The Title of Your Paper

Hongrui Li[1]

[1] *Department of Physics, University of California at San Diego, La Jolla, CA 92093*

In this short paper I introduce the basic definition and properties of quantum double group algebra and demonstrate an application of it using the example of Kitaev's quantum double model.

INTRODUCTION

Elementary excitation in quantum many-body system is a hot field and there are numerous research done on anyons which act as an excitation based on algebraic structure, such as quantum double algebra In this short paper, I am going to summarize the construction and property of quantum double algebra and some relation between Kitaev's quantum double model and the quantum double algebra.

Basic definitions

We use the convention introduced in Gould's article [1], where we let A be the group algebra of a finite group G over the complex field $\mathbb{C}$. Then A will become a co-commutative Hopf algebra (see Appendix) with coproduct, antipode and counit respectively defined by

$$\Delta(g) = g \otimes g,\ S(g) = g^{-1},\ \varepsilon(g) = 1,\ \forall g \in G. \tag{0.1}$$

the 1 we use above identifies the identity element of A extended from identity of G. Following theorem described in Appendix, A is a semi-simple algebra and we may decompose A as

$$A = \bigoplus_{\lambda} A_{\lambda} \tag{0.2}$$

Similar argument may apply to its dual $A^* = A^0$, under assumption given in appendix (eq.**??**). Then quantum double construction provides a method of imbedding A and A^0 in a quasi-triangular Hopf algebra, $D(A)$, which is spanned by $\{ab^* | a \in A, b^* \in A^*\}$. Therefore $D(A)$ will become a Hopf algebra with properties inherited from A and A^* by coproduct $\bar{\Delta}$, counit $\bar{\varepsilon}$ and antipode $\bar{S}$ to be

$$\bar{\Delta}(ab^*) = \Delta(a)\Delta_0(b^*)$$
$$\varepsilon(\bar{ab^*}) = \varepsilon(a)\varepsilon_0(b^*)$$
$$\bar{S}(ab^*) = S_0(b^*)S(a)$$

For simplicity we may intoduce basis of A^* as $\{a_s^*\}$ by

$$\langle a_s^*, a_t \rangle = \delta_{st}$$

Then

$$R = \sum_s a_s \otimes a_s^* \tag{0.3}$$

has inverse $R^{-1} = (\bar{S} \otimes I)R$, then we comes to the definition of quantum double:

Theorem 1 *$D(A)$ with canonical element R constitutes a qasi-triangular Hopf-algebra called the quantum double of A*

Then following similar argument of simplicity on A, we may reach a conclusion where A^* and $D(G)$ are all semi-simple and we then can decompose $D(G)$ as direct sum of simple two-sided ideals $D(G)_\Lambda$:

$$D(G) = \bigoplus_{\Lambda} D(G)_{\Lambda} \tag{0.4}$$

KITAEV'S QUANTUM DOUBLE MODEL

Let G be a finite group, and $\mathcal{H} = \mathbb{C}[G]$ the corresponding group algebra. Since by the definition of group algebra, we may also treat $\mathcal{H}$ as a Hilbert space spanned by orthonormal basis $\{|g\rangle \, |g \in G\}$. Therefore the dimension of $\mathcal{H}$ as Hilbert space is $\dim \mathcal{H} = |G|$. We may start the model from introducing 4 types of linear operators, $L^g_\pm$ and $T^h_\pm$ acting on $\mathcal{H}$, where

$$L^g_+ |z\rangle = |gz\rangle \tag{0.5}$$
$$L^g_- |z\rangle = |zg^{-1}\rangle \tag{0.6}$$
$$T^h_+ |z\rangle = \delta_{h,z} |z\rangle \tag{0.7}$$
$$T^h_- |z\rangle = \delta_{h^{-1},z} |z\rangle \tag{0.8}$$

So commutation relation reads

$$L^g_+ T^h_+ = T^{gh}_+ L^g_+ \tag{0.9}$$
$$L^g_- T^h_+ = T^{hg^{-1}}_+ L^g_- \tag{0.10}$$
$$L^g_+ T^h_- = T^{hg^{-1}}_+ L^g_+ \tag{0.11}$$
$$L^g_- T^h_- = T^{gh}_- L^g_- \tag{0.12}$$

Then we may relate these operator to a orientable 2-D surface. In his paper, Kitaev used the example as follows [2]: In the graph there are three basic elements: face p, arrow j and one of endpoints (vertices) s. Then we may define operator $L^g(j,s) = L^g_\pm(j)$ by the orientation of arrow j: if s is the origin of arrow j then $L^g(j,s) = L^g_-(j)$, otherwise $L^g(j,s) = L^g_+(j)$. Similarly, we may let $T^h(j,p) = T^h_-(j)$ if p is on the left adjacent face, otherwise $T^h(j,p) = T^h_+(j)$. Then similar to Toric code,

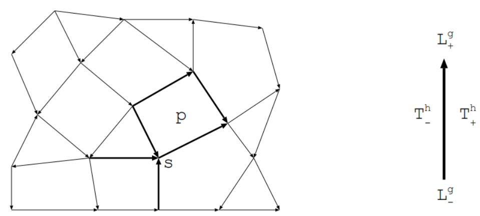

FIG. 1: Lattice and orientation rules for operator $L^g_\pm$ and $T^h_\pm$

we may define local gauge transformation and magnetic charge operator as follows:

$$A_g(s,p) = A_g(s) = \prod_{j \in \text{stars}(s)} L^g(j,s) \tag{0.13}$$

$$B_g(s,p) = \sum_{h_1 \cdots h_k = h} \prod_{m=1}^{k} T^{h_m}(j_m, p) \tag{0.14}$$

with $\{j_m\}$ are boundary arrows of p in counterclockwise order starting from vertex s (so ending at s also). Moreover, the sum is taking over all possible $\{h_i\} \subset G$, whose product is h. The group is not necessarily abelian so the order here matters. We may also define symmetric combinations of A_g and B_h by

$$A(s) = |G|^{-1} \sum_{g \in G} A_g(s,p) \tag{0.15}$$

$$B(p) = B_1(s,p) \tag{0.16}$$

Note that $A(s)$ and $B(p)$ are projection operators that commute with each other. Then we can define the system's Hamiltonian,

$$H_0 = \sum_s (1 - A(s)) + \sum_p (1 - B(p)) \tag{0.17}$$

which is extremely similar to Toric code's Hamiltonian. Without surprise, we can classify elementary excitation by which term the excitation fails to keep its minimum value.

Elementary Excitations

Similar to Toric code, which is a $\mathbb{Z}_2$ version of our definition, we may expect excitations come with pairs. Therefore we may need to project to space of two-particle excitations, $\mathcal{L}(a,b)$, where $a = (s,p)$ and $b = (s',p')$ are described by site and face occupied by the particles. Note that operator $A_g(a)$ and $B_h(a)$ commute with $A(r)$ and $B(l)$ for all $r \neq s$ and $l \neq p$, which constitute the projector operator which project states into $\mathcal{L}(a,b)$ so $A_g(a)$ and $B_h(a)$ also commute with projector onto subspace $\mathcal{L}(a,b)$. Moreover, the relation between $A_g \equiv A_g(a)$ and $B_h \equiv B_h(a)$ is

$$\begin{aligned} A_f A_g &= A_{fg}, \\ B_h B_i &= \delta_{h,i} B_h, \\ A_g B_h &= B_{ghg^{-1}} A_g. \end{aligned} \tag{0.18}$$

Therefore these element can generate an algebra $\mathcal{D}(a) \in \mathbf{L}(\mathcal{N})$, where $\mathcal{N}$ is the Hilbert space consists of all possible states and $D_{(h,g)} = B_h A_g$ form a linear basis of $\mathcal{D}(a)$.

Relation to Quantum Double Algebra

Although we have defined a new algebra $\mathcal{D}(a)$ in last section, the algebra does not depend on a. Meanwhile it is the embedding $\mathcal{D}(a) \equiv \mathcal{D} \to \boldsymbol{L}(\mathcal{N})$ that depends on a, which is the position of one elementary excitation. The algebra is the quantum double of group G, which we briefly discussed in first section, and denoted by $\mathbf{D}(G)$. Using relation in equation 0.18, we may obtain the multiplication rules

$$D_{\boldsymbol{m}} D_{\boldsymbol{n}} = \Omega^{\boldsymbol{k}}_{\boldsymbol{mn}} D_{\boldsymbol{k}} \tag{0.19}$$

with $\Omega^{(h,g)}_{(h_1,g_1)(h_2,g_2)} = \delta_{h_1, g_1 h_2 g_1^{-1}} \delta_{h,h_1} \delta_{g, g_1 g_2}$, which determins the structure of quantum double algebra. By the property of quantum double algebra (0.4), we may decompose the representation as

$$\mathcal{D} = \bigoplus_d \mathbf{L}(\mathcal{K}_d)$$

where d runs over all representations of $\mathcal{D}$ and can be interpreted as particle type.

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[1] M.D. Gould. Quantum double finite group algebras and their representations. *Bulletin of the Australian Mathematical Society*, 48(2):275–301, 1993. 1, 3
[2] A.Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2 – 30, 2003. 1

APPENDIX

Some useful fact of Hopf Algebra

In order to introduce the construction of quantum double group algebra, we need to start from some basic definitions. We use the convention introduced in Gould's article [1], where we let A be a Hopf algebra with identity $1 \in A$, co-unit $\varepsilon : A \to \mathbb{C}$, coproduct $\Delta : A \to A \otimes A$ and bijective antipode $S : A \to A$. We will write the coproduct

$$\Delta(a) = \sum_a a^{(1)} \otimes a^{(2)}, \ a \in A. \qquad (0.20)$$

We also need to mention the dual of Hopf algebra A, denoted by A^* and bilinear form $\langle , \rangle$ defined by $\langle a^*, a \rangle := a^*(a)$, $\forall a^* \in A^*$ and $a \in A$. Then we assume $A^0 := \{a^* \in A^* | \ker a^* \text{ contains a cofinite two sided ideal of } A\} \subset A^*$ is dense in A^*, in other words

$$(A^0)^\perp \equiv \left\{a \in A | \langle b^*, a \rangle = 0, \forall b^* \in A^0 \right\} = \{0\}$$

Then we may begin a definition:

Definition 1 *A Hopf algebra A is called quasi-triangular if there exists an invertible element*

$$R = \sum_i a_i \otimes b_i \in A \otimes A$$

satisfying $\Delta^T(a)R = R\Delta(a)$, $\forall a \in A$, and $(\Delta \otimes I)R = R_{12}R_{23}$, $(I \otimes \Delta)R = R_{13}R_{12}$ with $R_{12} = \sum_i a_i \otimes b_i \otimes 1$ and $R_{13} = \sum_i a_i \otimes 1 \otimes b_i$, etc.

Then we have

Theorem 2 *A^0 becomes a Hopf algebra with multiplication m^0, unit u^0, coproduct Δ^0, antipode S^0 and counit ε^0 defined by*

$$m^0 = \Delta^* \Big|_{A^0 \otimes A^0},$$
$$u^0 = \varepsilon^* \Big|_{A^0}$$
$$\Delta^0 = m^* \Big|_{A^0}$$
$$S^0 = S^* \Big|_{A^0}$$
$$\varepsilon^0(a^*) = \langle a^*, 1 \rangle$$

with $m : A \otimes A \to A$ is the multiplication map on A and $m^*, \Delta^*, \varepsilon^*, S^*$ are the natual dual maps of m, Δ, ε and S.

Moreover,

Theorem 3 *A finite dimensional Hopf algebra A is semi-simple if and only if there exists $x \in A$, such that*

$$ax = \varepsilon(a)x, \ \forall a \in A$$

and such $x \in A$ is called a left integral.