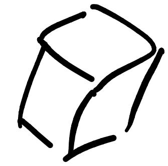
<u>Recap:</u> $W = \frac{1}{4}\sum_{\langle ij\rangle} |i\rangle\langle j| = \sum_\lambda \lambda \underbrace{P_\lambda}$ 

$$W^t = \sum_\lambda \lambda^n P_\lambda \qquad P_\lambda P_{\lambda'} = \delta_{\lambda\lambda'} P_\lambda$$

$$\{\lambda\} = \{\underset{|u\rangle}{1}, 0, 0, 0, -\tfrac{1}{2}, -\tfrac{1}{2}\}$$

Rate of approach : $\left(-\frac{1}{2}\right)^t$.

<u>Projection ops</u> : <u>CLAIM</u> : $P_a = \frac{d_a}{|G|}\sum_{g\in G} \chi_a(g)^* D(g)$

Given a rep $D(g)$,

$$D = \bigoplus_a R_a \otimes \underbrace{V_a^D}_{\dim V_a^D = m_a^D} \xrightarrow{P_a} R_a \otimes V_a^D .$$

<u>whence?</u> $\begin{cases} P_a P_b = P_a \delta_{ab} \\ \sum_a P_a = \mathbb{1} \end{cases}$.

<u>Group algebra</u> Reg. rep of $G = \mathrm{Span}_{\mathbb{C}}\{\underline{g}_i, g_i \in G\}$

a general element is $\underline{x} = \sum_{i=1}^{|G|} x_i \underline{g}_i$

<u>A product:</u> $\underline{g}\,\underline{x} = \sum_{i=1}^{|G|} x_i \underline{g g_i}$

A rep of G is also a rep of group alg. $\mathbb{C}[G]$.

$$D(\underline{x}) = D\Big(\sum_{i=1}^{|G|} x_i \underline{g_i}\Big) = \sum_{i=1}^{|G|} x_i D(\underline{g_i})$$

(like $\mathbb{C}[x]$)

$$\mathbb{C}[G] \ni e^a_{ij} \equiv \frac{d_a}{|G|} \sum_{g\in G} \big(D^a_{ij}(g)\big)^* \underline{g}$$

claim: $\underline{h}\, e^a_{ij} = \underline{\underline{D(h)^T_{ik}}}\ \underline{\underline{e^a_{kj}}}$. (6) $M^T_{ik} = M_{ki}$.

Pf: $\underline{h}\, e^a_{ij} = \frac{d_a}{|G|} \sum_g \big(D^a_{ij}(g)\big)^* \underline{hg} = \dots$

$g = h^{-1}g'$

$g' = hg$

claim: $e^a_{ij}\, e^b_{kl} = \delta^{ab} \delta_{jk}\, e^a_{il}$ ☆

Pf: $$y = \underbrace{\frac{d_a}{|G|} \sum_{h\in G} D^a_{ij}(h)^* \big(D^b(h)^T\big)_{km}}_{\overset{\text{G.O.F}}{=}\ \delta^{ab}\delta_{jk}\delta_{im}} e_{ml}$$

$l=j$

$i=k$

claim: ☆ is the mult. law for matrices $(E_{ij})_{kl} =$ 0 unless $i=k,\ j=l$.

Regular rep $\quad R_{reg} = \bigoplus_a R_a^{\oplus d_a}$

$$|G| = \sum_a d_a^2$$

$$\mathbb{C}[G] = \text{span}\,\{ e^a_{ij} \,,\ \ i,j = 1..d_a,\ a = 1..\#\text{ of irreps} \}$$

Take a rep. of G. $\quad |v\rangle \in V$ carrier space.

$$|v, aik\rangle \equiv \hat{D}(\underline{e}^a_{ik})|v\rangle \xrightarrow{g} \hat{D}(g)|v\,aik\rangle$$

$$= \hat{D}(g)\hat{D}(\underline{e}^a_{ik})|v\rangle$$

$$\overset{\text{rep}}{=} \hat{D}(g\,\underline{e}^a_{ik})|v\rangle$$

$$= \hat{D}(\underline{e}^a_{jk} D^a(g)_{ji})|v\rangle$$

$$= \underbrace{\hat{D}(\underline{e}^a_{jk})|v\rangle}\, D^a(g)_{ji}$$

$$= |V, ajk\rangle D^a(g)_{ji}$$

TRANSFORMS as R^a !

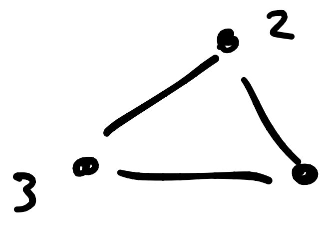
$$\Rightarrow P^a = \sum_i e^a_{ii}$$

$$= \frac{d_a}{|G|} \sum_{g\in G} \chi^a(g)^* \underline{g}$$

$$\xrightarrow{\text{rep}} P^a_D = \frac{d_a}{|G|} \sum_{g\in G} \chi^a(g)^* D(\underline{g})$$

$P_a P_b = \delta_{ab} P_b \Leftarrow$ row $\perp$, $\quad \sum_a P_a = e \Leftarrow$ def. of e.

— $G = D_3 \cong S_3, \quad R = \underline{3}$



S_3	$\underline{1}$	$\underline{1}'$	$\underline{2}$
e	1	1	2
(12)	1	-1	0
(123)	1	1	-1

$$P_1 = \frac{d_1}{6} \sum_{g \in D_3} \chi_1(g)^* D(g)$$

$$= \frac{1}{6} \sum_{g \in D_3} D(g) = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$P_{1'} = \frac{1}{6}\Big(D(e) + \big(D(123) + D(321)\big) - \big(D(12) + D(13) + D(23)\big)\Big)$$

$$= 0$$

$$P_2 = \frac{2}{6}\Big(2D(e) - \big(D(123) + D(321)\big) + 0 \Big)$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$$H_\Delta = -t \sum_{i=1}^{3} \big(|i\rangle\langle i+1| + h.c. \big) = -t \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= t P_2 - 2t P_1$$

$\underline{2}$ has energy t

$\underline{1}$ has energy $-2t$

examples from classical mech.

N particles of mass m in d dims near eqbm

$$\ddot{x}^{ia} = -\sum_{bj} H^{ia,jb}x^{jb} + \cancel{O(x^2)} \to 0$$

$x^{ia} \equiv$ deviation from eqbm of ath particle in dir i.

$$H = H^T = \partial\partial V$$

$A = ia = 1 \dots Nd.$, $X^A(t) = x^A e^{i\omega t}$

$$\Rightarrow \quad \underline{\underline{H}}^{AB}x^B = \omega^2 x^A \qquad \text{normal modes.}$$

Goal: find normal modes w/o even writing down H^{AB}.

(for symmetric cases.)

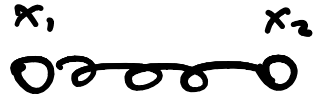
Suppose: H has symmetry G.

$x^A \in$ dN-dim'l rep of G. R

$$R = \bigoplus_a V_a^R \otimes R_a \qquad \dim V_a^R = m_a^R$$

$\Rightarrow$ we need to diagonalize $\underline{m_a \times m_a}$.

$\underline{d=1,\ N=2}$.



$$D(e) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$D(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$G = S_2 = \mathbb{Z}_2 \qquad \begin{array}{c|cc} & 1 & 1' \\ \hline e & 1 & 1 \\ g & 1 & -1 \end{array}$$

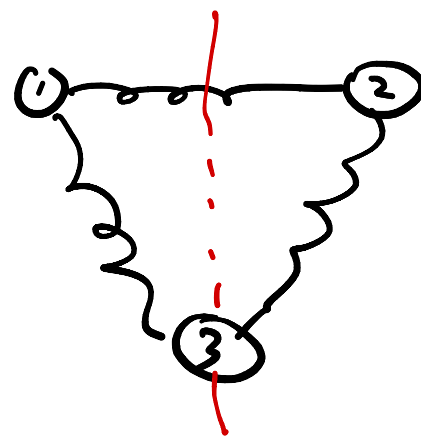
$$\chi_R\begin{pmatrix} e \\ (12) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \text{regular rep.}$$

$$\underline{R = 1 \oplus 1'}.$$

$\psi_1 = \begin{pmatrix} X \\ X \end{pmatrix}$ has $E=0$.

$\psi_{1'} = \begin{pmatrix} X \\ -X \end{pmatrix}$ has $E = 2$.

$\underline{d=2\ \ N=3}$. $\quad G = S_3 = D_3$



[Rotate by $\frac{2\pi}{3}$ & relabel $1 \to 2 \to 3$]

[reflect $x \to -x$ relabel $1 \leftrightarrow 2$.

$$\vec{X} = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \end{pmatrix} \in \underline{6} \text{ of } S_3$$

$$\chi_6\begin{pmatrix} e \\ (12) \\ (123) \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

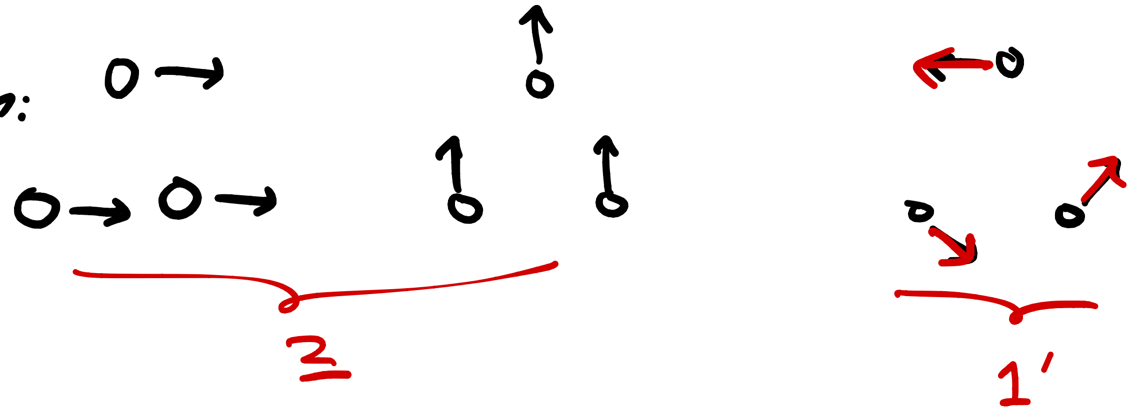
$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \to \begin{pmatrix} -x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

$$R_6 = \underline{1} \oplus \underline{1}' \oplus \underline{2} \oplus \underline{2}$$

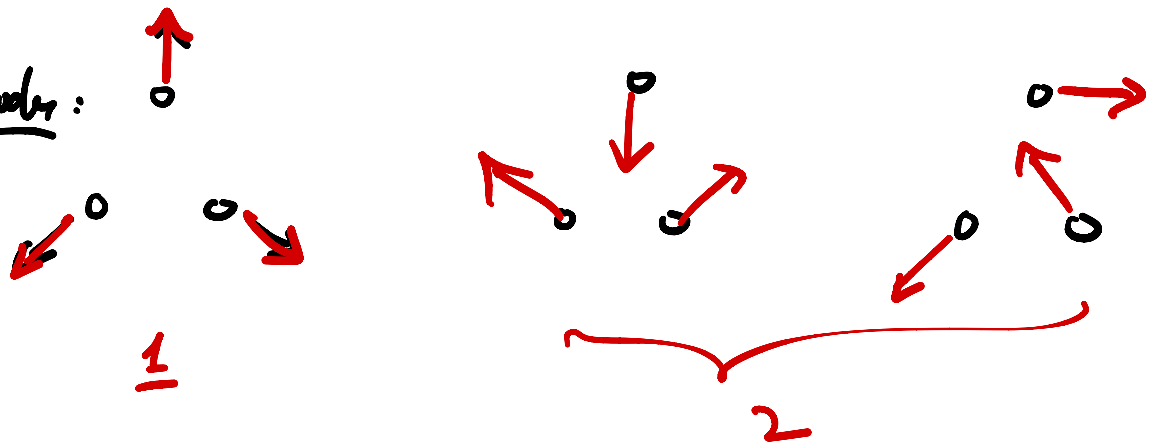
$\underline{\text{also:}}$ $R_6 = \underline{2} \otimes \underline{3} \Rightarrow \chi_6 = \chi_2\chi_3 = \chi_2(\chi_2 + \chi_1) = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$.

$\underline{2}$ of S^3 is $\left\langle \frac{2\pi}{3} \text{rot on} \begin{pmatrix} x \\ y \end{pmatrix},\ x \to -x \right\rangle$

zero modes:



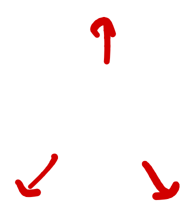
non zero modes:



$$P^{1} = \frac{1}{6}\sum_g \underbrace{D(g)}_{6\times 6 \text{ matrix}} \quad \text{has rank } 1 \qquad \underline{\underline{\mathrm{tr}\, P^{1} = 1}}.$$

(evals of $P^2 = P$ are 0 or 1.)

$$P^{1}\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}x^1\\y^1\\x^2\\y^2\\x^3\\y^3\end{pmatrix}$$



$$R_6 = \underline{1}\oplus\underline{1}'\oplus\underline{2}\oplus\underline{2}$$

$$P^{1'} = \frac{1}{6}\sum_g (-1)^g D(g) \quad \text{has rank } 1.$$

$$P^{2} = \frac{2}{6}\sum_g \chi_2^*(g)\, D(g) \quad \text{has rank } 4.$$

$$P_{trans} = |u\rangle\langle u|\otimes\begin{pmatrix}1 & \\ & 0\end{pmatrix} + |u\rangle\langle u|\otimes\begin{pmatrix}0 & \\ & 1\end{pmatrix} \quad \text{has rank } 2$$

$P_* = P^2 - P_{\text{trans}}$ has rank 2.

$$\text{Orthogonalize}[\{P_* v_1 , P_* v_2\}] = (\quad), (\quad)$$

Comment on Degeneracies: QM on $\mathcal{H}$ in a rep D of G.

$$[H, D(g)] = 0 \quad \forall g \in G.$$

$$\mathcal{H} = \text{span}\{ |a, i, \ell\rangle , \ a \in \text{irreps of } G, \ i = 1..d_a, \ \ell = 1.. m_a \}$$

$$\langle a, i, \ell | H | a', i', \ell' \rangle = \delta_{aa'} \delta_{ii'} \underbrace{\overbrace{H^a_{\ell\ell'}}^{a}}_{m_a \times m_a}$$

(Wigner Eckhart)

not group theory.

In general
$$\langle a, i, \ell | a', i', \ell' \rangle = \underline{O_{\ell\ell'}} \delta_{aa'} \delta_{ii'}$$

which multiplets can collide?

Copies of the same irrep experience level repulsion.

Recall: Block of H between colliding levels

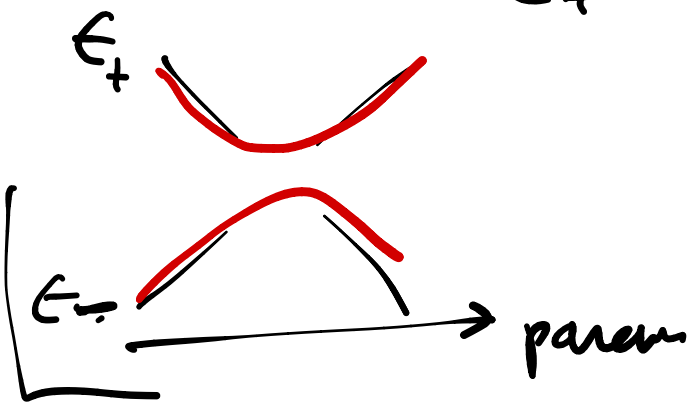
is $h = d_0 \mathbb{1} + \vec{d}\cdot\vec{\sigma}$ (2×2 hermitian)

has $\epsilon_\pm = d_0 \pm \sqrt{|\vec{d}|^2}$

$\epsilon_+ = \epsilon_- \iff d_x = d_y = d_z = 0$

3 real conditions

$\Rightarrow$ at codim 3

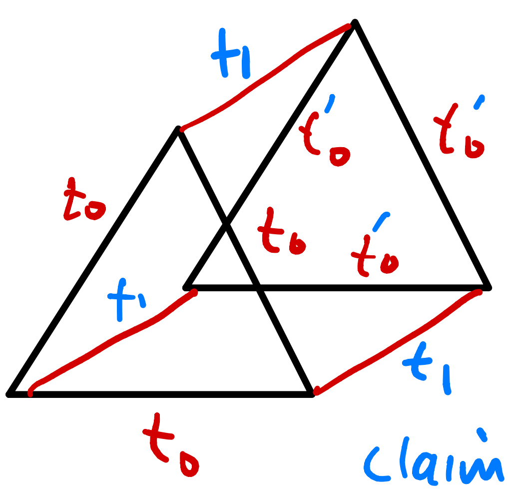


$\Rightarrow$ degeneracies occur between different irreps

$\langle\ |H|\ \rangle \propto \delta_{aa'}$

at codim 1.

eg:



$t_0 \neq t_1 \Rightarrow D_3$ sym.

(if $t_0 = t_0'$ "C_{3h}" sym)

If $t_1 = \frac{3}{2} t_0$ $\cancel{\left(= \frac{3}{2} t_0'\right)}$

claim:

A 1' collides w/ a 2.

Algebra of classes

$$\underline{C}_\alpha = \frac{1}{n_\alpha} \sum_{g \in C_\alpha} \underline{g} \quad \longleftarrow$$

a conj. class of G.

$|C_\alpha| = n_\alpha$

$$\underline{h}^{-1}\, \underline{C}_\alpha\, \underline{h} = \underline{C}_\alpha$$

$$\equiv \quad C_\alpha \in Z(\mathbb{C}[G])$$

"center of group alg."

$\boxed{\begin{array}{l} g \in G \quad \cancel{\text{"}g_1 + g_2\text{"}} \\ \underline{g} \in \mathbb{C}[G] \quad x_1 \underline{g}_1 + \underline{g}_1 x_2 \quad \checkmark \end{array}}$

claim:

An arbitrary element of $Z(\mathbb{C}[G])$ is

$$\underline{X} = \sum_\alpha x_\alpha \underline{C}_\alpha \, .$$

$\boxed{\text{Pf: ?}}$

- $\underline{C}_\alpha \underline{C}_\beta = \underline{C}_\beta \underline{C}_\alpha \in Z(\mathbb{C}[G])$

commutative product between conj. classes!

$$\underline{C}_\alpha \underline{C}_\beta = \sum_\gamma c_{\alpha\beta}^{\gamma}\, \underline{C}_\gamma$$

- $\Rightarrow \underline{\underline{c_{\alpha\beta}^{\gamma} = c_{\beta\alpha}^{\gamma}}} \, . \quad \in \frac{\mathbb{Z}}{|G|}$

Recall: fusion of irreps

$$R_a \otimes R_b = \bigoplus R_c^{\oplus m_{ab}^c}$$

$$\Rightarrow \quad \chi_a \chi_b = \sum_c m_{ab}^c \chi_c \qquad \underline{m_{ab}^c = m_{ba}^c}.$$

<u>Q</u>: Rel'n between $c_{\alpha\beta}^{\gamma}$ & m_{ab}^c?

$\gamma, \beta, \alpha = 1 \ldots$ # of conj. classes

$a, b, c = 1 \ldots$ # of irreps.