# A Conjecture on the Representation of Massless Particles in Higher Dimensions

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In a paper on the little group representations of massless one-particle states in dimensions $d \geq 4$, Weinberg (2010) explores some examples of tensor and spinor fields that describe such states, and makes a conjecture using the language of Young tableaux that summarizes the findings from these examples. A proof of the conjecture is given by Distler (2010). We go through the general ideas and implications of the conjecture and its proof in this brief summary.

## INTRODUCTION

We generally assume that the laws of nature are invariant under transformations of the Lorentz group $\mathsf{SO}(3,1)$ in a $d=4$ spacetime. A particular subgroup of $\mathsf{SO}(3,1)$, called the little group, consists of elements that leave the spatial momentum $\mathbf{k}$ invariant, and its representations describe one-particle states of a field that transform under the Lorentz group [1–3]. This result may be generalized for $\mathsf{SO}(d-1,1)$. Given a field with known transformation properties under $\mathsf{SO}(d-1,1)$, the question of how its particles transform under the little group boils down to the group theory question: how do the $\mathsf{SO}(d-1,1)$ irreps decompose under the little group?

For massive particles, we can work in the rest frame where $\mathbf{k}=0$. It becomes clear the little group is the rotation group $\mathsf{SO}(d-1)$, and the particles can furnish any representation of this group that is contained in the $\mathsf{SO}(d-1,1)$ representation of the field. For massless particles, the little group is the Euclidean group $\mathsf{E}(d-2)$, which contains an invariant $(d-2)$-dimensional abelian group $\mathsf{T}$, such that $\mathsf{SO}(d-2) \cong \mathsf{E}(d-2)/\mathsf{T}$ [2, 4]. In this case, the conserved $(d-1)$-momentum cannot be set to zero. Since this is the only conserved quantity given, $\mathsf{T}$ must in general be represented trivially by the particle to avoid introducing new conserved quantities. This requirement places a restriction on which $\mathsf{SO}(d-2)$ representations that are contained in the representation of the field can be furnished by the particle [2].

Instead of giving a general statement on the allowed $\mathsf{SO}(d-2)$ representions of massless particles for a field that furnishes a given representation of the Lorentz group, Weinberg [2] examines the transformation properties of particles for several types of fields, and makes a conjecture on obtaining these representations from arbitrary representations of the fields using Young tableaux. A proof of the conjecture is given by Distler [5].

## DECAPITATION CONJECTURE

As mentioned above, not all representations of $\mathsf{SO}(d-2)$ that are contained in a representation of $\mathsf{SO}(d-1,1)$ furnished by the field are associated with a trivial representation of $\mathsf{T}$. To approach this problem, consider a local field operator $\psi^n(x)$ that transforms in the representation $R$ of $\mathsf{SO}(d-1,1)$. The superscript $n=\mu\nu\cdots$ is the set of tensor indices of the field. If a massless particle described by this field furnishes an irrep $R'$ of $\mathsf{SO}(d-2)$, the matrix element of the field should satisfy:

$$0 \neq \langle 0| \psi^n(0) |\mathbf{k},\sigma\rangle \equiv u^n_\sigma \tag{1}$$

where $\sigma$ labels the states in $R'$. Equation 1 says a massless particle state in $R'$ can be created by the field $\psi^n$ from vacuum. These $u^n_\sigma$ can be used to construct the field [2, 3].

Equation 1 reformulates our quest into one of finding what components of $u^n_\sigma$ is non-vanishing, but there is one more requirement for $u^n_\sigma$, that any Lorentz transformation $W$ is trivially represented in the $\mathsf{T}$ subgroup as $d_{\sigma,\bar\sigma}(W)=\delta_{\sigma,\bar\sigma}$. The $\mathsf{SO}(d-1,1)$ generators satisfy

$$i[J^{\mu\nu},J^{\rho\sigma}] = \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu} \tag{2}$$

where $\mu,\nu,\rho,\sigma = 0,1,2,\cdots,d-1$. The generators of the little group are $J^{ij}$ and $K^i \equiv J^{i\ d-1} - J^{i0}$ with $i,j = 1,2,\cdots,d-2$. Because $[K^i,K^j]=0$ and $[J^{ij},K^k] \propto K$, $K^i$ form the aforementioned invariant abelian subalgebra $\mathsf{t}$. $K^i$ must annihilate $u^n$, i.e. $\sum_{\bar\sigma} K^i_{\sigma\bar\sigma} u^n_{\bar\sigma}=0$, since $\mathsf{t}$ is trivially represented by the particle. The action of $K^i$ on $u$ in $R'$ is summarized below [2]

$$0 = \sum_{\bar\sigma} K^j_{\sigma\bar\sigma} u^{+\cdots}_{\bar\sigma} = i u^{j\cdots}_\sigma + \cdots \tag{3}$$

$$0 = \sum_{\bar\sigma} K^j_{\sigma\bar\sigma} u^{i\cdots}_{\bar\sigma} = 2i\delta_{ij} u^{-\cdots}_\sigma + \cdots \tag{4}$$

$$0 = \sum_{\bar\sigma} K^j_{\sigma\bar\sigma} u^{-\cdots}_{\bar\sigma} = 0 + \cdots \tag{5}$$

where $u^{\pm\cdots} \equiv (u^{0\cdots} \pm u^{d-1\cdots})/2$, and the remaining terms on the right-hand side are the action of $K^i$ on the remaining indices of $u$.

We now have the necessary tools to find the allowed particles for a given field representation $R$. Consider the example where the field is a symmetric traceless rank-$N$ tensor $\psi^{\mu_1\mu_2\cdots\mu_N}$. We denote a component of the matrix element $u$ with $N_+$ + indices, $N_-$ − indices, and $M = N - N_+ - N_-$ other indices $u^{i_1 i_2\cdots i_M(N_+,N_-)}$. Equations 3-5 then become

$$\begin{aligned} 0 &= u^{i_1 i_2\cdots i_M j(N_+-1,N_-)} \\ &+ 2\sum_{r=1}^{M} \delta_{j i_r} u^{i_1 i_2\cdots i_{r-1} i_{r+1}\cdots i_M(N_+,N_-+1)} \end{aligned} \tag{6}$$

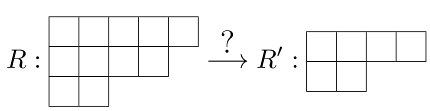
With a few manipulations and the traceless property of $u$, this leads to $u^{i_1 i_2 \cdots i_{M-1}(N_+, N_- + 1)} = 0$, which means any component of $u$ with at least one index equal to $-$ must vanish. This also means only the first term in Equation 6 remains. Now that we have $N_- = 0$, the remaining term implies $u$ also vanishes for any $M \leq N - 1$. As a result, the only non-zero component is $u^{++\cdots}$, with all indices equal to $+$. This component transforms trivially under $\mathsf{SO}(d-2)$.

Following a similar procedure, a completely antisymmetric rank-$N$ tensor $\psi^{\mu_1 \mu_2 \cdots \mu_N}$ can be shown to have non-vanishing $u$ components of the form $u^{i_1 i_2 \cdots i_{N-1} +}$, which transforms as a completely antisymmetric rank-$(N-1)$ tensor under $\mathsf{SO}(d-2)$. A few more examples on the Weyl tensor and spinor fields were also considered, but I will not cover them here [2].

In a Young tableau of the representation furnished by a tensor field, each box is associated with a tensor index. The tensor is antisymmetric under exchange of any two indices in the same column, and symmetric under exchange of two columns of the same height.

Weinberg notes that for all examples he considered, the non-vanishing components of $u^{\mu\nu\cdots}_{\sigma}$ have no $-$ indices, and have one $+$ in each column, which can be moved to the top by the antisymmetric property. Since the $+$ component is unaffected by the $\mathsf{SO}(d-2)$ rotations, he speculates that by removing the top row, the rest of the Young tableau describes the representation of $\mathsf{so}(d-2)$ furnished by the massless particle.

For example, the symmetric traceless rank-$N$ tensor has a Young tableau with a single row of width $N$. Removing this row gives the trivial representation in $\mathsf{SO}(d-2)$. For the completely antisymmetric rank-$N$ tensor, the Young tableau is a single column of height $N$. Removing the top row leaves a completely antisymmetric rank-$(N-1)$ tensor under $\mathsf{SO}(d-2)$.




## PROOF

Now consider the algebra of the little group, $\mathsf{iso}(d-2) = \mathsf{so}(d-2) \ltimes \mathsf{t}$. As we have previously stated, for a given field representation $R$, the allowed massless particle states require that the invariant abelian subalgebra $\mathsf{t}$ be trivially represented on $R'$, which is the $\mathsf{so}(d-2)$ representation furnished by the particle.

We first note that the generator of $\mathsf{so}(1,1)$ is $J^{0\ d-1}$. We can check that $[K^j, J^{0\ d-1}] = iK^j$ for all $j = 1, 2, \cdots, d-2$. This means $K^i \in \mathsf{t}$ effectively raise the weight of $\mathsf{so}(1,1)$. Since the latter is a subalgebra of $\mathsf{so}(d-1,1)$, $K^i$ must also annihilate the highest weight state of $R$. Additionally, $J^{0\ d-1}$ commutes with $\mathsf{so}(d-2)$. We can therefore conclude that the little algebra $\mathsf{iso}(d-2)$ acting on the highest weight state of $R$ gives an irrep $R'$ of $\mathsf{iso}(d-2)$, with $K^i$ trivially represented as desired. This also means the highest weight state of $R'$ is the highest weight state of $R$.

The above statements suggest that $R'$, as an irrep of $\mathsf{iso}(d-2)$ by virtue of being an irrep of $\mathsf{so}(d-2)$, must also be an irrep of $\mathsf{so}(1,1) \times \mathsf{so}(d-2)$, since both share $\mathsf{so}(d-2)$ as a subalgebra. Thus we can focus on finding irreps of $\mathsf{so}(1,1) \times \mathsf{so}(d-2)$ from now on.

The Dynkin diagram of $\mathsf{so}(d-2)$ can be obtained by omitting the leftmost node in the Dynkin diagram of $\mathsf{so}(d-1,1)$, so $\mathsf{so}(d-2)$ has all but one of the simple roots of the full algebra. Let $\alpha^i$ be the simple roots of $\mathsf{so}(d-1,1)$ and $\mu^i$ be the corresponding fundamental weights. We define the Dynkin label associated with the highest weight $\mu = \sum_i n^i \mu^i$ of $R$ to be

$$n^i = \frac{2\alpha^i \cdot \mu}{(\alpha^i)^2}$$

We can therefore denote $R$ as $(n^0, n^1, n^2 \cdots n^r)$, where $r + 1 = \frac{d}{2}$ (even $d$), $\frac{d-1}{2}$ (odd $d$) is the number of simple roots for $\mathsf{so}(d-1,1)$. We choose $\alpha^1, \cdots, \alpha^r$ to be the simple roots of $\mathsf{so}(d-2)$.

In general, we can write the decomposition of $R$ under $\mathsf{so}(1,1) \times \mathsf{so}(d-2)$ as

$$R = \bigoplus_j (\lambda_j) \otimes R_j \tag{7}$$

where $R_j$ are the irreps of $\mathsf{so}(d-2)$ and $\lambda_j$ are the weights of the corresponding 1-dimensional irreps of $\mathsf{so}(1,1)$, which are unaffected by actions of $\mathsf{so}(d-2)$. We choose the labels $j$ such that $\lambda_j$ are ordered from highest to lowest.

As we have found earlier, $R$ and $R'$ have the same highest weight state, so following Equation 7 we conclude that $R' = (\lambda_1) \otimes R_1$, with the $\mathsf{so}(d-2)$ representation being

$$R_1 = (n^1, n^2 \cdots n^r)$$

where we have removed the first Dynkin label.

For $\mathsf{SO}(d-1,1)$, the Dynkin labels are the differences in lengths between adjacent rows, starting from the top. The $r+1$ Dynkin labels imply a maximum of $r+1$ rows. Suppose the Young tableau of $R$ has rows with lengths $l_0, l_1 \cdots l_r$. For odd $d$, the Dynkin labels are [5]

$$n^i = l_i - l_{i+1},\ i = 0, 1 \cdots r-1 \tag{8}$$

$$n^r = 2l_r \tag{9}$$

For even $d$, they are

$$n^i = l_i - l_{i+1},\ i = 0, 1 \cdots r-2 \tag{10}$$

$$n^{r-1} + n^r = 2l_{r-1} \tag{11}$$

$$|n^{r-1} - n^r| = 2l_r \tag{12}$$

With these relations, we can see that removing the first Dynkin label $n^0$ leads directly to a Young tableau with the first row of length $l_0$ removed. Thus we have proven the "decapitation conjecture." Additionally, Distler finds the $\mathsf{so}(1,1)$ weight is $\lambda_1 = 2l_0$, twice the length of the removed row [5].

## CONCLUSIONS

In an attempt to find a general statement for the allowed types of massless particles given a field in arbitrary representation of $\mathsf{SO}(d-1,1)$, Weinberg (2010) makes a conjecture which states that for tensor fields, the Young tableau of the $\mathsf{SO}(d-2)$ representation furnished by massless particles can be obtained by removing the top row of the Young tableau of the $\mathsf{SO}(d-1,1)$ representation furnished by the field describing the particles [2].

The proof to the conjecture is given by Distler (2010), which also generalizes the statement to cover spinor representations of the field [5]. These results provide a systematic way to determine the types of massless particles allowed for a field that transforms in a given representation of the Lorentz group $\mathsf{SO}(d-1,1)$, in arbitrary dimensions $d$.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