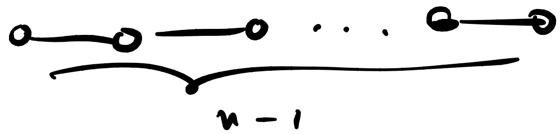
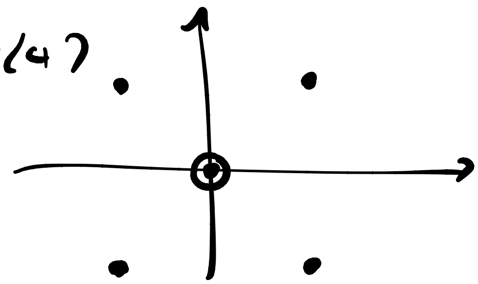
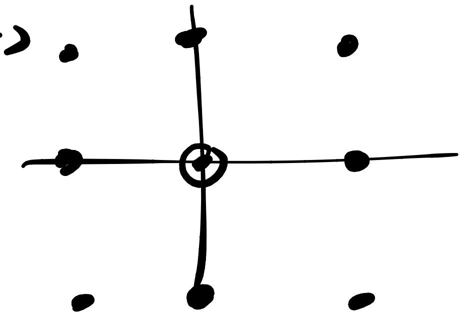
A_{n-1}  $\to su(n)$

B_n  $\to so(2n+1)$

D_n  $\to so(2n)$

eg: $SO(4)$  $so(5)$ 

$C_n =$  $\to sp(2n)$

$Sp(2n) = \{ 2n \times 2n$ matrices preserving an AS form $\omega : V \otimes V \to \mathbb{C} \}$

$V = \underline{2n}$

$M \in Sp(2n)$
if $\omega(Mv, Mw) = \omega(v,w) \quad \forall\, v,w \in V$

$$\left(\omega(v,w) \equiv \sum_{i=1}^{2n} \omega_{ij} v^i w^j \right)$$

Lie alg:

$M = e^{iX} \quad (X = X^{+})$

$$\Rightarrow \underline{\omega(Xv, w) + \omega(v, Xw) = 0.}$$

choose $\omega = Y \otimes \mathbb{1} = \left(\begin{array}{c|c} 0 & -i\mathbb{1}_n \\ \hline i\mathbb{1}_n & 0 \end{array}\right)$

$\Rightarrow \quad \underline{\underline{Y X^T Y + X = 0.}}$

expand $\quad X = \sigma^i \otimes 6^i + \mathbb{1} \otimes ∂$

<u>use</u>: $Y(\sigma^i)^T Y = -\sigma^i$

$\Rightarrow \quad X = \underset{\substack{\uparrow \text{ real} \\ n\times n \;\; 3 \text{ symm.}}}{\sigma^i \otimes S^i} + \underset{\substack{\uparrow \\ 1 \text{ A.S. imaginary } n\times n}}{\mathbb{1} \otimes A}$

$\dim sp(2n) = 3\,\frac{n(n+1)}{2} + 1\,\frac{n(n-1)}{2} = n(2n+1).$

<u>choose a Cartan subalg</u>: $\quad H_m = \sigma^3 \otimes h_m$

$\left((h_m)_{ij} \equiv \delta_{mi}\delta_{mj} \right) \qquad = \left(\begin{array}{c|c} h_m & 0 \\ \hline 0 & -h_m \end{array}\right)$

<u>wts of $\underline{2n}$</u>: $\quad$ The state $|i\rangle$ is an evec of H_m

w/ eval $\begin{cases} \delta_{im} & i \leq n \\ -\delta_{im} & i > n \end{cases}$

$\Rightarrow$ wts are $\underline{(\pm e^i)_m}$.

<u>roots of $sp(2n)$</u>: $\pm e^i \pm e^j \quad i \neq j$

But <u>also</u> $\pm 2e^i$.

eigenvectors w/ evals ± 1 under H_1

are $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$ are related by an $Sp(2n)$ transf.

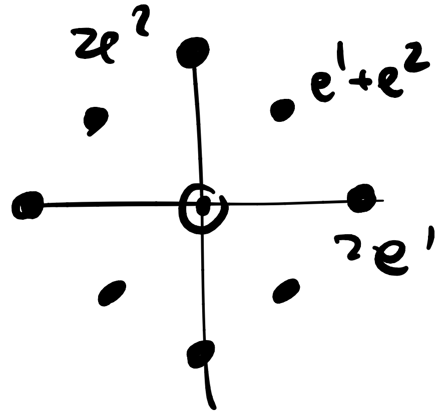
eval $+1$ eval -1

<u>positive roots</u>: $\{ e^i \pm e^j, \; i > j, \; 2e^i \;\; i = 1..n \}$

<u>Simple roots</u>: $\underbrace{e^i - e^{i+1}}_{\text{short}} \; (i = 1..n-1), \; \underbrace{2e^n}_{\text{long}}$.

$\sqrt{\alpha^2} = \sqrt{2}$ $\sqrt{\alpha^2} = 2$.

Sp(4)



$= SO(5)$

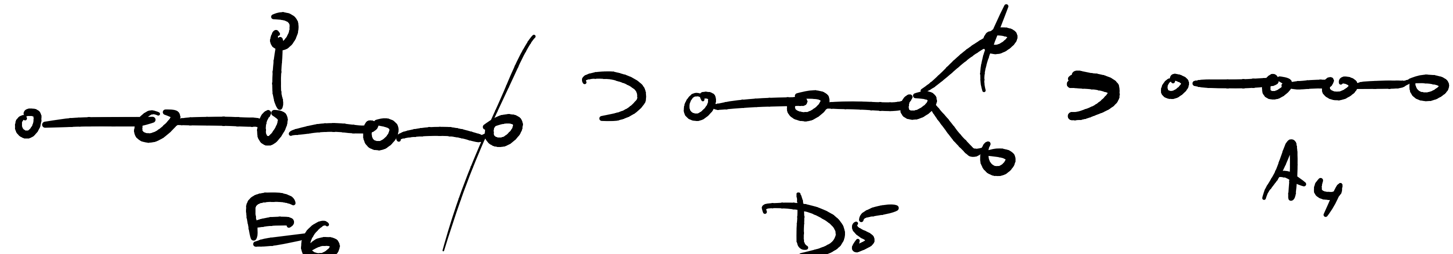
∎

3.8 Regular subalgebras

$\mathfrak{h}$ $\equiv$ shares Cartan generators & roots.

$\underline{\underline{SU(2)_{12}}} \subset SU(3)$.

eg:



$E_6 \supset D_5 = SO(10) \supset A_4 = SU(5)$.

not every regular subalg. arises as a subdiagram:

eg: $G_2 \supset SU(3)$ w/ simple roots $\alpha_2,\ 3\alpha_1+\alpha_2$

(G_2: simple roots $=\alpha_1, \alpha_2$)

eg: $SU(4) \supset G_2$

$$15 = 14 + \underset{H_3}{\underline{1}}$$

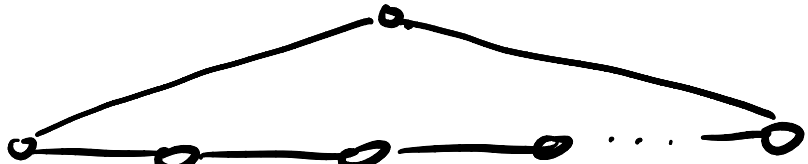
A <u>maximal</u> subalgebra of $\mathfrak{g}$ has rank = rank $\mathfrak{g}$.

<u>Trick</u>: add to the Dynkin diagram <u>the lowest root</u> α_0.

remove a node from "extended Dynkin diagram" → maximal regular subalg.

$\hat{A}_n$

$\alpha^0 = -e^1 + e^n$

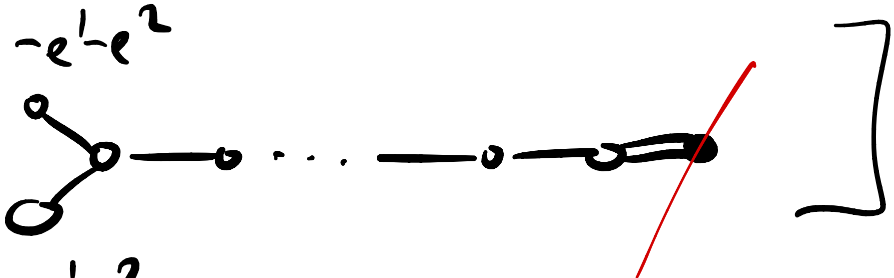


$e^1 - e^2 \quad e^2 - e^3 \qquad\qquad e^{n-1} - e^n$

$\alpha^1 = \qquad\qquad\qquad \alpha^{n-1}$

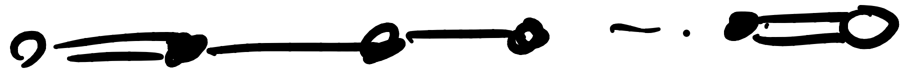
$\hat{B}_n$

$\alpha^0 = -e^1 - e^2$

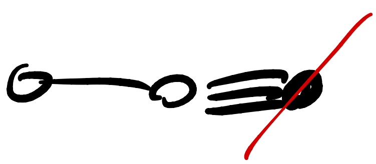


$\hat{D}_n =$ $\alpha^0 = -e^1 - e^2$

$\hat{C}_n =$ $-2e^1$



$\hat{G}_2$:

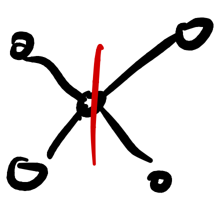


$\Rightarrow$ $\underline{G_2 \supset SU(3)}$

⋮

$SO(2n+1) \supset SO(2n)$

$\hat{D}_4$:



$\Rightarrow$ $\underline{SO(8) \supset SU(2)^4}$.

3.9 Spinor (projective) reps of $SO(N)$

$$\left(\phi \to \mathbb{Z}_2 \to \underbrace{Spin(N)}_{} \to SO(N) \to \phi \right)$$

"Majorana zeromodes" : ("Clifford algebra")

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{1} \qquad \gamma_i^\dagger = \gamma_i \qquad i = 1..2n$$

↑ suggests some connection to $SO(2n)$?

$$\begin{cases} c_a \equiv \frac{1}{2}(\gamma_{2a-1} + i\gamma_{2a}) \\ c_a^\dagger \equiv \frac{1}{2}(\gamma_{2a-1} - i\gamma_{2a}) \end{cases} \qquad a = 1..n.$$

$$\xRightarrow{\text{clifford}} \quad \{c_a, c_b^\dagger\} = \delta_{ab}\mathbb{1}, \quad \{c_a, c_b\} = 0$$

(ordinary canonical fermion anticommutation algebra)

Representations: vacuum $|0\rangle$ s.t. $c_a|0\rangle = 0 \;\; \forall a$.

$c_a^\dagger|0\rangle$, $c_a^\dagger c_b^\dagger|0\rangle$ ... $c_1^\dagger \cdots c_n^\dagger|0\rangle$
$= |1\rangle$ "plenum".

$N_a \equiv c^\dagger_a c_a$ # operator (no sum)

$$N_a \, c^\dagger_b |0\rangle = \delta_{a,b} \, c^\dagger_b |0\rangle .$$

$$\mathcal{H} = \text{span}\{ |0\rangle, c_a^\dagger |0\rangle, c_a^\dagger c_b^\dagger |0\rangle \dots \}$$

is 2^n dim'l $\quad = \text{span}\{ |s_1 \dots s_n\rangle \}$

$s_a = +\frac{1}{2}$ unoccupied

$\quad\;\; -\frac{1}{2}$ occupied

i.e. $c_a^\dagger c_a |s_1 .. s_n\rangle = (\frac{1}{2} - s_a) |s_1 .. s_n\rangle$

Let $T^{ij} \equiv \frac{1}{2}\sqrt{-1}\,\gamma_i \gamma_j \quad i \neq j$

$= \frac{1}{4}\sqrt{-1}\,[\gamma_i, \gamma_j]$

$(T^{ij})^\dagger = T^{ij}$

claim: these satisfy $so(2n)$ alg.

$$A \to \Gamma_A \equiv \frac{1}{2} A_{ij} T^{ij}$$

$A_{ij} = -A_{ji}$.

satisfy

$$[\Gamma_A, \Gamma_B] = \Gamma_{[A,B]} .$$

is a rep of $so(N)$.

$i = 1 \dots N$

$$\left[e^{i A_{ij} \hat{T}_{ij}} \in SO(N). \right.$$

$\Rightarrow \mathcal{H}$ carries a rep of $so(2n)$.

<u>claim:</u> It is reducible.

$$\gamma_{2n+1} = \gamma_F \equiv C\,\gamma_1 \gamma_2 \cdots \gamma_{2n}$$

choose C s.t. $\gamma_F^{+} = \gamma_F$, $\gamma_F^2 = \mathbb{1}$.

$$\{\gamma_F, \gamma_i\} = 0 \quad \forall\, i = 1..2n.$$

$$\Rightarrow [\gamma_F, T^{ij}] = 0.$$ $\Rightarrow \gamma_F$ is an intertwiner.

$\gamma_F^2 = 1 \Rightarrow$ evals are ± 1. each eigenspace is an irrep.

<u>Cartan-Weyl</u> : Cartan subalg of $so(2n)$ is

$$\{ H_a = \tfrac{i}{2}\, \gamma^{2a-1}\gamma^{2a} \quad a = 1..n \}$$

$$N_a = c_a^{+} c_a = \tfrac{1}{2}\big(\underbrace{1 + i\,\gamma^{2a-1}\gamma^{2a}} \big)$$

$\boxed{\text{evals of } i\gamma^i\gamma^j \text{ are } \pm 1.}$

evals of H_a on $|s_1 \ldots s_n\rangle$.

$$H_a |s_1 .. s_n\rangle = s_a |s_1 .. s_n\rangle.$$ $\Rightarrow$ wts are $\tfrac{1}{2}(\pm e^1 \pm e^2 \cdots \pm e^n)$.

$$\gamma_F = \text{sign}(H_1 \cdots H_n) = \text{sign}(s_1 \cdots s_n)$$
$$= (-1)^{\#\text{ of } - \text{signs}}$$

highest wt of irrep w/ $\gamma_F = +1$ is $\frac{1}{2}\sum_{a=1}^{n} e^a = \mu^n$

" " " " $\gamma_F = -1$ is $\frac{1}{2}\sum_{a=1}^{n-1} e^a - \frac{1}{2}e^n = \mu^{n-1}$

(⟹ these are irreps) fund wts of $SO(2n)$ ✓

Raising & lowering ops:

$$\begin{cases} H_a = \frac{1}{2} i \gamma^{2a-1}\gamma^{2a} = c_a^\dagger c_a - \frac{1}{2} \\ E_{ab} \equiv c_a^\dagger c_b \qquad (E_{ab})^\dagger = E_{ba} \\ E'_{ab} \equiv c_a^\dagger c_b^\dagger \qquad (E'_{ab})^\dagger = c_b c_a \\ \qquad = -E'_{ba} \end{cases}$$

$$E_{12} \left| -\tfrac{1}{2}, +\tfrac{1}{2} \cdots \right\rangle \propto \left| \tfrac{1}{2}, -\tfrac{1}{2} \cdots \right\rangle$$

changes weight by $e^1 - e^2$.

$$[H_a, E_{bc}] = (\delta_{ab} - \delta_{ac}) E_{bc} = (e_b - e_c)_a E_{bc}$$

$\Rightarrow e_b - e_c \quad b \neq c$ is a root vector.

roots of $SU(n)$!

$[\sum_a H_a, E_{bc}] = 0.$ $\quad \sum_a H_a = \sum_a c_a^\dagger c_a$

$=$ total particle #.

$SU(n) \subset SO(2n)$

$=$ subgroup preserving $\sum_a H_a$ the # of particles.

$=$ " respecting the pairing $c_a = \frac{1}{2}(\gamma^{2a-1} + i\gamma^{2a})$

Q: how do the spinor reps $(2^{n-1})_\pm$ decompose under $SO(2n) \supset SU(n)$?

$+$: even # of particles

$-$: odd # of particles.

# of particles	states	rep of $SU(n)$ $\quad n=5$
0	$\|0\rangle$	$\underline{1}$
1	$A_a c_a^\dagger \|0\rangle$	$\underline{n} = \underline{5}$
2	$A_{ab} c_a^\dagger c_b^\dagger \|0\rangle$	$\frac{n(n-1)}{2} = \underline{10}$
3	$A_{abc} c_a^\dagger c_b^\dagger c_c^\dagger \|0\rangle$	$\frac{n(n-1)(n-2)}{3!} = \overline{10}$

$$|1\rangle = c_1^\dagger c_2^\dagger \cdots c_n^\dagger |0\rangle$$

$$= \frac{1}{n!} \epsilon_{a_1 \ldots a_n} c_{a_1}^\dagger \cdots c_{a_n}^\dagger |0\rangle$$

$\tilde{A}_a c_a |1\rangle$ has $n-1$ particles

$\tilde{A}_{ab} c_a c_b |1\rangle$ " $n-2$ "

$A_{abc} \equiv \epsilon_{abcde} \tilde{A}_{de}$.

# of particles	states	rep of $SU(n)$ $\quad n=5$
0	$\lvert 0\rangle$	$\underline{1}$
1	$A_a c_a^\dagger \lvert 0\rangle$	$\underline{n} = \underline{5}$ ←
2	$A_{ab} c_a^\dagger c_b^\dagger \lvert 0\rangle$	$\frac{n(n-1)}{2} = \underline{10}$
3	$A_{abc} c_a^\dagger c_b^\dagger c_c^\dagger \lvert 0\rangle$ $= \epsilon_{abcde} \tilde{A}_{de} c_d c_e \lvert 1\rangle$	$\frac{n(n-1)(n-2)}{3!} = \overline{10}$ ←
4	$\tilde{A}_d c_d \lvert 1\rangle$	$\underline{n} = \overline{5}$
5	$\lvert 1\rangle$	$\underline{1}$ ←

$$\underline{16}_- = \overset{5 \text{ particles}}{\underline{1}} \oplus \overset{3}{\underline{10}} \oplus \overset{1}{\underline{5}}$$

$$\underline{16}_+ = \underset{4}{\overline{5}} \oplus \underset{2}{\overline{10}} \oplus \underset{0}{1}.$$

$Sp(2n) = \{ M$ preserving $\omega \in \Lambda^2 \underline{2n} \}$

$SO(N) = \{ M$ " $\delta \in \underline{\underline{Sym^2 \underline{N}}} \}$

$Sym^2 N = \underset{\substack{\uparrow \\ \underline{\text{trace.}}}}{\underline{1}} \oplus (\underline{\text{traceless}})$ in $SO(N)$

in $Sp(2n)$ $\quad \Lambda^2 \underline{2n} = \underset{\substack{\uparrow \\ \omega}}{\underline{1}} \oplus (\ldots)$.

$E'_{ab} = \underline{\underline{c^\dagger_a c^\dagger_b}}$ "Cooper pair operator".

$\boxed{a \neq b \,! \quad (c^\dagger_a)^2 = 0}$

$[H_a, E'_{bc}] = (\delta_{ab} + \delta_{ac}) E'_{bc}$

$= (e_b + e_c)_a E'_{bc}$.

$\underline{\text{count}}$: $\underset{\text{cartan}}{n} + \underset{E_{ab}\text{s}}{n(n-1)} + \underset{E'_{ab}, (E'_{ab})^\dagger}{\frac{n(n-1)}{2} \cdot 2} = n(2n-1) = \dim SO(2n)$ ✓

the rest of the roots $\pm e_a \pm e_b$.

$[H_a, (E'_{bc})^\dagger] = -(e_b + e_c)(E'_{bc})^\dagger$.

$\underline{SO(2n+1)}$: $\underline{\text{same } \mathcal{H}}$.

$$\gamma_F \equiv \gamma_{2n+1} \,.$$

$\gamma_F = \gamma_F^\dagger$, $\gamma_F^2 = \mathbb{1}$

$\{\gamma_F, \gamma_i\} = 0$

$i = 1..2n$

$$\Rightarrow \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

$\underline{i,j = 1..2n+1}$, $T^{ij} = \frac{i}{2}\gamma^i\gamma^j$

$j = 1..2n+1$

represent $SO(2n+1)$.

now $[\gamma_F, T_{ij}] \neq 0$

for $j = 2n+1, i$.

now $\mathcal{H}$ of dim 2^n is an irrep of $SO(2n+1)$.

$$\rightarrow \mu^n = +\frac{e^1}{2} + \frac{e^2}{2} \;\cdots\; + \frac{e^n}{2}\,.$$

$\underline{\text{Matrix Rep}}$: $\mathcal{H} = \text{span}\{|s_1 s_2 .. s_n\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \cdots \otimes |s_n\rangle\}$

$= \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$

each $\mathcal{H}_a$ is a qubit.

$$\underline{\underline{\vec{\sigma}_a}}\, |s_1 .. s_n\rangle = \sum_{s_a'} (\vec{\sigma})_{s_a s_a'} |s_1 \ldots s_a' \ldots s_n\rangle$$

or $\vec{\sigma}_a = \mathbb{1} \otimes \cdots \otimes \underbrace{\vec{\sigma}}_{a\text{th site}} \otimes \cdots \otimes \mathbb{1}$.

Cartan generators are $H_a = \frac{1}{2} Z_a$.

(evals are $s_a = \pm \frac{1}{2}$)

<u>notice</u>: $(T^{ij})^2 = \frac{\mathbb{1}}{4}$. on this rep.

<u>Goal</u>: write the $so(2n+1)$ generators in terms of these Pauli matrices.

$$E_a \equiv \left(T_{2a-1,2n+1} - i T_{2a,2n+1} \right)$$

$$= i \, \frac{\gamma^{2a-1} - i\gamma^{2a}}{2} \cdot \gamma_F$$

$$= i \, c_a^\dagger \, \underline{\underline{\gamma_F}}$$ <u>acts like σ_a^+</u>.

claim: $\{E_a, E_b\} = \{ i c_a^\dagger \gamma_F, i c_b^\dagger \gamma_F \}$

$$\propto \underline{\underline{\{ c_a^\dagger, c_b^\dagger \}}} = 0.$$

$$[\sigma_a^+, \sigma_b^+] = 0.$$