Winter 21 phys 239: "Algebra & Topology in Physics"

Last time: $\mathfrak{g} \supset su(2)_\alpha$

For each pair of roots $\pm\alpha$

$$\left\{ \begin{array}{l} J^3_\alpha = \frac{\alpha\cdot H}{\alpha^2} \\ J^\pm_\alpha = E_{\pm\alpha}/\sqrt{\alpha^2} \end{array} \right.$$

$$\left([H_i, E_\alpha] = \underset{\substack{\uparrow \\ \text{root}}}{\alpha} E_\alpha \right)$$

Given a state $\underline{|\mu\rangle}$, wrt $su(2)_\alpha$ it has

eg spin $j = 3/2$

$2j+1 = 4$

	m	p	q	$p+q = 2j$
$J^+ \curvearrowright 0$	$3/2$	0	3	3
J^-	$1/2$	1	2	3
J^-	$-1/2$	2	1	3
J^-	$-3/2$	3	0	3

$J^- \curvearrowright 0$

$$\boxed{\begin{aligned} m &= -\frac{(p-q)}{2} \\ &= \frac{\alpha\cdot\mu}{\alpha^2} \end{aligned}}$$

$$H_i|\mu\rangle = \mu_i|\mu\rangle$$

eg: $SU(3)$

8 traceless hermitian 3x3 matrices:

$$\lambda_1 = X_{12} = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} \qquad \lambda_2 = Y_{12} = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix} \qquad \lambda_3 = Z_{12} \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$$

$$\lambda_4 = \underline{\underline{X_{13}}} = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & & 0 \end{pmatrix} \qquad \lambda_5 = Y_{13} = \begin{pmatrix} 0 & 0 & -i \\ & 0 & \\ i & & 0 \end{pmatrix}$$

$$\lambda_6 = X_{23} = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \qquad \lambda_7 = Y_{23} = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{Z_{13} + Z_{23}}{\sqrt{3}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \frac{1}{\sqrt{3}}$$

$$\lambda_A = \lambda_A^{\dagger} \quad , \quad \mathrm{tr}\,\lambda_A = 0.$$

$$[\lambda_1, \lambda_2] = 2i\,\underline{\lambda_3} \qquad SU(2)_{12} < SU(3)$$

$$[\lambda_4, \lambda_5] = 2i\,\lambda_{[4,5]} = i(\lambda_3 + \sqrt{3}\lambda_8)$$

$$\longrightarrow \underline{\underline{f_{ABC}}} \text{ of } SU(3).$$

$$T_A = \lambda_A/2 \quad \text{generate } \underline{3} \text{ of } SU(3) \qquad \left[\mathrm{tr}\, T^A T^B = \tfrac{1}{2}\delta^{AB} \right.$$

Cartan Subalgebra: $H_1 \equiv \lambda_3/2$ $H_2 \equiv \lambda_8/2$.

weights of $\underline{3}$: evals of $\underline{\underline{H_{1,2}}}$ rank(SU(3)) = 2

evec $\begin{pmatrix}1\\0\\0\end{pmatrix}$ has evals $\mu^1 = \left(\frac{1}{2}, \underset{?}{\underline{\underline{\frac{1}{\sqrt{3}}}}}\frac{1}{2}\right)$

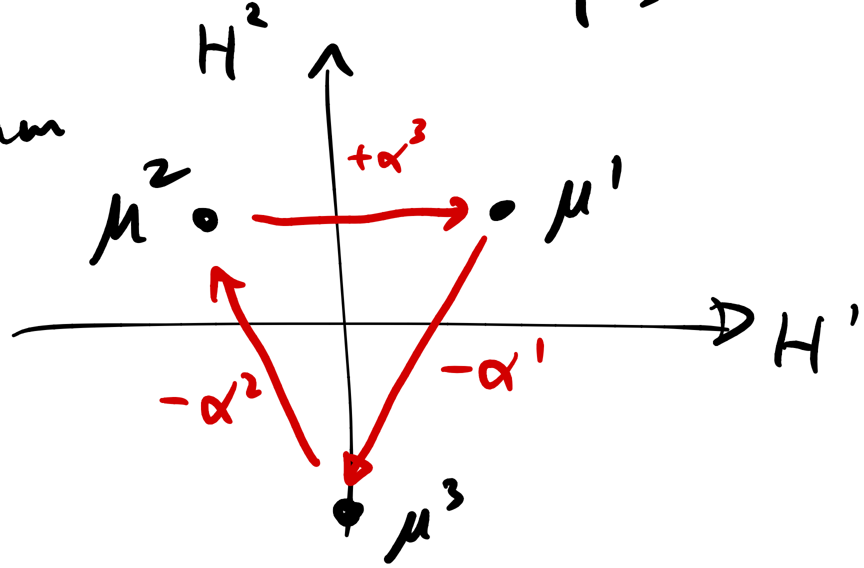
evec $\begin{pmatrix}0\\1\\0\end{pmatrix}$ " " $\mu^2 = \left(-\frac{1}{2}, \frac{1}{\sqrt{3}}\frac{1}{2}\right)$

evec $\begin{pmatrix}0\\0\\1\end{pmatrix}$ " " $\mu^3 = \left(0, -\frac{2}{\sqrt{3}}\frac{1}{2}\right)$

check: $\frac{1}{2}\delta^{88} \overset{!}{=} \mathrm{tr}\left(\frac{\lambda_8}{2}\right)^2 = \frac{1}{4}\cdot\left(\frac{1+1+4}{3}\right)$

$= \frac{6}{4}\frac{1}{3} = \frac{2}{4} = \frac{1}{2}$ ✓

weight diagram for $\underline{3}$

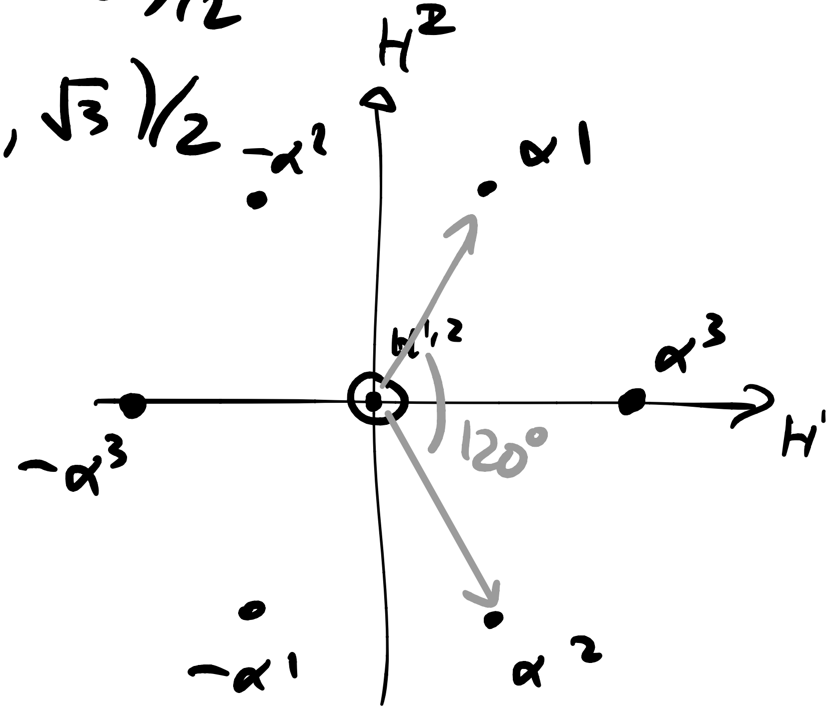


Roots are a property of the algebra: are differences of weights.

Roots:

$$\alpha^3 = \mu^1 - \mu^2 = (1, 0) \qquad \text{and } -\alpha^{a=1..3}$$

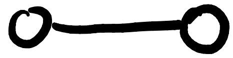
$$\begin{cases} \alpha^1 = \mu^3 - \mu^2 = (1, -\sqrt{3})/2 \\ \alpha^2 = \mu^1 - \mu^3 = (1, \sqrt{3})/2 \end{cases}$$



$[H_i, H_j] = 0$.
have weight zero.

$$\begin{cases} \dfrac{\alpha^1 \cdot \alpha^2}{|\alpha^1||\alpha^2|} = -\dfrac{1}{2} \\ \dfrac{(\alpha^1)^2}{(\alpha^2)^2} = 1 \Rightarrow \text{single line.} \end{cases}$$

Dynkin Diagram:



$$\begin{cases} E_{\pm\alpha^3} = (\lambda_1 \pm i \lambda_2)/\sqrt{8} \\ E_{\pm\alpha^2} = (\lambda_4 \pm i \lambda_5)/\sqrt{8} \\ E_{\pm\alpha^1} = (\lambda_6 \mp i \lambda_7)/\sqrt{8} \end{cases}$$

eg $E_{\alpha^1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$$E_{-\alpha^1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = E^{+}_{\alpha^1}.$$

Claim: $[H_i, E_{\pm\alpha^a}] = \pm \alpha^a_i E_{\pm\alpha^a}$.

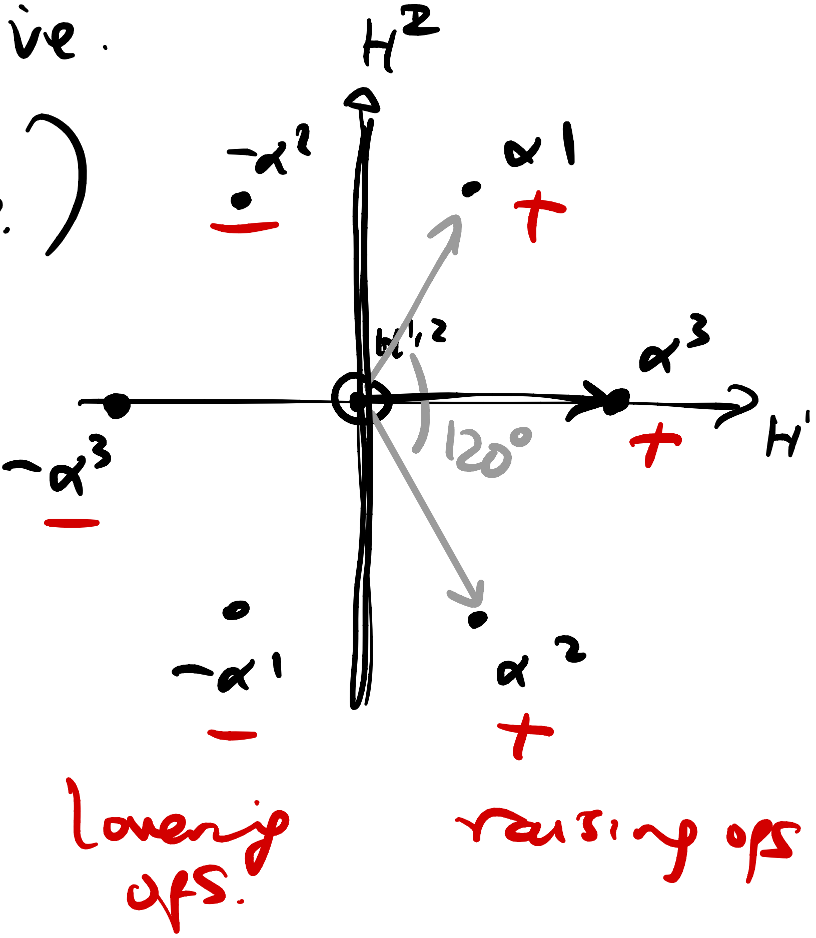
Positive & simple roots : CONVENTION

- choose an order for H_1, H_2.
- a root $\alpha > 0$ if its first nonzero entry is +ive.

(– $\alpha_{1,2,3} > 0$, $-\alpha_{1,2,3} < 0$.)

- $\alpha > \beta$ if $\alpha - \beta > 0$.
- same convention for wts.

$\Rightarrow \exists$ highest weight.



A root α is **Simple**: if $\alpha > 0$

and $\alpha \neq a\beta + b\gamma$ $\beta, \gamma, a, b > 0$.

eg: simple roots
$$\begin{cases} \alpha^1 = (1, \sqrt{3})/2 \\ \alpha^2 = (1, -\sqrt{3})/2 . \end{cases}$$

$\alpha^3 = \alpha^1 + \alpha^2$ is not simple.

There are r simple roots, they span the root space.

(**Heuristic**: The simple roots are closest to the bdy of the cone of positive roots)

Build an irrep: Consider a highest wt state $|\phi\rangle$

$$H_i|\phi\rangle = (\mu_\phi)_i|\phi\rangle.$$

highest wt $\Rightarrow$ $E_\alpha|\phi\rangle = 0$ $\forall$ positive roots α.

$$|\{\alpha_k\}\rangle_\phi = \underline{\underline{E_{-\alpha_1}E_{-\alpha_2}\cdots|\phi)}}$$

simple roots

is another state in the rep.

(could be zero)

ANALOGY: $|\phi) \longrightarrow |0)$ Fock vacuum

$E_\alpha|\phi) = 0 \longrightarrow a_\alpha|0) = 0$

$E_\alpha \longrightarrow a_\alpha$ annihilate

$E_{-\alpha} \longrightarrow a^\dagger_\alpha.$ creation.

$[H_i, E_\alpha] = \alpha_i E_\alpha \longrightarrow |\{\alpha_k\}\rangle_\phi$

has weight

$\mu_\phi + \sum_k \alpha_k$.

IRREP w highest wt ϕ :

$$\underline{R_\phi = \text{span}\{|\{\alpha\}\rangle_\phi\}}$$

$$\langle\{\beta\}|\{\alpha\}\rangle = \langle\phi|\underbrace{E^\dagger_{-\beta_n}\cdots E^\dagger_{-\beta_1}}_{E_{+\beta_n}\cdots E_{\beta_1}} E_{-\alpha_1} E_{-\alpha_2}\cdots|\phi\rangle$$

$\underline{\beta, \alpha > 0.}$

$\underline{E_{\beta_i}|\phi\rangle = 0.}$

$\langle 0|\, a \;\; \underline{a^\dagger} \;\; |0\rangle$

$[a, a^\dagger] = 1$

$[E_{\beta_1}, E_{-\alpha_1}]$

$= N_{\beta_1,-\alpha_1} E_{\beta_1-\alpha_1}$

(only nonzero if $\beta_1-\alpha_1$ is a root.)

Q: which ϕ can be the H.W. for a finite-dim'l (unitary) rep?

$E_\alpha|\phi\rangle = 0 \quad \alpha > 0. \Rightarrow \phi+\alpha$ is not a weight.

$\Rightarrow |\phi\rangle$ is H.W. for $SU(2)_a$ $\forall a$.

$$\underline{\frac{2\alpha^a\cdot\phi}{(\alpha^a)^2}} = -(\cancelto{0}{p^a} - q^a) = q^a \quad \begin{matrix}\in\mathbb{Z}\\ \geq 0.\end{matrix}$$

r conditions on ϕ.

$$\boxed{\frac{2\alpha^a\cdot\phi}{(\alpha^a)^2}\in\mathbb{Z}_{\geq 0}\ \forall a.}$$

$\{$possible HW states$\}$ $\longleftrightarrow$ $\{$ choices of $q^a\}$

A basis for H.W. vectors $\{\mu^b\}_{b=1}^{r}$

$$\frac{2\alpha^a\cdot\mu^b}{(\alpha^a)^2} \overset{!}{=} \delta^{ab} \qquad \mu^b \equiv \text{fundamental weights}$$

$\Lambda_R \equiv \{n_a\alpha^a, n_a\in\mathbb{Z}\}$

↑ simple roots

Root lattice.

$\Lambda_W = \{m_b\mu^b, m_b\in\mathbb{Z}\}$

$= \Lambda_R^*$. weight lattice..

Any highest weight is

$$\mu = \sum_{b=1}^{r} \underline{\underline{q_b}}\, \mu^b$$

$\uparrow$ $q_b \in \mathbb{Z}_{\geq 0}$

Dynkin indices of the rep.

Build irreps of SU(3):

$$\begin{cases} \alpha^1 = (1, \sqrt{3})/2 \\ \alpha^2 = (1, -\sqrt{3})/2 \end{cases}$$ Simple roots. $(\alpha^a)^2 = 1$.

fund. weights: $\mu^a \cdot \alpha^b = \frac{1}{2}\delta^{ab}$

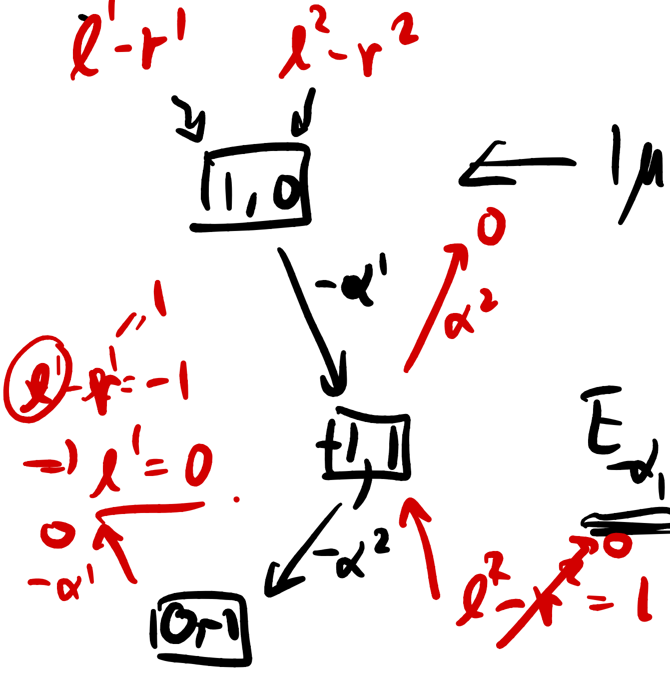
$$\Rightarrow \mu^{1,2} = \left(\frac{1}{2}, \pm\frac{\sqrt{3}}{6}\right).$$

$R_{\mu^1} = R_{(1,0)}$

$= R_{1\cdot\mu^1 + 0\cdot\mu^2}$:

$$2\frac{\alpha^a \cdot \mu^1}{(\alpha^a)^2} = q^a = \#\text{ of times we can lower w/ } E_{-\alpha^a}$$

$= (1, 0)$



$\leftarrow |\mu^1\rangle$ $\quad$ $\underline{\text{wt}}$ $\quad \mu^1$

$$\downarrow$$

$$2\frac{\alpha^a \cdot (\mu^1 - \alpha^1)}{(\alpha^a)^2} = (l^a - r^a)$$

$\underline{\underline{E_{-\alpha_1}|\mu^1\rangle}}$ $\quad \mu^1 - \alpha^1$

$\searrow$ $= (1,0) - (2,-1)$

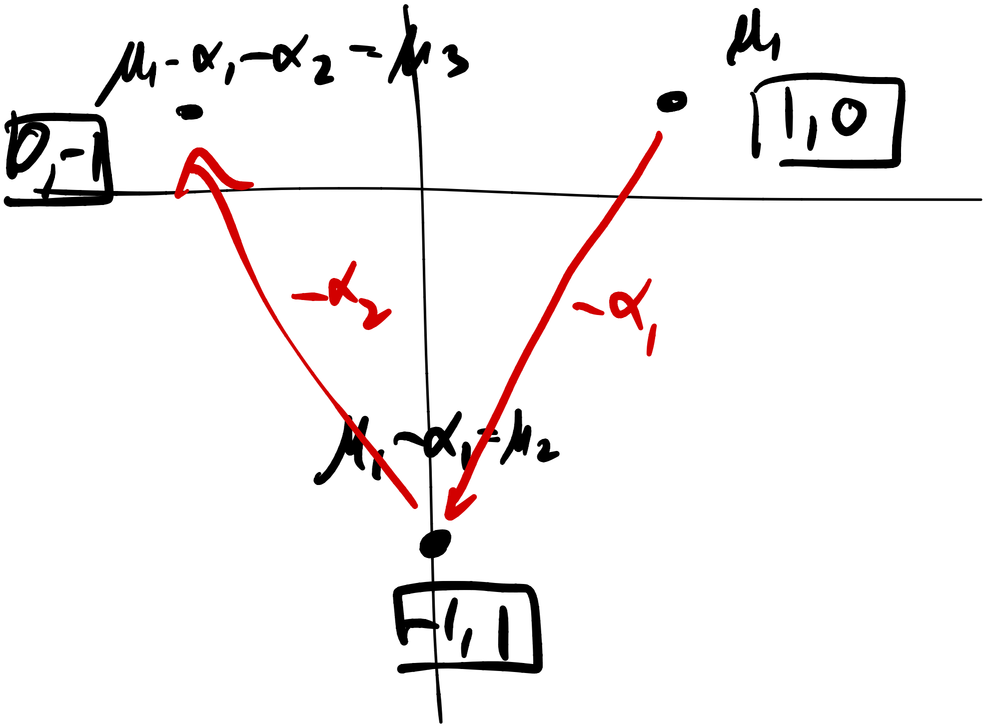
$E_{-\alpha_2} E_{-\alpha} |\mu^1\rangle$ $\quad \mu^1 - \alpha^1 - \alpha^2 = (-1, 1)$

$l^1 - r^1 = 0 \Rightarrow l^1 = 0$

$l^2 - \underset{=1}{\tilde{r}^2} = -1 \Rightarrow l^2 = 0$

$$\frac{2\alpha^a(\mu^1-\alpha^1-\alpha^2)}{(\alpha^a)^2} = (-1,1) - \underline{\underline{(-1,2)}}$$

$$= (0,-1)$$



$$A_{ab} = 2\frac{\alpha^a \cdot \alpha^b}{(\alpha^a)^2} \overset{SU(3)}{=} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}_{ab}$$

Cartan matrix. ($\Leftarrow$ simple roots)

is what we subtract from $(l^a - r^a)$ when acting w/ $E_{-\alpha^b}$.

Q: how does $|\alpha^b\rangle$ transform under $SU(2)_{\alpha^a}$?

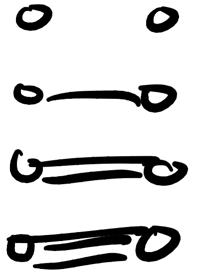
Properties of Cartan matrix :

- $A_{ab} = \ell^a - r^a$ for $|\alpha^b)$

  $= \Big(\underline{\underline{\underline{2}}}$ eval of $\frac{\alpha^a \cdot H}{(\alpha^a)^2} \Big) \in \mathbb{Z}$

- $A_{bb} = 2.$
- $A_{ba} \neq A_{ab}$ if roots have different lengths $(\alpha^a)^2 \neq (\alpha^b)^2.$
- $A_{b\neq a} \in \{0, -1, -2, -3\}$

  encode $\theta_{\alpha^a\alpha^b}$, $\frac{(\alpha^a)^2}{(\alpha^b)^2}.$



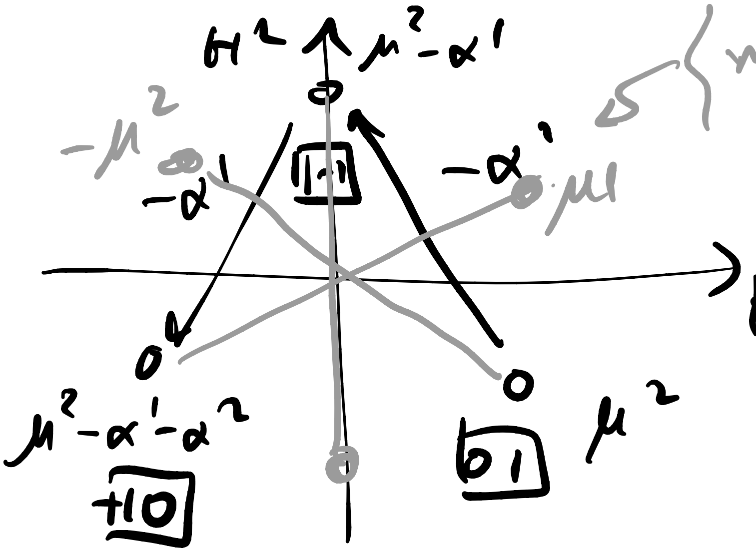
- invertible since the r α^a are lin indep

$R_{\mu^2} = R_{(0,1)}$:

$$\begin{tikzcd}
\boxed{0\,1} \arrow[d, "-\alpha^2"'] & |\mu^2\rangle & \overset{\text{wt}}{\mu^2} \\
\boxed{1,-1} \arrow[d, "-\alpha^1"'] & E_{-\alpha^2}|\mu^2\rangle & \mu^2 - \alpha^2 \\
\boxed{-1,0} & E_{-\alpha^1}E_{-\alpha^2}|\mu^2\rangle & \mu^2 - \alpha^2 - \alpha^1
\end{tikzcd}$$

subtract $A_{2b} = (-1,2)_b$

subtract $A_{1b} = (2,-1)$



$$\underset{\equiv \underline{3}}{\{\text{wts of } R_{10}\}} = \underbrace{\{-\text{wts of } R_{01}\}}_{= \underline{3}}.$$

<u>claim</u>: The rep whose wts are $\{-\mu\}$ (wts of R are $\{\mu\}$) is $\bar{R}$.

<u>Pf</u>: $D_{\bar{R}}(g) = D_R(g)^*$.

$$D_R(g) = e^{i\theta\cdot T_R}$$

$$D_{\bar{R}}(g) = D_R(g)^* = e^{-i\theta\cdot T_R^*}$$

$\Rightarrow$ If T_R^A generate R

$(-T_R^A)^*$ " $\bar{R}$.

Cartan generators of $\bar{R}$: $-H_i^*$ has evals $\underline{-\mu_i}$.

A general rep of SU(3): $R_{n\mu^1+m\mu^2} = R_{(n,m)}$

$n, m \in \mathbb{Z}_{\geq 0}$.

$$\overline{R} = \overline{R}_{(m,n)}.$$

Weyl Group:

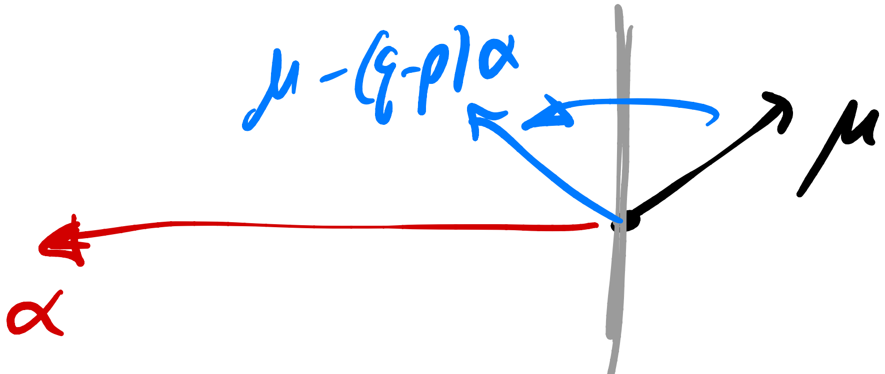
Recall that SU(2) reps are symmetrical under $\underline{m \to -m}$.

for each $SU(2)_\alpha \subset \mathfrak{g}$.

$$J_z^\alpha = \frac{\alpha\cdot H}{\alpha^2}, \quad J_z^\alpha|\mu\rangle = \frac{\alpha\cdot\mu}{\alpha^2}|\mu\rangle$$

The state w/ eval $-m$ in the same $SU(2)_\alpha$ irrep has weight

$$\mu - (q-p)\alpha \quad \text{where } q-p = \frac{2\alpha\cdot\mu}{\alpha^2}.$$



indep of $\alpha \to \lambda\alpha$, $\lambda \in \mathbb{R}\setminus 0$

If μ is a wt, α a root

$$\boxed{W_\alpha : \mu \to \mu - \frac{2\alpha\cdot\mu}{\alpha^2}\alpha}$$

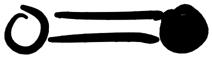
is also a root in the same $SU(2)_\alpha$ irrep as μ

orbits of simple roots under $\mathcal{W} = \langle W_\alpha , \alpha \text{ simple} \rangle$

= all the roots.

Weyl chamber: fund. domain for the action of $\mathcal{W}$

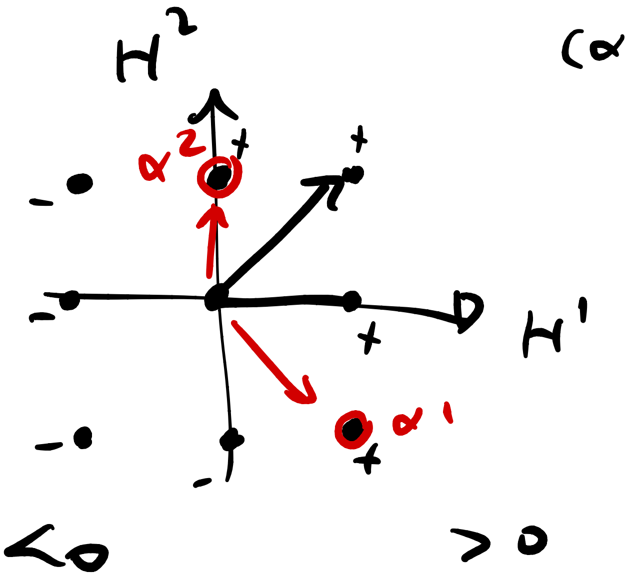
SO(5) :



$$|\cos\theta_{12}| = \frac{1}{\sqrt{2}} .$$

$$\frac{(\alpha^1)^2}{(\alpha^2)^2} = 2 .$$

$$\theta_{12} = + 135^\circ .$$



$SO(4) = SO(3) \oplus SO(3)$. $[J_{ij}, J_{kl}] = i(\delta_{ik} J_{jl} + \cdots$

$so(3,1)$: $[J_{\mu\nu}, J_{\rho\sigma}] = i(\eta_{\mu\rho} J_{\nu\sigma} + \cdots\cdots)$

$\simeq sl(2) \oplus sl(2)$

$$\boxed{e^{i\underline{\eta}\cdot J}}$$

$\underline{J_{0i} \to i J_{ii}}$