Recall: Group algebra $\mathbb{C}[G] \ni \sum_i x_i \underline{g}_i$, $x_i \in \mathbb{C}$.

$=$ regular rep w/ multiplication

$$\underline{C}_\alpha \equiv \frac{1}{n_\alpha} \sum_{g \in C_\alpha} \underline{g} \quad \in Z(\mathbb{C}[G]).$$

$$\underline{C}_\alpha \underline{C}_\beta = \underline{C}_\beta \underline{C}_\alpha = \sum_\gamma C_{\alpha\beta}^{\ \ \gamma} \underline{C}_\gamma$$

$n_c$ matrices:

$$(C_\alpha)^\gamma{}_\beta \equiv C_{\alpha\beta}^{\ \ \gamma}$$

can be simultaneously diagonalized.

claim: $\underline{C}_\alpha P^a = \lambda_\alpha^a P^a$

$$P^a = \frac{d_a}{|G|} \sum_{g \in G} \chi_a^*(g)\, \underline{g}, \quad \lambda_\alpha^a = \frac{\chi_\alpha^a}{\chi_e^a}$$

Remarks: ① find $\chi_\alpha^a$ from $C_{\alpha\beta}^{\ \ \gamma}$: $\chi_e^a = \mathrm{tr}_a D(e) = d_a$.

fix normalization by $\sum_\alpha n_\alpha |\chi_\alpha^a|^2 = |G|$.

② $\Rightarrow \chi_\alpha^a \in$ algebraic integers. (ie. $x^n + a_{n-1}x^{n-1} + \dots a_0 = 0$, $a_i \in \mathbb{Z}$.)

Pf: evals of $M$ are roots of $p(\lambda) = \det(M - \lambda \mathbb{1}) = 0$. $\quad n_\alpha (C_\alpha)^\gamma{}_\beta \in \mathbb{Z}$ ∎

"Fusion (semi-)rings" for each finite $G$.

① $\underline{C}_\alpha \underline{C}_\beta = \sum_\gamma C_{\alpha\beta}{}^\gamma \underline{C}_\gamma$ classes $\alpha, \beta, \gamma = 1 \ldots n_c$

② $R_a \otimes R_b = \bigoplus_c R_c^{\oplus m_{ab}{}^c}$

$\Rightarrow \underbrace{\chi_a{}^\alpha \chi_b{}^\alpha}_{} = \sum_c m_{ab}{}^c \chi_c{}^\alpha$   $a, b, c = 1 \ldots n_c$

$= \chi_{R_a \otimes R_b}(\alpha)$   $= m_{ba}{}^c$

$= \chi_{R_b \otimes R_a}(\alpha)$

CLAIM: $(m_a)_b{}^c$ can simult. diagonalized.

$\chi_0{}^\alpha \chi_b{}^\alpha = \chi_b{}^\alpha$. like $\underline{C}_e \underline{C}_\beta = \underline{C}_\beta$.

trivial rep.

$m_{0b}{}^c = \delta_b{}^c$   $C_{e\beta}{}^\gamma = \delta_\beta{}^\gamma$.

$m_a = S \Lambda_a S^{-1}$   $(m_a)_b{}^c = S_b{}^\alpha (\Lambda_a)_\alpha{}^\beta (S^{-1})_\beta{}^c$

w/ $(\Lambda_a)_\alpha{}^\beta = \delta_\alpha^\beta \lambda_a^\beta$.

trick: $m_0 = \mathbb{1}$. $\Rightarrow S = m_0 S$

$$S_a{}^\alpha = \sum_c m_{a0}{}^c S_c{}^\alpha = \sum_\beta S_0{}^\beta \lambda_a{}^\beta \underbrace{(S^{-1})_\beta{}^c S_c{}^\alpha}_{\delta_\beta{}^\alpha}$$

$$= S_0{}^\alpha \lambda_a{}^\alpha$$

$$\Longrightarrow \lambda_a{}^\alpha = \frac{S_a{}^\alpha}{S_0{}^\alpha}.$$

$$\Rightarrow \boxed{m_{ab}{}^c = \sum_\alpha \frac{S_a{}^\alpha S_b{}^\alpha (S^{-1})_\alpha^c}{S_0{}^\alpha}}$$

"verlinde formula".

who is $S$?

$$(m_a)_b{}^c \chi_c(\alpha) = \chi_a(\alpha)\chi_b(\alpha)$$

$$M_b{}^c \quad V_c \quad = \lambda \quad V_b$$

$\alpha$ labels evals & evecs.

{ evecs of $m_a$ are $\chi_b{}^\alpha = S_b{}^\alpha$.
{ evals of $m_a$ are $\chi_a{}^\alpha$.

$$\Rightarrow \quad m_{ab}{}^{c} = \sum_{\alpha} \frac{\chi_a{}^{\alpha}\,\chi_b{}^{\alpha}\,(\chi^{-1})_{\alpha}{}^{c}}{\chi_0{}^{\alpha}} .$$

---

analogously,

$$C_{\alpha\beta}{}^{\gamma} = \sum_{a} \frac{n_\gamma}{d_a |G|}\, \chi_a{}^{\alpha}\, \chi_a{}^{\beta}\, \bar{\chi}_a{}^{\gamma} ,$$

$$\bar{\chi}_a{}^{\gamma} \equiv (\chi_a{}^{\gamma})^*$$

---

Re Pf of $C_\alpha P^a = \lambda_a{}^{\alpha} P^a$ :

$$\Lambda_a{}^{\alpha} \equiv \frac{1}{n_\alpha} \sum_{g\in C_\alpha} D^a(g) = \sum_{g\in G} f_\alpha D^a(g)$$

$$\Rightarrow \Lambda_a{}^{\alpha} D^a(g) = D^a(g) \Lambda_a{}^{\alpha} \qquad f_\alpha = \begin{cases} 1/n_\alpha & \text{if } g\in C_\alpha \\ 0 & \text{else.} \end{cases}$$

$$\overset{\text{Schur}}{\Longrightarrow} \Lambda_a{}^{\alpha} = \frac{\chi_a{}^{\alpha}}{d_a} \mathbb{1}_a .$$

$$C_\alpha C_\beta = C_{\alpha\beta}^{\gamma} C_\gamma \Rightarrow \Lambda_a{}^{\alpha} \Lambda_a{}^{\beta} = C_{\alpha\beta}{}^{\gamma} \Lambda_a{}^{\gamma} .$$

A rep of $G$ is a rep of $\mathbb{C}[G]$.

## Real vs. Complex representations:

### what is reality?

Given a rep $R$ of $G$ $\quad D(g) \quad \chi_R(g) = \mathrm{tr}\, D(g)$

$\bar{R} \quad D^*(g) \quad \chi_{\bar{R}}(g) = \chi_R(g)^*$.

A rep $R$ is <u>NOT COMPLEX</u> if $\bar{R} \sim R$.

i.e. if $\exists S \mid D(g)^* = S D(g) S^{-1} \quad \forall g \in G$.

If, moreover, $\exists$ a basis where $D(g)_{ij} \in \mathbb{R}$ then $R$ is <u>REAL</u>

else $R$ is <u>pseudoREAL</u>.

eg: $\underline{2}$ of $SU(2)$

$$U = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \in SU(2)$$

$U^\dagger U = \mathbb{1}$

$\underbrace{\det U = |a|^2 + |b|^2 = 1}_{S^3 \subset \mathbb{R}^4}$.

$$\bar{\underline{2}} : U^* = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$$

claim

$$= \epsilon^{-1} U \epsilon \,, \quad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

$$\Rightarrow \underline{2} \sim \bar{\underline{2}} \,.$$

**But**: $\nexists$ no real basis : a real $2\times 2$ unitary matrix is specified by 2 real #s.

$\Rightarrow \underline{2}$ is pseudoreal.

---

if $\bar{R} \sim R \Rightarrow \chi_{\bar{R}}(g) = \chi_R(g)^* = \chi_R(g)$.

If $\chi_R(g) \neq \chi_R(g)^*$ for any $g \in G$, $\Rightarrow R$ is complex.

If $\chi_{\bar{R}}(g) = \chi_R(g) \ \forall g$. $\longrightarrow R \sim \bar{R}$. not complex

Assume $R$ is unitary.

take $x, y$ in $R$.

$$G: \left.\begin{aligned} x &\mapsto D(g)x \\ y &\mapsto D(g)y. \end{aligned}\right\}$$

<u>claim:</u> $\exists$ <u>G-invariant</u> <u>bilinear</u> $S$ $\Longleftrightarrow$ $R$ is not complex

$$y^T S x$$

$\boxed{\Rightarrow}$ If $\exists$ $S$ w/

$$\left[\begin{aligned} y^T S x &\xrightarrow{G} y^T \underbrace{D^T(g)\, S\, D(g)}\, x \\ &\overset{!}{=} y^T \underbrace{S}\, x. \quad \forall x, y. \end{aligned}\right.$$

$$\Rightarrow D^T(g)\, S\, D(g) \overset{!}{=} S \Rightarrow S D(g)^{-1} S^{-1} = D(g^T)^{-1} = D(g)^*$$

$$\Rightarrow R \sim R^*$$

$\boxed{\Leftarrow}$ If $D^*(g) = S D(g) S^{-1}$

$$\Rightarrow \underline{D(g)^T = S D(g)^{\dagger} S^{-1}}$$

$$\Rightarrow y^T S x \longmapsto y^T D(g)^T S D(g) x$$

$$= y^T \underbrace{S D(g)^{\dagger} \underbrace{S^{-1} S}\, D(g)} x = y^T S x$$

$\blacksquare$

summary: $R$ is not complex $\iff$

$$R \otimes R = \underset{\substack{\uparrow \\ S}}{1} \oplus \dots$$

<u>why care?</u> $\mathcal{L}(\phi) \overset{?}{\ni} m^2 \phi^2$

if so, $\Rightarrow$ massive

$\xi < \infty$.

suppose: $\phi$ are in a rep $R$ of $G$

$R \sim \bar{R} \iff$ particles are their own <u>antiparticles</u>. ("majorana")

are we allowed to add

$\Delta \mathcal{L} \ni \phi^T \phi$ ?

<u>what is S?</u> $R \sim \bar{R} \Rightarrow \underline{S D(g) S^{-1} = D^*(g)}$.

$\forall g$ including $g^{-1}$ $\left[ \begin{array}{l} \underline{D(g^{-1}) = D(g)^\dagger} \\ \Rightarrow D(g)^{*T} = \underline{(S^{-1})^T D(g)^T S^T} \end{array} \right.$

$$\Rightarrow D(g) = (S^{-1})^T D(g^{-1})^T S^T = (S^{-1})^T S D(g) S^{-1} S^T$$

$$= (S^{-1} S^T)^{-1} D(g) S^{-1} S^T.$$

$\Rightarrow$ $S^{-1}S^T$ is an intertwiner

$\xrightarrow{\text{Schur}}$ $S^{-1}S^T = \eta \mathbb{1}$ ie $\boxed{S^T = \eta S}$

$$S = (S^T)^T = (\eta S)^T = \eta\,\eta S = \eta^2 S$$

CLAIM:

$$\Rightarrow \eta = \begin{cases} +1 & S^T = S \quad S \text{ symmetric} \quad \text{REAL} \\ -1 & S^T = -S \quad S \text{ A.S.} \quad \text{PSEUDOREAL.} \end{cases}$$

Note: invertible AS matrix is even dim'l

$$\begin{pmatrix} 0 & a & & & \\ -a & 0 & & & \\ & & 0 & b & \\ & & -b & 0 & \\ & & & & \ddots \end{pmatrix}$$

Note: $S \propto$ unitary, $S^\dagger S \propto \mathbb{1}$.

Pf: $\forall g$ $S = D(g)^T S D(g)$, $S^\dagger = D(g)^\dagger S^\dagger D(g)^*$

$$\Rightarrow S^\dagger S = D^\dagger(g) S^\dagger \underbrace{D(g)^* D(g)^T}_{\mathbb{1}} S D(g)$$

$$= D^\dagger(g)\, S^\dagger S\, D(g) \xrightarrow{\text{Schur}} S^\dagger S \propto \mathbb{1}.$$

rescale $\Rightarrow$ $S^\dagger S = \mathbb{1}$.

*claim: if $\eta = +1$, $W \equiv \sqrt{S}$ is also unitary & symmetric.

Pf: $S = e^{iH}$

$S$ unitary $\Leftrightarrow H = H^\dagger$

$S = S^T \Leftrightarrow H = H^T + 2\pi n \mathbb{1}$, $n \in \mathbb{Z}$.

$\Rightarrow n = 0$.

$\Rightarrow W = e^{iH/2}$ is unitary & symmetric.

Contrastingly: if $\eta = -1$.

$S \stackrel{?}{=} e^{iH}$ $\quad S^T = -S$ requires

$\Rightarrow \sqrt{S} = W = e^{iH/2}$ $\quad H^T \stackrel{?}{=} H + \pi \mathbb{1}$

$(W)^T = e^{iH^T/2} = e^{i\pi/2} e^{iH/2} = iW$

X

$$W^{-1} = W^{\dagger} = W^{*} \Rightarrow$$

$$W^{2} D(g) W^{-2} = D(g)^{*} \Rightarrow$$

$$W D(g) W^{-1} = W^{-1} D(g)^{*} W$$

$$= W^{*} D(g)^{*} (W^{-1})^{*}$$

$$= (W D(g) W^{-1})^{*} \in R$$

<u>Frobenius-Schur Indicator</u>:

let $S_X \equiv \frac{1}{|G|} \sum_{g \in G}' D^{T}(g) X D(g)$

is G inv't since

$$S_X \mapsto D^{T}(h) S_X D(h)$$

$$= \sum_{g} D(h)^{T} D(g) X D(g) D(h) = S_X$$

$\forall h \in G.$

So: $y^{T} S_{X} x$ is a G-inv't bilinear.

$= 0$ if $R$ is complex.

take $\underline{\underline{(X)_{jk} = \delta_{ij}\,\delta_{\ell k}}}$ $\quad \begin{array}{cc} & i \rightarrow \\ \ell \downarrow & \begin{pmatrix} 0 & \vert & 0 \\ \hline & 1 & \\ 0 & \vert & 0 \end{pmatrix} \end{array}$

$$\Rightarrow (S_X)_{jk} = \sum_g (D^T X D)_{jk}$$

$$= \sum_g (D^T(g))_{ji}\,(D(g))_{ik} = \sum_g D(g)_{ij} \cdot D(g)_{\ell k}$$

$$\xrightarrow[j=\ell]{\text{contract}} \quad \sum_{g\in G} D(g)_{ij} \cdot D(g)_{jk} = \sum_{g\in G} D(g^2)_{ik} \qquad \underline{\forall\, ijk\ell.}$$

$$\xrightarrow{i=k} \quad \sum_g \chi(g^2) \qquad \left( = 0 \text{ if } R \text{ is complex} \right)$$

If $R$ is not complex $\quad S_X^T = \eta\, S_X$

$$\Rightarrow S_X^T = \sum_{g\in G} D(g)^T \underline{X^T}\, D(g) = \eta \sum D(g)^T \underline{X}\, D(g)$$

$$\Rightarrow (S_X)_{jk} = \sum_{g\in G} D(g)^T_{j\ell}\, D(g)_{ik} = \eta \sum D(g)^T_{ji}\, D(g)_{\ell k}$$

$$\to \sum_{g\in G} D(g)_{\ell j}\, D(g)_{ik} \overset{!}{=} \eta \sum D(g)_{ij}\, D(g)_{\ell k}.$$

$$\xrightarrow{j=l} \sum_{g\in G} \chi(g) D(g)_{ik} = \eta \sum_{g\in G} D(g^2)_{ik}$$

$$\xrightarrow{i=k} \eta \underbrace{\sum_{g\in G} \chi(g)^2}_{} = \sum_{g\in G} \chi(g^2)$$

If $\eta \neq 0$, $\chi(g) = \chi(g^*) \rightarrow \sum_{g\in G} \chi(g)\chi(g)$

$$= \sum_{g} \chi^*(g)\chi(g)$$

$$= |G|.$$

if $R$ is an irrep.

$\Downarrow$

FS indicator is

$$\eta_R \equiv \frac{1}{|G|} \sum_{g\in G} \chi_R(g^2) = \begin{cases} 0, & R \text{ complex} \\ 1, & R \text{ is real} \\ -1, & R \text{ is pseudoreal} \end{cases}$$

$$= \frac{1}{|G|} \sum_{\alpha} n_\alpha \chi_R(g_\alpha^2)$$

<u>note</u>: • $[g_1] = [g_2] \Rightarrow [g_1^2] = [g_2^2]$.

• trivial rep : $\eta_R = 1$.

**Q**: how many sol'ns of $g^2 = e$ in $G$?

$$\eta_a = \frac{1}{|G|} \sum_{h \in G} \underline{\underline{\sigma(h)}}\, \chi_a(h)$$

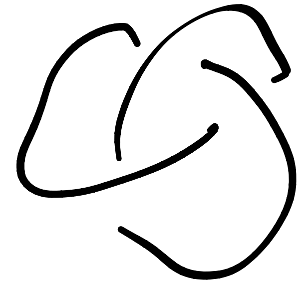
$\sigma(h) \equiv$ # of sol'ns of $g^2 = h$ in $G$.

$$\sum_a (\text{BHS})\, \chi_a(h') \Longrightarrow$$

$$\sigma(h) \propto \sum_a \frac{\eta_a}{d_a} .$$

**Q**: How many homomorphisms from $K \longrightarrow G$

w/ $K = \langle a, b \mid a^2 = b^3 \rangle$

$= \pi_1 \left( \text{complement in } \mathbb{R}^3 \text{ of } \right.$  $\left. \right)$

## Induced reps :

given a rep $D^W(h): W \to W$ of $H \subset G$

make a rep of $G \equiv \mathrm{Ind}_H^G(W)$.

carrier space is $W \times V_{G/H}$

$$V_{G/H} = \mathrm{span}\{\,|x\rangle,\ x \in G/H\}.$$

$G$ acts on $V_{G/H}$ by

$$x = \{g_1, g_2 \ldots\} \longrightarrow \{gg_1, gg_2 \ldots\}.$$

Pick a representative $a_x$ of $x \in G/H$.
$\in G$

$$a_x \longrightarrow g a_x = a_{gx} h$$

<u>for $h \in H$</u>: $D(h)|n, 0\rangle = |m, 0\rangle \left(D^W(h)\right)_{mn}$

$\uparrow$
coset containing e

<u>for reps</u>:
$$D(a_x)|n, 0\rangle = |n, x\rangle.$$

$$D(g)\,|n,x\rangle = \underbrace{D(g)\,D(a_x)}_{D(\underbrace{g a_x}_{= a_{gx} h})}\,|n,0\rangle$$

$$= D(a_{gx})\,\underbrace{D(h)\,|n,0\rangle}$$

$$= \underline{D(a_{gx})}\,|m,0\rangle\,\left(D^W(h)\right)_{mn}$$

$$= |m, gx\rangle\,\left(D^W(h)\right)_{mn}.$$

is a Rep: $D(g_1)\,D(g_2)\,|n, x\rangle$

$$= D(g_1 g_2)\,|n, x\rangle.$$