Recap: $\mathbb{C} \ni \chi_R(g) = \mathrm{tr}_{V_R} D_R(g) = \chi_R(hgh^{-1})$

characters of irreps $\chi_a^{\alpha} \equiv \mathrm{tr}\, D_a(g_\alpha)$ $\quad g_\alpha \in C_\alpha$

form a basis for class functions on G.

$$\text{Any } R = \bigoplus_{\substack{\text{irreps} \\ a}} R_a \otimes \underbrace{V_a^R}_{\dim V_a^R = m_a^R} = \bigoplus_a R_a^{\oplus m_a^R}$$

$$m_a^R = \langle \chi_a, \chi_R \rangle \equiv \frac{1}{|G|} \sum_\alpha n_\alpha \chi_a^*(\alpha) \chi_R(\alpha)$$
$$= \sum_{g \in G} \chi_a^*(g) \chi_R(g) .$$

Reconstruct an irrep from its character:

$$\chi_{\underline{2}} \begin{pmatrix} (1) \\ (12) \\ (123) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \leftarrow$$

$[(123), (132)] = 0.$ $\qquad \chi_2(123) = \chi_2(321) \overset{!}{=} -1$ ✓

$\underline{(123)^3 = 1}.$ $\Rightarrow$ evals of $D(123) \in \{1, \omega, \omega^2\}$

• $D(123) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$ $\qquad D(321) = \begin{pmatrix} \omega^2 & \\ & \omega \end{pmatrix}.$

$\mathrm{tr}\, D(12) = \chi_2(12) \overset{!}{=} 0$ $\qquad (12)(123)(12) = (321)$

$\Rightarrow$ • $D(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^x$ $\quad \rightarrow D(23) = D(123) D(12) D(312) = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix} \ldots$

G.O.T : $\sum_g D^a(g^{-1})_{ij} D^b(g)_{kl} = \frac{|G|}{d_a} \delta^{ab} \delta_{il} \delta_{jk}$

eg: $a = \underline{2}$, $b = \underline{1}$ $\Rightarrow \forall i j \quad \sum_g D^a(g)_{ij} = 0.$

$a = \underline{2}$, $b = \underline{1}'$ $\Rightarrow \forall i j \quad \sum_g (-1)^g D^a(g)_{ij} = 0$ $\blacksquare$

Building the character table w/o irreps :

Let N = # of irreps = # of conj. classes.

given: $n_\alpha = |C_\alpha|$. $\left(\sum_\alpha n_\alpha = |G| \right)$

$\chi_a^\alpha \equiv \chi_a(\alpha)$ is an $N \times N$ matrix, satisfying :

$\forall \alpha, \beta$: $\sum_{a=1}^{N} \chi_a(\alpha)^* \chi_a(\beta) = \delta_{\alpha\beta} |G| / n_\alpha$ $\quad (N^2)$

$+$

AND $\forall a, b$: $\sum_{\alpha=1}^{N} n_\alpha \chi_a(\alpha)^* \chi_b(\alpha) = \delta_{ab} |G|$ $\quad (N^2)$

$2N^2$

S_4 character table :

① find C_α & n_α.

$$n_\alpha = \frac{n!}{\prod_j j^{k_j} \underline{\underline{k_j!}}}$$

	n_α
e	1
(12)	6
$(12)(34)$	3
(123)	8
(1234)	6

② $|G| = \sum_a d_a^2 = 1 + 1 + a^2 + b^2 + c^2 \overset{!}{=} 24$

$= 1^2 + 1^2 + 2^2 + 3^2 + 3^2$

we know: $d_1 = 1$, $d_{1'} = 1$

③

		n_α	$\underline{1}$	$\underline{1'}$	$\underline{2}$	$\underline{3}$	$\underline{3'}$
.	e	1	1	1	2	3	3
$\mathbb{Z}_2$	(12)	6	1	-1			
$\mathbb{Z}_2$	$(12)(34)$	3	1	1			
$\mathbb{Z}_3$	(123)	8	1	1			
$\mathbb{Z}_4$	(1234)	6	1	-1			

③' If $\underline{g_\alpha^n = e}$. $(\chi_{1d}(g_\alpha))^n = 1$. $\Rightarrow \chi_{1d}(g_\alpha) \in \{1, \omega, \omega^2 \ldots\}$

$\omega = e^{2\pi i/n}$.

④ S_4 has a $\underline{4}$ is reducible since $|u\rangle = \sum_j |j\rangle$ is $\underline{\text{invariant}}$ (in $\underline{1}$.)

$$\chi_4\begin{pmatrix} e \\ (12) \\ (12)(34) \\ (123) \\ (1234) \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$\chi_1 + \chi_3$$

is reducible because

$$\langle \chi_4, \chi_4 \rangle = \frac{1}{24}\left(1\cdot 4^2 + 6\cdot 2^2 + 3\cdot 0^2 + 8\cdot 1^2 + 6\cdot 0^2\right)$$

$$= \frac{16+24+8}{24} = 2.$$

$$\chi_4 = \chi_x + \chi_y \Rightarrow \langle \chi_x+\chi_y, \chi_x+\chi_y \rangle$$

$$= \langle \chi_x, \chi_x \rangle + \langle \chi_y, \chi_y \rangle + \cancel{\langle \chi_x, \chi_y \rangle} + \cancel{\langle \chi_y, \chi_x \rangle}$$

$$= 2.$$

			n_α	$\underline{1}$	$\underline{1}'$	$\underline{2}$	$\underline{3}$	$\underline{3}'$
		e	1	1	1	2	3	3
$\mathbb{Z}_2$		(12)	6	1	-1		1	
$\mathbb{Z}_2$		$(12)(34)$	3	1	1		-1	
$\mathbb{Z}_3$		(123)	8	1	1		0	
$\mathbb{Z}_4$		(1234)	6	1	-1		-1	

⑤ $\underline{1}' \otimes \underline{3} = ?$ is a 3d rep.

$\chi_{\underline{1}'\otimes\underline{3}} = \chi_{1'} \otimes \chi_3$ has norm 1.

$$\langle \chi_{1'\otimes 3}, \chi_3 \rangle = \frac{1\cdot 3^2 + 6\cdot 1(-1) + 3\cdot 1(-1) + 0 + 6(-1)1}{24}$$

$$= \frac{9-6+3-6}{24} = 0.$$

			n_α	$\underline{1}$	$\underline{1}'$	$\underline{2}$	$\underline{3}$	$\underline{3}'$
·		e	1	1	1	2	3	3
$\mathbb{Z}_2$		(12)	6	1	-1	$\cancel{w} \to 0$	1	-1
$\mathbb{Z}_2$		$(12)(34)$	3	1	1	x	-1	-1
$\mathbb{Z}_3$		(123)	8	1	1	y	0	0
$\mathbb{Z}_4$		(1234)	6	1	-1	$\cancel{z} \to 0$	-1	1

⑥ $\sum_a |\chi_a(\alpha)|^2 = |G|/n_\alpha$.

r_2^2: $4 + |w|^2 = 24/6 = 4 \Rightarrow w = 0.$

r_3^2: $4 + |x|^2 = 24/3 = 8$ $\quad |x|^2 = 4$

r_4^2: $2 + |y|^2 = 24/8 = 3 \Rightarrow |y|^2 = 1$

r_5^2: $4 + |z|^2 = 24/6 = 4 \Rightarrow |z|^2 = 0.$

⑦ For S_n g^{-1} has the same cycle structure as g.

$$\left(\text{eg } (1234)^{-1} = (4321) \right)$$

$$\Rightarrow [g^{-1}] = [g]. \Rightarrow \chi(g^{-1}) = \chi(g)^* = \chi(g).$$

$$\Rightarrow x = \pm 2, \quad y = \pm 1$$

$\underline{r_1 \perp r_3}$: $1 + 1 + 2x - 3 - 3 \overset{!}{=} 0$

$$\Rightarrow \underline{x = 2}.$$

$\underline{r_1 \perp r_4}$: $1 + 1 + 2y = 0 \Rightarrow y = -1.$

S_4

		n_α	$\underline{1}$	$\underline{1'}$	$\underline{2}$	$\underline{3}$	$\underline{3'}$
	e	1	1	1	2	3	3
$\mathbb{Z}_2$	(12)	6	1	-1	0	1	-1
$\mathbb{Z}_2$	$(12)(34)$	3	1	1	2	-1	-1
$\mathbb{Z}_3$	(123)	8	1	1	-1	0	0
$\mathbb{Z}_4$	(1234)	6	1	-1	0	-1	1

$\mathbb{Z}_3$			
	1	ω	ω^2
	ω	ω^2	1
	ω^2	ω	1

$\underline{x^n = 1}$.

$$x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots$$

$a_j \in \mathbb{Z}$.

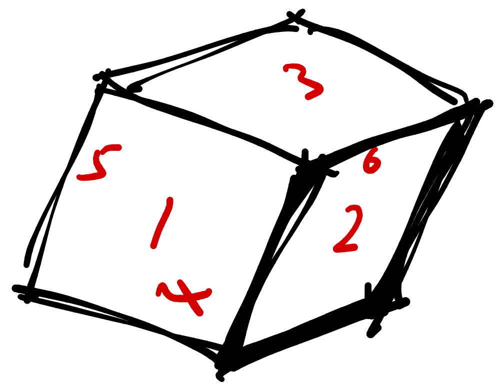
x is an "algebraic integer".

$\exists\, a, \alpha$

__eg__: for A_5 $\quad \chi_a(\alpha) = \frac{1+\sqrt{5}}{2}$.

(rotational symmetry of buckyball.)

CUBE example.

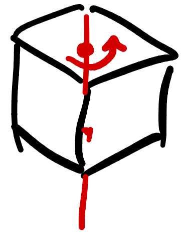


$$a_i(t+1) = \sum_{\substack{\text{nbrs} \\ j \sim i}} \frac{a_j(t)}{4}$$

Group of rotations $(\in SO(3))$ which map cube to itself?

$\equiv O$.

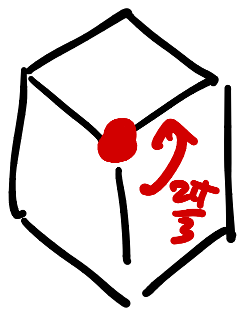
1 · e fixes all faces

6+3 ·  $\frac{\pi}{2}, \frac{3\pi}{2}$ or π $\times 3$ axes fix 2 faces

$\underbrace{\frac{\pi}{2}, \frac{3\pi}{2}}$: order 4, 2×3

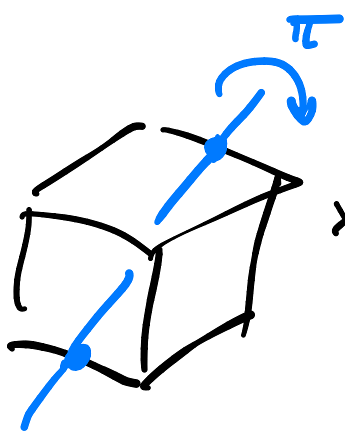
$\underbrace{\pi}$: order 2, $\times 3$

(1234) $(13)(24) = (1234)^2$

8 ·  "large diagonal" = axis through 2 distant vertices. $\times 4$ fixes 0 face.

(123)

order 3: $\frac{2\pi}{3}, \frac{4\pi}{3}$

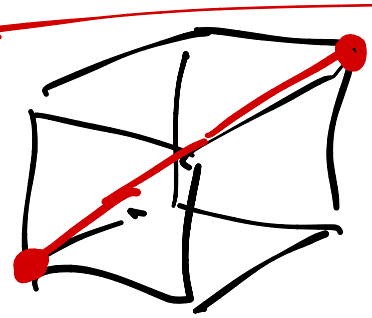
6 · 

$\times 6$ opposite edges. fixes 0 faces.

(12)

24

4 large diagonals $\in$ 4 of S_4.



$\boxed{O = S_4}$

Actual sym of a perfect cube is $O_h \equiv O \times \mathbb{Z}_2$

$\mathbb{Z}_2 = \langle \underline{\underline{P}} \mid P^2 = e \rangle$. $\qquad \underline{|O_h| = 48}$.

$$D_3(P)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\det = -1 \quad \underline{\notin SO(3)}$.

whereas

$O \subset SO(3)$

$G \times \mathbb{Z}_2$	even gerade R_a^+	odd ungerade R_a^-
C_α	χ_a^α	χ_a^α
PC_α	χ_a^α	$-\chi_a^\alpha$

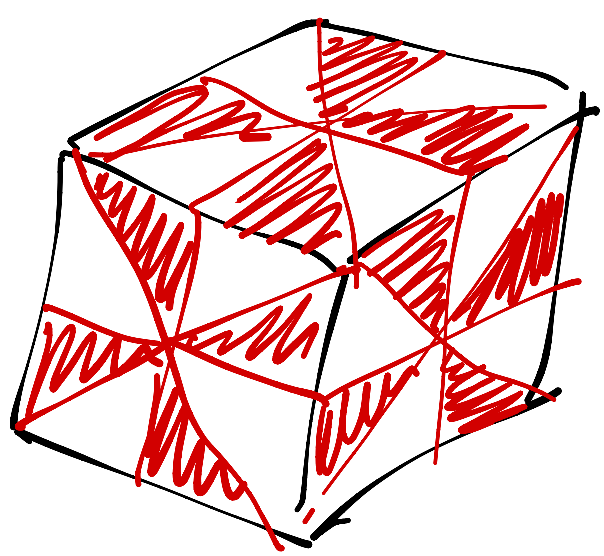
<u>WARNING about nomenclature</u>:

A rep t of $G \longrightarrow t_g, t_u$ of $G \times \mathbb{Z}_2$.

$\mathbb{Z}_n$ = "C_n"

D_n = "C_{nv}"

An object w O but not O_h symmetry :



"CHIRAL CUBE"

$$a_j(t+1) = W_j^{\ k} a_k(t).$$

$$[W, D(g)] = 0 \quad \forall g \in S_4.$$

$$\chi_{\underline{F}} \begin{pmatrix} (1) \\ (12) \\ (12)(34) \\ (123) \\ (1234) \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} = \underline{\underline{\chi}}\, m^F$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} m_1^F + \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} m_{1'}^F + \begin{pmatrix} 2 \\ 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} m_2^F + \ldots$$

with indices $\chi_{\underline{F}}(\alpha) = \chi_\alpha^{\ a}\, m_a^F$

$$m_a^F = (\chi^{-1})_a^{\ \alpha}\, \chi_{\underline{F}}(\alpha)$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}_{a} \qquad \underset{6d}{\underline{F}} = \underline{1} \oplus \underline{2} \oplus \underline{3}'$$

$$\chi^{-1} = \frac{1}{24} \begin{pmatrix} 1 & 6 & 3 & 8 & 6 \\ 1 & -6 & 3 & 8 & -6 \\ 2 & 0 & 6 & -8 & \cdots \\ \cdots \end{pmatrix}$$

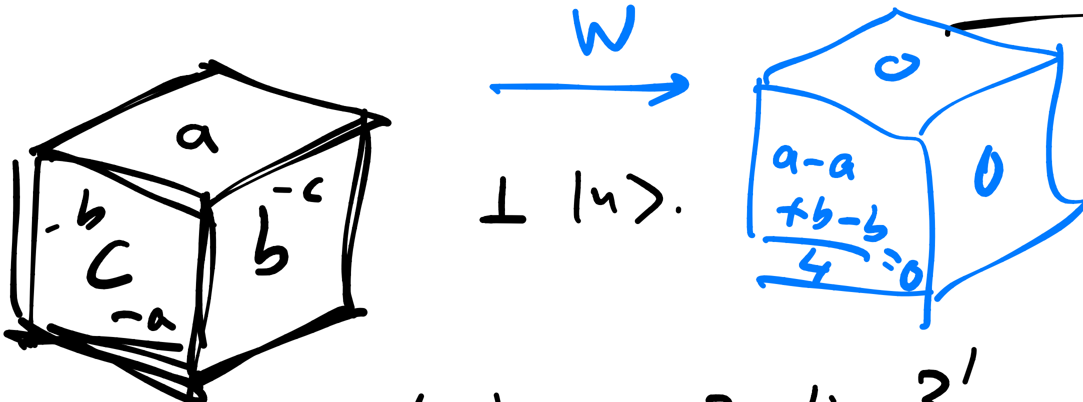
evals & evecs:

$|4\rangle = \frac{1}{\sqrt{6}} \sum_j |j\rangle \in \underline{1}$ has eval 1 under $W = \frac{1}{4} \sum_{\langle ij \rangle} |i\rangle\langle j|$

$= \underline{W^t}$.

$$|\psi\rangle = \sum_j \psi_j |j\rangle$$

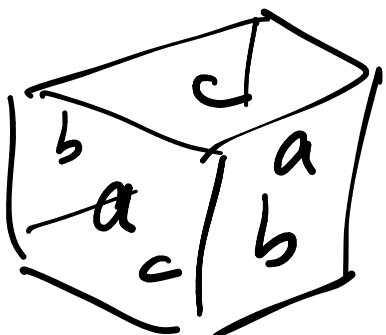
other evecs of W: $0 = \langle u|\psi\rangle \iff \sum_j \psi_j = 0$.



$\perp |u\rangle$.

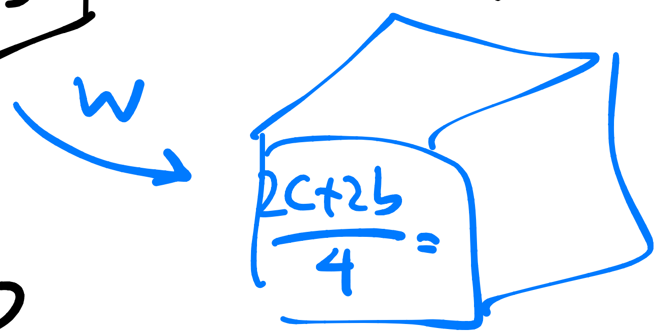
has eval 0.

is an invariant subspace of dim $\underline{3'}$.



$\underline{a+b+c=0}$

This is the $\underline{2}$.



$-\frac{a}{2}$.

has eval $-\frac{1}{2}$.

$$P_\lambda^2 = P_\lambda$$

$$W = \sum_\lambda \lambda P_\lambda \qquad \{\lambda\} = \{1, \underbrace{0, 0, 0}_{\underline{3'}}, \underbrace{-\tfrac{1}{2}, -\tfrac{1}{2}}_{\underline{2}}\}.$$

$$W^n = \sum_\lambda \underline{\underline{\lambda^n}} P_\lambda$$

$\underline{1 \text{ wins}}$

error $\sim \left(\frac{1}{2}\right)^t$