3.2 Irreps of SO(n) by tensor methods

Reps of SO(n) so far: $\underline{n}$ fundamental, vector

$\frac{n(n-1)}{2}$ adjoint

$v^i \mapsto R^{ij} v^j$ $\boxed{R^T R = \mathbb{1}}$ $R^{ik} R^{jl} \delta_{ij} = \delta_{kl}$

$\underline{n \otimes n}$: $T^{ij} \longmapsto \underline{\underline{D(R)}}_{ij,kl} T^{kl}$

$= \underbrace{R^{ik} R^{jl}}_{n^2 \times n^2 \text{ matrix}} T^{kl}$.

Reducible?

$A^{ij} \equiv T^{ij} - T^{ji}$ $\frac{n(n-1)}{2}$ dim'l invariant subspace

$S^{ij} \equiv T^{ij} + T^{ji}$ $\frac{n(n+1)}{2}$ dim'l subspace

$n \otimes n = \Lambda^2 n \oplus \mathrm{Sym}^2 n$

But $\exists$ invariant symbol δ_{ij}

$S \equiv S^{ii} \equiv S^{ij} \delta_{ij} \longmapsto \underbrace{R^{ik} R^{jl} \delta_{ij}}_{\delta_{kl}} S^{kl} = S.$

1d invt subspace

$$n \otimes n = 1 \oplus \frac{n(n-1)}{2} \oplus \underbrace{\frac{n(n+1)}{2} - 1}_{\text{traceless symmetric}}$$

for $\underline{SU(3)}$: $\quad 3 \otimes 3 = \underline{1} \oplus \underline{3} \oplus \underline{5}$.

Another invt symbol for $SU(n)$

$$\epsilon_{i_1 \ldots i_n} M^{i_1 j_1} \cdots M^{i_n j_n} = \epsilon^{j_1 \ldots j_n} \det M$$

$$\Rightarrow \underline{\epsilon^{i_1 \ldots i_n} R^{i_1 j_1} \cdots R^{i_n j_n}} = \det R\, \epsilon^{j_1 \ldots j_n} = \underline{\epsilon^{j_1 \ldots j_n}}.$$

$A^{i_1 \ldots i_p}$ antisymmetric tensor. $\quad A^{i_1 \ldots i_p} \mapsto R^{i_1 j_1} \ldots R^{i_p j_p} A^{j_1 \ldots j_p}$

$$\to B^{\widetilde{i_{p+1}} \ldots i_n} = \epsilon_{\widetilde{i_1} \ldots i_n} A^{i_1 \ldots i_p} \quad \Rightarrow B \text{ is a tensor}$$

$A^{ij} \in n \otimes n \qquad \underline{A^{ij} = \epsilon^{ijk} A^k}$.

$$\underline{\Lambda^2 \underline{3} \cong \underline{3}}.$$

$\underline{\text{What is } \frac{n(n-1)}{2} \text{ AS rep}?}$ $\quad A_{ij} \mapsto R_{ij} R_{kl} A_{jl}$

$$= R_{ik} A_{kl} (R_{lj})^T$$

adjoint. $\quad = (R A R^{-1})_{ij}$

$R = e^{\theta^A J^A}$

$A = \sum_B A^B J^B$

$A \mapsto A + \theta^A [J^A, A] + \dots$

$= A + \theta^A \underbrace{[J^A, J^B]}_{f^{ABC} J^C} A^B$

$\delta A^C = \theta^A f^{ABC} A^C = \theta^A (f_A)_C{}^B A^B$

generators of adj. rep.

3.3 Casimirs

semisimple $\Rightarrow$

$\kappa_{AB} = \mathrm{tr}\, X_A X_B$ is invertible

$(\kappa^{-1})_{AB} \kappa_{BC} = \delta_{AC}$.

$C_2 \equiv (\kappa^{-1})_{AB} X_A X_B$

$\kappa^{AB} = \kappa^{BA}$ But f_{ABC} is AS.

claim: $[C_2, X_A] = 0 \quad \forall A$.

$\xRightarrow{\text{Schur}} C_2 = \mathbb{1}\, c_2(R)$ on each irrep R.

eg: $G = SU(2)$. $C_2 = J^2 = J_x^2 + J_y^2 + J_z^2$.

3.4 Cartan-Weyl Method. $\circlearrowleft \mathfrak{g}$

$$\mathrm{tr}\, X^A X^B = \lambda\, \delta^{AB} \qquad [X^A, X^B] = i f^{ABC} X^C$$

Chose a maximal ~~set~~ subset of commuting hermitian generators $\{H_i\}$

$$[H_i, H_j] = 0 \qquad H_i = H_i^{\dagger} \qquad i = 1 \ldots r$$

$r \equiv \mathrm{rank}\, \mathfrak{g}$

$\equiv$ Cartan subalgebra

for $SU(2)$: J_3 $\quad$ $\mathrm{rank}(SU(2)) = 1$.

In any rep R can diagonalize $\{H_i\}$

$$H_i |\mu\rangle = \mu_i |\mu\rangle \qquad \mu_{i=1\ldots r}$$

weight vector

for $SU(2)$: $J_3 |j, m\rangle = m |j, m\rangle$

$m \in -j, -j+1, \ldots, j-1, j$.

Rest of the alg: Diagonalize ad_{H_i} on the rest

$$\star \quad [H_i, E_\alpha] = \alpha_i E_\alpha \quad \star$$

in adj. rep: $H_i |H_j\rangle = |[H_i, H_j]\rangle = 0$

have weight 0.

$$H_i |E_\alpha\rangle = |[H_i, E_\alpha]\rangle \overset{!}{=} \alpha_i |E_\alpha\rangle.$$

the nonzero weights of the adj. rep

$\equiv$ roots

$$(\star)^\dagger \Rightarrow [H_i, E_\alpha^\dagger] = -\alpha_i E_\alpha^\dagger$$

$$\Longrightarrow E_\alpha^\dagger = E_{-\alpha}.$$

PROMISE: $\exists$ only one generator of $\mathfrak{g}$ for each α.

SU(2): $J_\pm \equiv \frac{1}{\sqrt{2}}(J_1 \pm i J_2)$

$$\star \quad \underbrace{[J^3, J^\pm] = \pm J^\pm}_{\text{"}J^\pm \text{ are eigenvectors of } \mathrm{ad}_{J^3}\text{"}} \qquad \alpha = \pm 1.$$

$$\mu^2 = \mu \cdot \mu = \sum_{i=1}^{r} \mu_i^2$$

SU(2): wts of adjoint rep: $\boxed{-1, 0, 1}$

$|J^-\rangle \; |J^3\rangle \; |J^+\rangle$

$r = 1$.

Raising & Lowering.

for any $\boxed{\text{finite-dim'l}}$ rep of SU(2)

$\exists$ a state with highest weight

= largest eval of J^3

$\Downarrow$

call $|j j\rangle$. $\qquad$ $J^+ |j j\rangle = 0$

$$J^- |j, j\rangle = N_j |j, j-1\rangle$$

since $J^3 (\underbrace{J^- |jj\rangle}) = (\underbrace{[J^3, J^-]}_{= -J^-} + J^- \underbrace{J^3}_{j}) |jj\rangle$

$$= (j-1)(\underbrace{J^- |jj\rangle})$$

$\exists x$ s.t. $(J^-)^x |j, j\rangle = 0$

$x \in \mathbb{Z}_+$

$\langle j, j \beta | j, j, \alpha\rangle = \delta_{\alpha\beta}$

$$J^- |j j \alpha\rangle = N_j(\alpha) |j, j-1, \alpha\rangle$$

$$\langle j j-1 \beta | j j-1 \alpha\rangle \, N_j^*(\alpha) N_j(\beta)$$

$$= \langle j j \beta | J^+ J^- | j j \alpha\rangle$$

lowering preserves orthogonality α, β.

$\Rightarrow$ each α is an inv't subspace.

$$= \langle jj\beta | [J^+, J^-] | jj\alpha \rangle$$

$$= \langle jj\beta | J^3 | jj\alpha \rangle$$

$$= j\, \delta_{\alpha\beta}.$$

$$\begin{cases} J_+ J_- = J^2 - (J_3^2 - J_3) \\ J_- J_+ = J^2 - (J_3^2 + J_3) \end{cases}$$

$$J^2 = \sum_i^3 J_i^2 .$$

on an irrep:

$$0 = J^+ |jj\rangle$$

$$0 = \| J_+ |jj\rangle \|^2 = \langle jj | J_- J_+ | jj \rangle$$

$$= \langle jj | (\underbrace{J^2}_{= c_2(j)\mathbb{1}} - \underbrace{J_3(J_3+1)}_{j(j+1)}) | jj \rangle$$

$$\boxed{c_2(j) = j(j+1)} \qquad = (c_2(j) - j(j+1)) \underbrace{\langle jj | jj \rangle}_{\doteq 1}$$

$$|j, m\rangle \sim J_-^{j-m} |jj\rangle \quad \text{w/} \quad J_3 |jm\rangle = m |jm\rangle,$$

$$J_- |jm\rangle \propto |j, m-1\rangle$$

$$\| J_- |jm\rangle \|^2 = \langle jm | J_+ J_- | jm \rangle = \langle jm | (J^2 - J_3(J_3 - 1)) | jm \rangle$$

$$= (j(j+1) - m(m-1)) \langle jm | jm \rangle$$

$\stackrel{!}{=} 1$

$\Rightarrow J_3|jm\rangle = m|jm\rangle$

$$J_\pm |jm\rangle = \sqrt{\underbrace{j(j+1) - m(m\pm 1)}}\, |j, m\pm 1\rangle$$

if $j \in \frac{\mathbb{Z}}{2}$ $\quad J_-|j,-j\rangle = 0$

$$j(j+1) - m(m-1)\Big|_{m=-j} = 0.$$

if $j \notin \frac{\mathbb{Z}}{2}$, $\quad \{j, j-1, \ldots, -j+1, -j\}$

$2j+1$ states.

$$\| J_-^{n>2j} |jj\rangle \|^2$$

$$= j(j+1) - m(m-1) < 0.$$

neither finite-dim'l nor unitary

All finite-dim'l unitary IR reps of $SU(2)$:

span$\{|j,m\rangle\}$ $\quad j \in \mathbb{Z}/2$

dim		
1	$j=0$	singlet
2	$j=\frac{1}{2}$	fundamental, spinor
3	$j=1$	adjoint rep
4	$j=3/2$	
5	$j=2$	symmetric tensor $\mathrm{Sym}^2 3 \setminus$ trace
$\vdots$		

If $j \in \mathbb{Z}$: Also reps of $SO(3)$

(if $j \in \mathbb{Z} + \frac{1}{2}$ projective reps of $SO(3)$.)

$$e^{i\alpha\cdot X} \xrightarrow[X=X^\dagger]{\alpha\in\mathbb{R}} \text{compact group } SU(2)$$

$$\Big\downarrow \alpha\in i\mathbb{R}$$

noncompact group] $SL(2,\mathbb{R})$

Raising & lowering in general

$J^\pm \rightsquigarrow E_{\pm\alpha}$

$$H_i(E_{\pm\alpha}|\mu\rangle) = (\underbrace{[H_i, E_{\pm\alpha}]}_{\pm\alpha E_{\pm\alpha}} + E_{\pm\alpha}\mu_i)|\mu\rangle$$

$$= \underbrace{(\mu_i \pm \alpha_i)}_{\text{weight}} E_{\pm\alpha}|\mu\rangle$$

$$E_\alpha|E_{-\alpha}\rangle = |[E_\alpha, E_{-\alpha}]\rangle = |\beta_i H_i\rangle$$

$$= \beta_i |H_i\rangle$$

claim: $H_i|[E_\alpha, E_{-\alpha}]\rangle = 0$

$$\langle H_i|H_j\rangle = \lambda \delta_{ij}$$

$$\Rightarrow \beta_i = \frac{1}{\lambda}\langle H_i|E_\alpha|E_{-\alpha}\rangle$$

$$= \frac{1}{\lambda} \operatorname{tr} H_i [E_\alpha, E_{-\alpha}]$$

$$\overset{IBP}{=} -\frac{1}{\lambda} \operatorname{tr} \underbrace{[E_\alpha, H_i]}_{-\alpha_i E_\alpha} E_{-\alpha}$$

$$= \frac{\alpha_i}{\lambda} \operatorname{tr} E_\alpha E_{-\alpha} = \alpha_i \underbrace{\frac{\operatorname{tr} E_\alpha^\dagger E_\alpha}{\lambda}}_{=1}$$

$= \lambda \delta_{\alpha\alpha}$

$$\Rightarrow \boxed{[E_\alpha, E_{-\alpha}] = \alpha \cdot H}$$

(in $SU(2)$: $[J^+, J^-] = J^3$)

Each pair of roots $\pm\alpha \longleftrightarrow SU(2)_\alpha \subset \mathcal{G}$

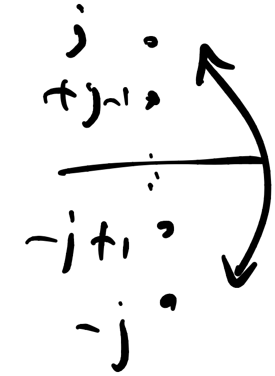
$$E^\pm_{(\alpha)} = \frac{E_{\pm\alpha}}{|\alpha|}, \quad E^3_{(\alpha)} \equiv \frac{\alpha \cdot H}{\alpha^2}$$

Claim: for each root α one E_α.

Pf: suppose E_α, E_α' $\qquad \langle E_\alpha | E_\alpha' \rangle = 0.$

$$E^-_{(\alpha)} |E_\alpha'\rangle = E_{-\alpha} |E_\alpha'\rangle$$

$$E^-|E'_\alpha\rangle = \beta_i |H_i\rangle$$



$$\Rightarrow \beta_i = \alpha_i \langle E'_\alpha | E_\alpha \rangle = 0$$

$$\Rightarrow \underline{E^-|E'_\alpha\rangle = 0}$$

is a lowest wt state for $SU(2)_\alpha$.

$$E_3 |E'_\alpha\rangle = \frac{\alpha\cdot H}{\alpha^2} |E'_\alpha\rangle = |E'_\alpha\rangle.$$

lowest eval of J^3 is always ≤ 0. Contradiction! ∎

Claim: If α is a root then $k\alpha$ is a root $\Leftrightarrow$ $k = \pm 1$.

Pf: $SU(2)_\alpha$ on $|E_{k\alpha}\rangle$ $\left(H_i |E_{k\alpha}\rangle = k\alpha_i |E_{k\alpha}\rangle \right)$

$$E^3_{(\alpha)} |E_{k\alpha}\rangle = k |E_{k\alpha}\rangle \xRightarrow{SU(2)_\alpha} k \in \mathbb{Z}/2.$$

If $k \in \mathbb{Z}$: $E_{-\alpha}^{k-1} |E_{k\alpha}\rangle \propto |E'_\alpha\rangle$ contradiction.

if $k \in \mathbb{Z} + \frac{1}{2}$ $\rightsquigarrow |E_{\alpha/2}\rangle$ use $SU(2)_{\alpha/2}$ ∎

$\underline{\underline{SU(2)_\alpha}}$ on a general rep $\Rightarrow$ $\underline{|\mu\rangle}$.

$$E_3|\mu\rangle = \frac{\alpha\cdot\mu}{\alpha^2}|\mu\rangle \xRightarrow{SU(2)} \underline{\frac{2\alpha\cdot\mu}{\alpha^2}\in\mathbb{Z}}.$$

$\exists\, p\in\mathbb{Z}_+$ s.t. $\underbrace{(E^+)^p|\mu\rangle}_{\text{highest wt for } SU(2)_\alpha}\neq 0$, $(E^+)^{p+1}|\mu\rangle = 0$

$$\Rightarrow \quad \frac{\alpha\cdot(\mu+p\alpha)}{\alpha^2} = \frac{\alpha\cdot\mu}{\alpha^2}+p \equiv j \quad (\in \mathbb{Z}/2) \qquad ★_1$$

$\exists\, q\in\mathbb{Z}_+$ s.t. $\underbrace{(E^-)^q|\mu\rangle}_{\text{lowest wt state for } SU(2)_\alpha}\neq 0$, $(E^-)^{q+1}|\mu\rangle = 0$.

$$\Rightarrow \quad \frac{\alpha\cdot(\mu-q\alpha)}{\alpha^2} = \frac{\alpha\cdot\mu}{\alpha^2}-q \equiv -j \qquad ★_2$$

$m_{lowest} = -m_{highest} = -j$.

$$★_1 + ★_2 \Rightarrow \frac{2\alpha\cdot\mu}{\alpha^2}+p-q = 0$$ "master eqn"

$$\frac{\alpha\cdot\mu}{\alpha^2} = -\frac{(p-q)}{2} \in \mathbb{Z}/2. \qquad \forall\, |\mu\rangle \quad \forall\, SU(2)_\alpha$$

$$\star_1 - \star_2 \Rightarrow p+q = 2j$$

$$p, q = p_{\alpha,\mu},\ q_{\alpha,\mu}.$$

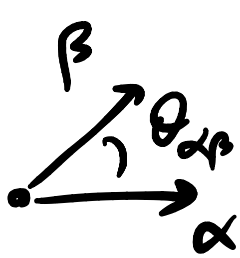
apply to adjoint rep: $|\mu\rangle = |\beta\rangle$ β is a root.

$$\Rightarrow \frac{\alpha\cdot\beta}{\alpha^2} = -\frac{1}{2}(p-q) \qquad ⊚_1$$

$|\mu\rangle = |\alpha\rangle$ wrt $SU(2)_\beta$

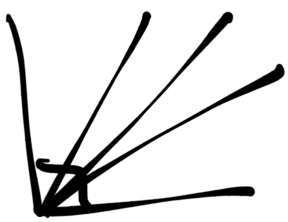
$$\Rightarrow \frac{\beta\cdot\alpha}{\beta^2} = -\frac{1}{2}(p'-q') \qquad ⊚_2$$

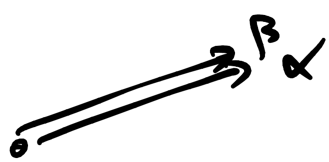
$$⊚_1 \cdot ⊚_2 \Rightarrow \frac{\mathbb{Z}}{4} \ni \frac{(p-q)(p'-q')}{4} = \frac{(\alpha\cdot\beta)^2}{\alpha^2\beta^2} = \cos^2\theta_{\alpha\beta} \in [0,1].$$

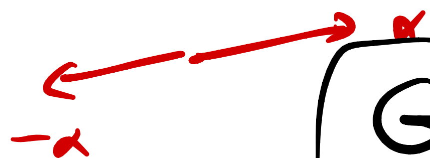


$\cos^2\theta_{\alpha\beta}$	$\lvert\cos\theta_{\alpha\beta}\rvert$	$(p-q)(p'-q')$	$\theta_{\alpha\beta}$
0	0	0	$\pi/2$
1/4	1/2	1	60, 120
1/2	$1/\sqrt{2}$	2	45, 135
3/4	$3/\sqrt{2}$	3	30, 150
1	1	4	~~0~~ or 180

always. ✓







$$G = e^{\mathfrak{g}}$$

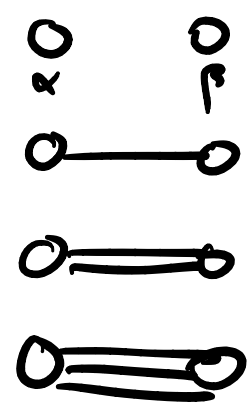
$SU(2)\otimes SU(2)$

$su(2)\oplus su(2)$

$$\frac{\Theta_1}{\Theta_2} \Rightarrow \frac{\alpha^2}{\beta^2} = \frac{p'-q'}{p-q}$$

$p-q$	$p'-q'$	$\theta_{\alpha\beta}$	$\frac{\alpha^2}{\beta^2}$
0	0	$\pi/2$	indeterminate
1	1	60, 120	1
1	2	45, 135	2
1	3	30, 150	3
1	4	0 or π	1
2	2	0 or π ✓	1

Dynkin notation



$$\{ e^{i\alpha\cdot X} \} = G \qquad X^A \text{ generators of the Lie alg, } \mathfrak{g}.$$

$$\equiv e^{\mathfrak{g}}$$

$$G \supset U(1)^r = \{ \underline{\underline{e^{i\alpha^i H^i}}} \}$$

Cartan of $\mathfrak{g}$ = span$\{ \underline{\underline{H^i}} \}$