mathbb{1} = 0}$ ∎

<u>Consequence:</u> If $S^{-1} D(g) S = D(g)$ $\forall g \in G$

$\xrightarrow{\text{Schur}}$ $\underline{S = \lambda\mathbb{1}}$. $\Rightarrow$ canonical form of irrep.

Grand Orthogonality Fact:

Let $D_{a/b}(g) : V_{a/b} \to V_{a/b}$ be an irrep of G (a compact group ✓)

& $\dim V_{a/b} = d_{a/b}$

<u>then</u>:

$$\frac{1}{|G|} \sum_{g \in G} \left(D^a(g^{-1}) \right)_{ij} \left(D^b(g) \right)_{kl} = \frac{1}{d_a} \delta_{jk} \delta_{il} \delta^{ab}$$

for unitary reps:

$$\frac{1}{|G|} \sum_{g \in G} \left(D^a(g) \right)^{+}_{ij} \left(D^b(g) \right)_{kl} = \frac{1}{d_a} \delta_{jk} \delta_{il} \delta^{ab}$$

Pf: Any op $M : V_a \to V_b$

let $$\Lambda^M = \sum_{g \in G} D^a(g^{-1}) M D^b(g) : V^b \to V^a$$

<u>claim</u>: is an intertwiner

$$D^a(g) \Lambda^M = \Lambda^M D^b(g) \quad \forall g \in G.$$

<u>why</u>: $D^a(g) \sum_{h \in G} D^a(h^{-1}) M D^b(h)$

$$= \sum_{h' = hg^{-1}} D^a(gh^{-1}) M D^b(h) = \sum_{h'} D^a(h'^{-1}) M D^b(h'g)$$
$$= \Lambda^M D^b(g).$$

$\Rightarrow$ either $\underline{\underline{\Lambda^M = 0}}$ or $\left(\begin{array}{l} d_a = d_b \\ \text{and } D^a(g) = \Lambda D^b(g) \Lambda^{-1} \\ \Rightarrow R_a = R_b . \end{array} \right.$

corollary $\Longrightarrow$ $\Lambda^M = \lambda \mathbb{1}$

$$\Lambda^M_{il} = \sum_g \left(D^a(g^{-1})\right)_{ij} \cdot \underline{\underline{M_{jk}}} \left(D^b(g)\right)_{kl} = \widetilde{\lambda(M) \delta_{il}} \delta^{ab}$$

take $M = \begin{cases} 0 \\ 1 \text{ in } jk \text{ entry} \end{cases}$

$$\Rightarrow \sum_g D^a(g^{-1})_{ij} \; D^b(g)_{kl} = \underline{\underline{\lambda_{jk} \delta_{il} \delta^{ab}}}$$

$\underline{\underline{\text{set } a=b}}$ $\underline{\underline{\text{contract}}}$ $i = l$ $\equiv$ multiply by δ_{il}
sum over i, l.

$$\sum_g \underbrace{D^a(g)_{kl} D^a(g^{-1})_{lj}}_{} = \lambda_{jk} \cdot d_a$$

$$D^a(\underbrace{g g^{-1}}_{e})_{kj} = \delta_{kj}$$

$$= |G| \delta_{jk} \quad \Rightarrow \quad \boxed{\lambda_{jk} = \frac{|G|}{d_a} \delta_{jk}}$$

∎