examples of projective reps:

1d SPTs $\quad$ gapped symmetric groundstate $|\psi\rangle$ of a local 1d hamiltonian w/ on-site G symmetry.

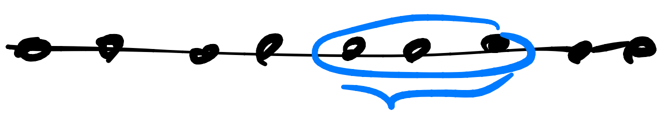
"symmetry-protected topological" state

$$U = \prod_i u_i$$

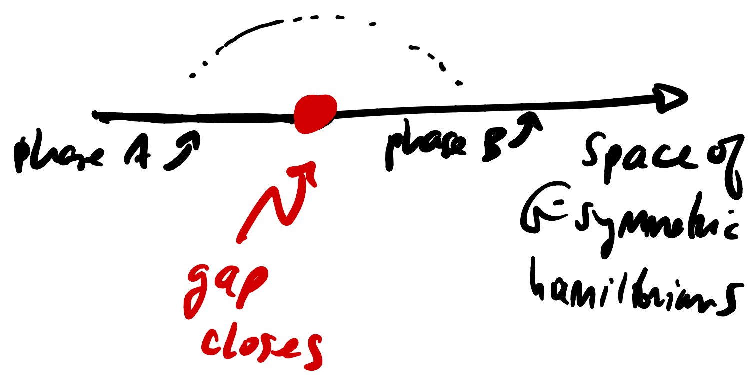
$$U|\psi\rangle = |\psi\rangle.$$

Space of all H

$$\mathcal{H} = \bigotimes_i \mathcal{H}_i$$



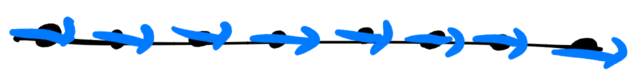
local: $H = \sum_i h_i$ ← acts near site i



TRIVIAL PHASE: $H_0 = -\sum_i X_i \qquad X|+\rangle = |+\rangle.$

g.s. is $|\text{trivial}\rangle = \bigotimes_i |+\rangle_i$

(no entanglement)



eg: $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle U_e, U_o \rangle \qquad H = -\sum_i Z_{i-1} X_i Z_{i+1}$

$$U_e = \prod_{i\,\text{even}} X_i \qquad U_o = \prod_{i\,\text{odd}} X_i$$

Let $S = \prod_i CZ_{i,i+1}$

$H = S H_0 S^\dagger$

$$\boxed{X_i CZ_{ij} X_i = CZ_{ij} Z_j}$$

$CZ_{ij} = CZ_{ji}$.

$CZ_{ij} \equiv e^{i\pi \frac{(1-Z_i)}{2}\frac{(1-Z_j)}{2}}$

$CZ|0s\rangle = |0s\rangle$

$CZ|1s\rangle = (-1)^s|1s\rangle$

$\bar{0}=1, \bar{1}=0.$

$Z_i|ts\rangle = (-1)^t|ts\rangle$
$Z_j|ts\rangle = (-1)^s|ts\rangle$
$t,s = 0,1$

g.s. of H is $\underline{|\psi\rangle = S\,|\text{trivial}\rangle}$

near one end of the chain: $\underset{1}{\cdot}\ \underset{2}{\cdot}\ \underset{3}{\cdot}\ \cdots$

$$\begin{aligned}
U_0|\psi\rangle &= (X_1 X_3 X_5 \cdots)\, S\,|\text{trivial}\rangle \\
&= (X_1 X_3 X_5 \cdots)\, CZ_{12}\, CZ_{23}\, CZ_{34} \cdots |\text{trivial}\rangle \\
&= CZ_{12}\, \cancel{Z_2}\, \cancel{Z_2}\, CZ_{23}\, CZ_{34}\, Z_4\, Z_4\, CZ_{45} \cdots |\text{trivial}\rangle \\
&= |\psi\rangle.
\end{aligned}$$

$$\begin{aligned}
U_e|\psi\rangle &= (X_2 X_4 \cdots)\, CZ_{12}\, CZ_{23}\, CZ_{34} \cdots |\text{trivial}\rangle \\
&= Z_1\, CZ_{12}\, CZ_{23}\, \cancel{Z_3}\, \cancel{Z_3}\, CZ_{34} \cdots |\text{trivial}\rangle \\
&= Z_1|\psi\rangle.
\end{aligned}$$

$$U_o U_e |\psi\rangle = (X_1 X_3 \cdots) Z_1 |\psi\rangle = - Z_1 |\psi\rangle$$

$\boxed{\text{"CLUSTER STATE"}}$

$$U_e \underbrace{U_o |\psi\rangle} = U_e |\psi\rangle = Z_1 |\psi\rangle.$$

$$U_e U_o = - U_o U_e$$ when acting on the end of the chain!

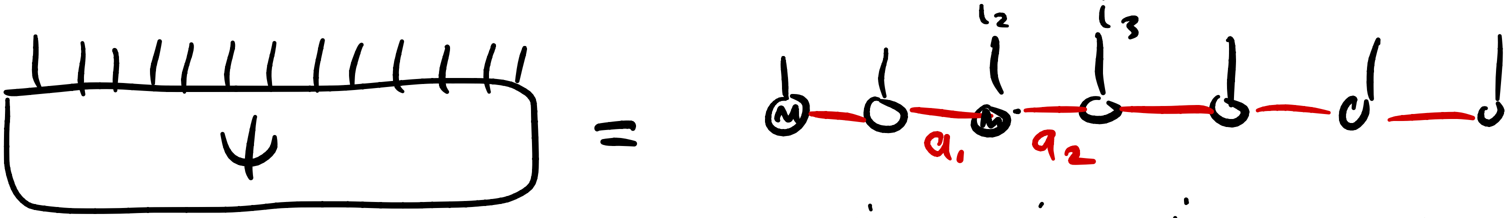
like $X Z = - Z X$

g.s. of end $\in \text{span}\{ |\psi\rangle, Z_1 |\psi\rangle \}$

$\Rightarrow$ degenerate edge states!

observable in spin 1 Heisenberg chain: $H = \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$.

More generally: Any gapped groundstate in 1d has a representation as a matrix product state



$$\psi_{i_1 \ldots i_L} |i_1 \ldots i_L\rangle = \sum M^{i_1}_{a_1 a_2} M^{i_2}_{a_2 a_3} M^{i_3}_{a_3 a_4} \cdots \quad |i_1 \ldots i_L\rangle$$

general state

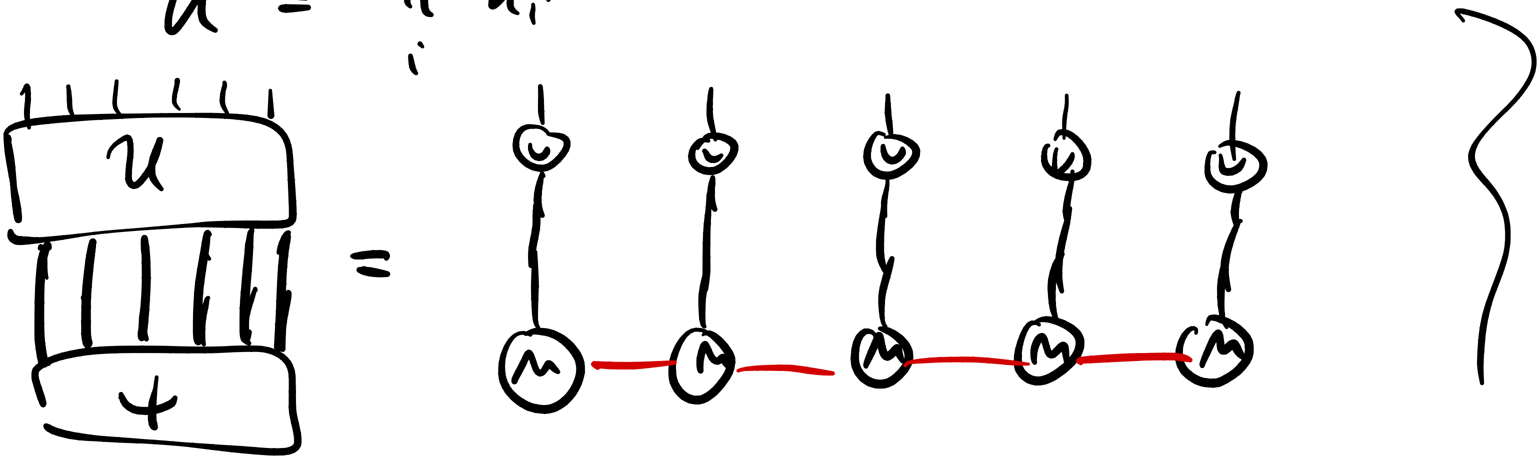
2^L complex #s

MPS :

$(L \gg 1)$ $a = 1..\chi$ bond dimension

$\gg \chi \times \chi \times 2 \times L$

$$\mathcal{U} = \prod_i \mathcal{U}_i$$

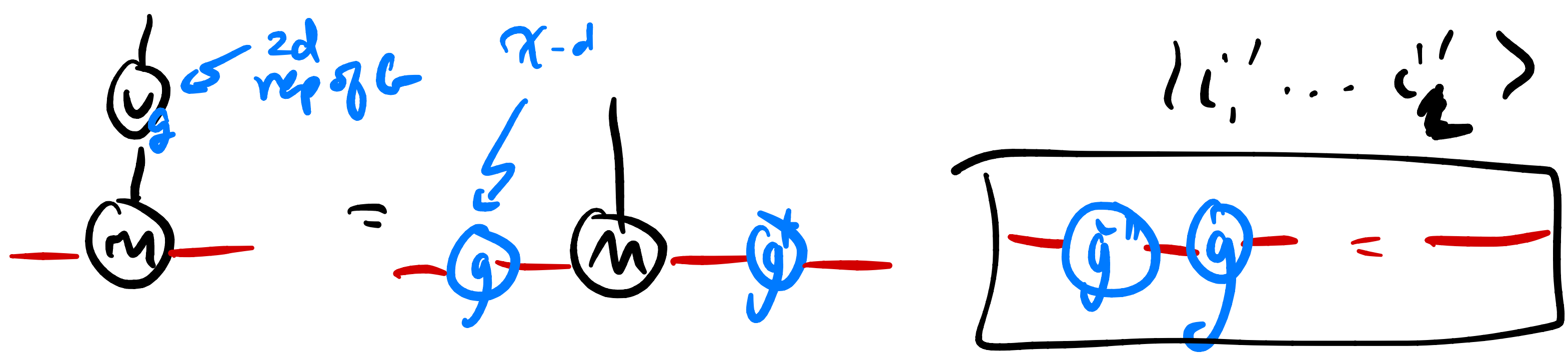


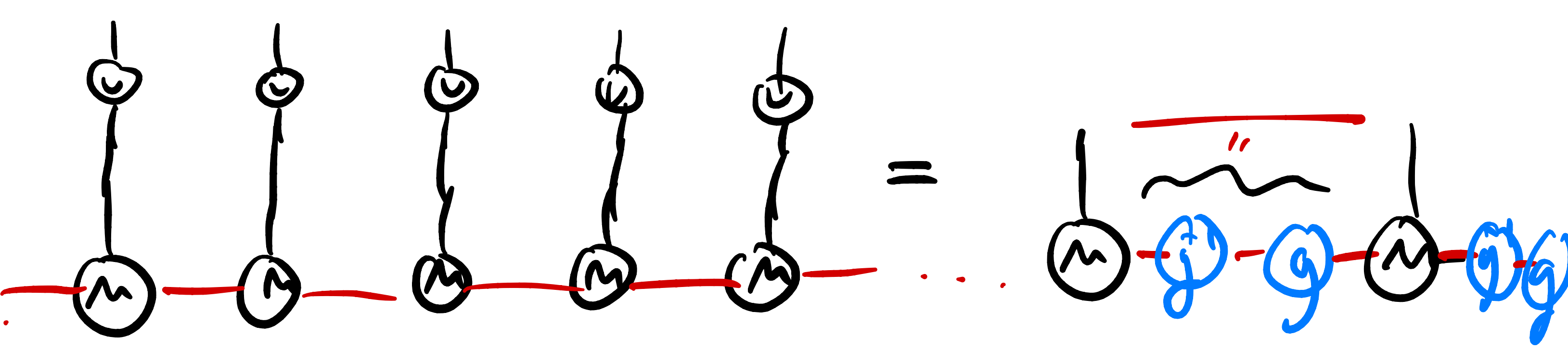
$$\mathcal{U}|\Psi\rangle =$$

$$\mathcal{U}\sum M^{i_1}_{a_1a_2} M^{i_2}_{a_2a_3}\cdots |i_1\dots i_L\rangle \overset{!}{=} |\Psi\rangle$$

$$= \sum M^{i_1}_{a_1a_2} M^{i_2}_{a_2a_3}\cdots u_{i_1i_1'} u_{i_2i_2'}\cdots$$

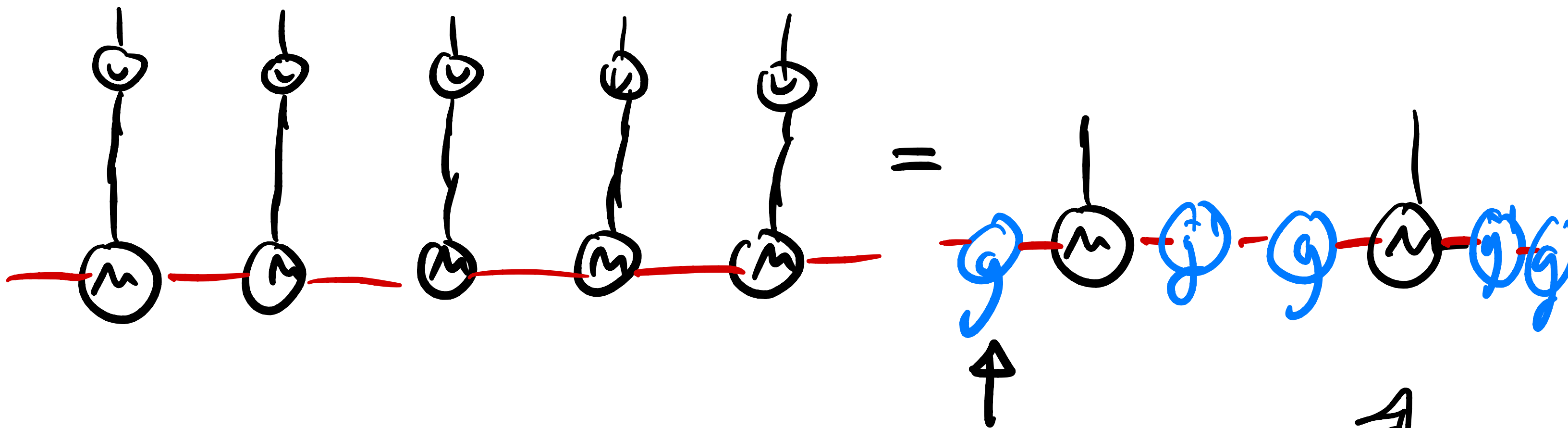
$$|i_1'\dots i_L'\rangle$$





i ———— j $= \delta_{ij}$

Now consider a bdy:

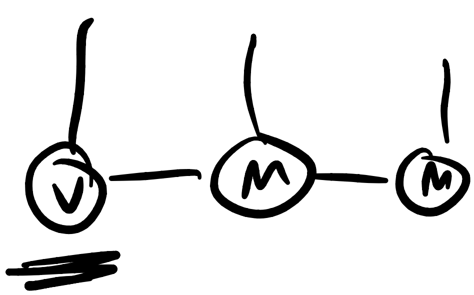


g was a $\chi \times \chi$ PROJECTIVE representation of G

$\rightarrow$ 1d $SPTS_G$ are classified by $H^2(G, U(1))$.

d-dim'l SPTS .... $H^{d+1}(G, U(1))$.

(w/ some assumptions)



3. Lie groups & Lie algebras

essential idea of rep. thy of Lie groups:

If $H = H^\dagger$ then $U(t) = e^{iHt}$ is unitary.

$$-i\partial_t U(t) = H U(t), \quad U(0) = \mathbb{1}.$$

$$U(\epsilon) = \mathbb{1} + i\epsilon H + O(\epsilon^2) \leadsto U(t) \text{ at finite } t.$$

generator of Lie algebra. element of Lie group.

Suppose $g(\vec{s}) \in G$ depends SMOOTHLY on $\vec{s}$, $d_G \equiv \dim G$ parameters.

$\equiv$ can do calculus.

Convention: $g(0) = e$.

in any rep $D(g(\vec{s}))\big|_{s=0} = D(e) = \mathbb{1}$. $\dim R \times \dim R$.

Near the identity

$$D(g(\vec{\epsilon})) \overset{\text{Taylor}}{=} \mathbb{1} + i\epsilon^A X^A + O(\epsilon^2)$$

$$X^A \equiv -i\,\partial_{s^A} D(\vec{s})\big|_{s=0}.$$

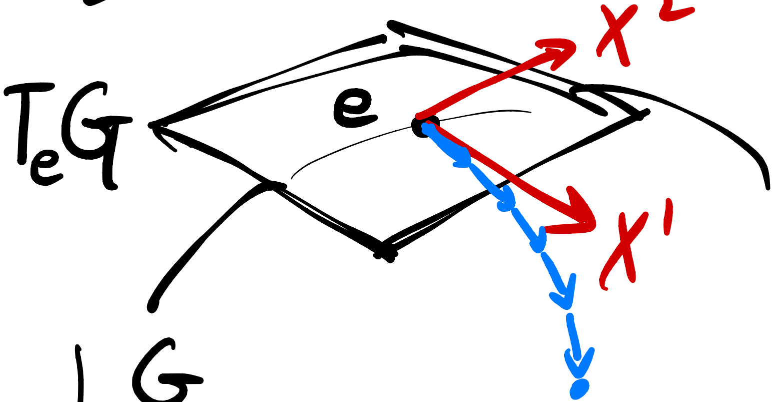
linearly indep. as $\dim R \times \dim R$ matrices.

i is there so that $\mathbb{1} = D^\dagger D \Rightarrow$

$$X^A = (X^A)^\dagger.$$

$\mathrm{span}_{\mathbb{C}}\{X^A\}$ is a vector space and an <u>algebra</u> $\equiv$ Lie algebra $\mathfrak{g}$ of G.

Geometrical aside:



Lie algebra generators $\hookrightarrow$ basis of tangent space of G at e.

$$\lim_{N\to\infty}\left(\mathbb{1} + \frac{\circledcirc}{N}\right)^N = e^{\circledcirc}$$

$$\circledcirc = i\,\vec{s}\cdot\vec{X} = i\,s^A X^A$$

$$D(\vec{s}) = \lim_{N\to\infty}\left(\mathbb{1} + \frac{i\vec{s}\cdot\vec{X}}{N}\right)^N = e^{i s^A X^A}.$$

<u>easy</u>:

$$D(\lambda_1\vec{s})\,D(\lambda_2\vec{s}) = D((\lambda_1+\lambda_2)\vec{s}\cdot X) = D(\lambda_2\vec{s})\,D(\lambda_1\vec{s})$$

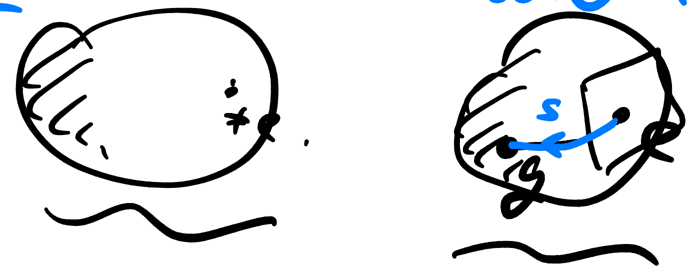
But $D(\vec{s})\,D(\vec{t}) \neq D(\vec{s}+\vec{t})$. $\underline{\underline{[\vec{s}\cdot\vec{X},\vec{t}\cdot\vec{X}] \neq 0}}$

$$D(\vec{s})\,D(\vec{t}) = e^{i\vec{s}\cdot X}\,e^{i\vec{t}\cdot X} \overset{!}{=} e^{i\vec{r}\cdot\vec{X}}$$

why? $D(\vec{s})$ form a rep. of

$$G_0 \equiv \{ g \in G \mid \text{path connected to } e \in G \} < G.$$

eg: $\det O = -1$ $\quad$ $\det O = 1$



G_0

If $g = e^{i\vec{s}\cdot\vec{X}}$: PATH $= \left\{ e^{i\lambda\vec{s}\cdot\vec{X}} \right\}_{\lambda\in(0,1)}$.

log (RHS):

$$i\vec{r}\cdot\vec{X} = \log(1+K) = K - \frac{K^2}{2} + O(\epsilon^3)$$

$$K \equiv e^{i\vec{s}\cdot\vec{X}} e^{i\vec{t}\cdot\vec{X}} - 1 = i\vec{s}\cdot\vec{X} + i\vec{t}\cdot\vec{X} - \tfrac{1}{2}(\vec{s}\cdot\vec{X})^2 - \tfrac{1}{2}(\vec{t}\cdot\vec{X})^2 - \vec{s}\cdot\vec{X}\;\vec{t}\cdot\vec{X} + O(\epsilon^3)$$

regard $t, s \Rightarrow r$ as $O(\epsilon)$.

$t \to \epsilon t$
$s \to \epsilon s$ $\Rightarrow r \to \epsilon r$

claim: $i\vec{r}\cdot\vec{X} \stackrel{?}{=} i\vec{s}\cdot\vec{X} + i\vec{t}\cdot\vec{X}$ if $[X^A, X^B] = 0$.

$$i\vec{r}\cdot\vec{X} = i\vec{s}\cdot\vec{X} + i\vec{t}\cdot\vec{X} - \tfrac{1}{2}[\vec{s}\cdot\vec{X}, \vec{t}\cdot\vec{X}] + O(\epsilon^3)$$

$$\Rightarrow [\vec{s}\cdot\vec{X}, \vec{t}\cdot\vec{X}] = -2i(\vec{r} - \vec{s} - \vec{t})\cdot\vec{X} \equiv i\vec{u}\cdot\vec{X}$$

$$u^A = s^A t^B f_{ABC}$$

where $\boxed{[X^A, X^B] = i f_{ABC} X^C}$ ☆

claim: This relation at $O(\epsilon^2)$ is the only requirement for $e^{is\cdot X} e^{it X} = e^{ir\cdot X}$.

Comment on BCH formula:

$$e^{-A} e^{A+B} = e^{\underbrace{\text{something}(A,B)}}.$$

$$= B - \tfrac{1}{2}[A,B] + \tfrac{1}{6}[A,[A,B]]^2 + \dots$$

let $\text{ad}_A B \equiv [A, B]$ "adjoint action of A".

is a derivation (like $d(fg) = df\, g + f dg$.)

$$\text{ad}_A (BC) = (\text{ad}_A B) C + B (\text{ad}_A C).$$

i.e. $[A, BC] = [A, B]C + B[A, C]$.

$\text{tr}[A,B] = 0 \implies \text{tr} \circ \text{ad} = 0$ (like $\oint dx \frac{df}{dx} = 0$).

$\text{ad}_X \sim$ matrix derivative along X.

claim: $\boxed{\exp \mathrm{ad} = \mathrm{Ad} \exp}$.

$$\mathrm{Ad}_g B \equiv g B g^{-1} \qquad g \in G.$$

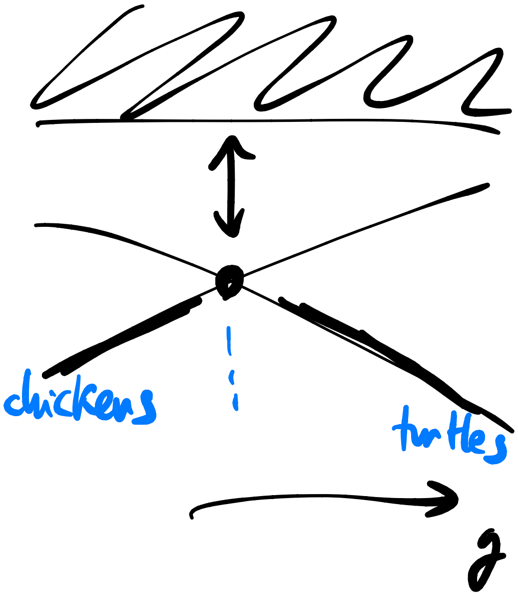
$$e^{s\,\mathrm{ad}_A} B = e^{sA} B e^{-sA} \equiv B(s)$$

$$\boxed{\frac{dB(s)}{ds} = [A, B(s)], \qquad B(0) = B.}$$

ODE with a unique solution.
(series exp. in s).

$$e^{\mathrm{ad}_A} B = \mathrm{Ad}_{e^A} B.$$

1st order



no universal physics.

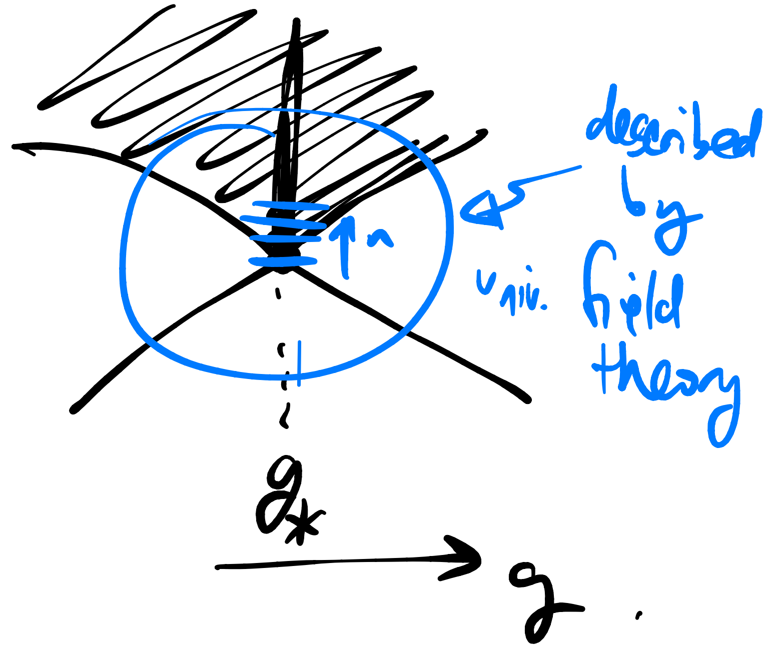
all garbage.

$m^2_{chickens} > 0$

$m^2_{chicks} > 0$, $m^2_{turtles} > 0$.

$m^2_{turtles} > 0$

2d order (continuous)



finite vol: $E_n \sim \frac{n}{L^z} \xrightarrow[L\to\infty]{} 0$

fixed pt field theory at g_* + relevant operator $(g-g_*)$

$$\mathcal{L} = (\partial\phi)^2 + \phi^4 + r\phi^2$$

$$(r = g - g_*)$$

Moonshine : sporadic groups $\longleftrightarrow$ 2d CFT.

" arise as symmetries of particular 2d CFTs.

monster moonshine

$$Z_{T^2} = \mathrm{tr}_{\mathcal{H}\text{ on } S^1}\, e^{-\beta H} = \mathrm{tr}\, q^{H}$$

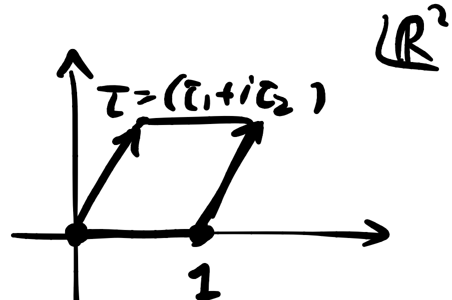
$$q = e^{-\beta} \longrightarrow e^{-(2\pi)^2/\beta} = \tilde{q}$$

$$\underline{q = e^{\tau}}$$

$$Z_{T^2}(q) = Z_{T^2}(\hat{q})$$

$=$ 

$$T^2 = \mathbb{R}^2/\mathbb{Z}\times\mathbb{Z}$$



$$\tau \to \frac{a\tau+b}{c\tau+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

gives <u>SAME</u> T^2.

$$j(\tau) = j\left(\frac{a\tau+b}{c\tau+d}\right) \cong Z_{T^2}\left(\text{CFT of 24 free bosons on } \mathbb{R}^{24}/\text{Leech lattice}\right)$$

$$= q^{-1} + \text{"}744\text{"} + \text{"}196884\text{"}\, q + \cdots$$

$\sim \dim R_a$ of Monster group