Symmetry Breaking and Superfluid Phases of Liquid ^{3}He from Laudau-Ginzburg Theory

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In this paper, we will briefly introduce the application of Landau-Ginzburg (L-G) theory and symmetry breaking in superfluid phases of liquid ^{3}He. From the solutions of L-G equation for B and A phase, we can identify the symmetry breaking in these two phases. Then we will briefly introduce the physical consequence of the broken relative symmetries in these two phases.

INTRODUCTION

Superfluidity of ^{3}He

A superfluid is a state of matter in which the matter behaves as a fluid with vanishing viscosity. The most famous example is ^{4}He, an isotope of He. Due to large zero-point motion of the atoms in ^{4}He, it still in a phase of liquid at absolute zero temperature. Since ^{4}He has zero nucleus spin, at a temperature of few K, the ensemble of the ^{4}He atoms forms a quantum Bose liquid. When $T < T_c = 2.2$ K, ^{4}He Bose-condenses and the correlated motion of the Bose condensate has no dissipation, therefore, it exhibits properties of superfluidity.

^{3}He also doesn't solidify at absolute zero temperature. It also can become a superfluid. However, ^{3}He has 1/2 nucleus spin, therefore, it becomes a quantum Fermi liquid at few K. In order to behave as a superfluid, ^{3}He atoms must first pair up and form Cooper pairs, which have integer spin. Then the ensemble of such bosonic Cooper pairs will Bose-condenses and results in superfluidity at low temperature.

Not only the complicity in the formation of Cooper pair, there is a more important and interesting consequence of the non-zero nucleus spin, namely, non-trivial internal structure, followed by additional degrees of freedom on low energy scale $E \ll T_c$. The Cooper pair has non-zero spin $S = 1$ and non-zero angular momentum of the orbital motion of the Cooper pair $L = 1$. The strong correlation of the spin and angular momenta of the Bose condensate results in magnetic and liquid-crystal-like ordering, which means not only the gauge symmetry, the $SO(3)_S$ for spin and $SO(3)_L$ for orbital rotation symmetries are also spontaneously broken in ^{3}He.

Superfluid Phase of ^{3}He

Since $S = 1$ and $L = 1$ are both three dimensional representation of $SO(3)_S$ and $SO(3)_L$ group respectively. Therefore the parameter space is $\mathbf{3} \otimes \mathbf{3}$, which is nine dimensional. Such large degrees of freedom enrich the classification of the superfluid phases of ^{3}He. Three of them are under extensive investigation[1]:

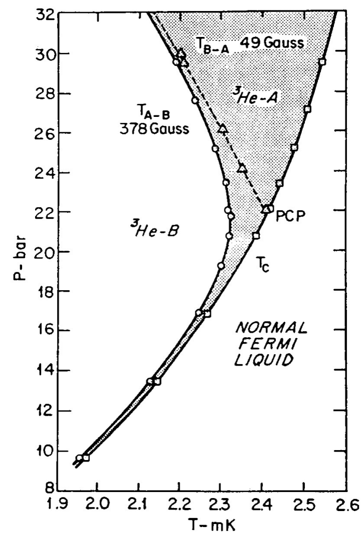

FIG. 1: Phase diagram for ^{3}He at the lowest magnetic field[2].

- The quasi-isotropic phase ^{3}He-B. The total angular momentum of the Cooper pair is zero, $J = 0$, where $\vec{J} = \vec{L} + \vec{S}$.

- The anisotropic phase ^{3}He-A. In this phase, the Cooper pair in this state has non-zero projection of the orbital angular momentum $m_l = 1$ on some axis $\hat{l}$, where $\hat{l}$ is not only an axis of spontaneous orbital anisotropy of this liquid, but also represents the direction of its spontaneous ferromagnetic moment. The projection of spin angular momentum is zero $m_s = 0$.

- There is another phase, namely, the ^{3}He-A_1 phase, which only exists if a magnetic field is present. Being different from the ^{3}He-A phase, it has $m_l = 1$ and also non-vanishing projection of spin angular momentum $m_s = 1$. The quantization axes for spin and orbital angular momentum are not necessarily parallel.

In this next section, we will mostly focus on B and A phase.

SYMMETRY BREAKING IN ^{3}HE SUPERFLUID

If we only consider the interactions involved in the formation of condensed states, then the free energy is invariant under the following transformation

$$G = SO(3)_L \times SO(3)_S \times U(1)_\phi, \quad (0.1)$$

where the two $SO(3)$ groups describe rotations in spin space and orbital space respectively, and $U(1)_\phi$ describes gauge transformation.

Symmetry breaking will result in different phases [3]:

- $G \supset SO(3)_L \times SO(3)_S \rightarrow SO(3)_{L+S}$, where $SO(3)_{L+S}$ describes some linear combinations of transformations from $SO(3)_L$ and $SO(3)_S$ groups. This symmetry breaking results in B phase.
- $G \supset SO(3)_L \times U(1)_\phi \rightarrow U(1)_{L_z+\phi}$, which results in A phase.

In this section, we will first briefly review the Landau-Ginzburg theory and then introduce these two phases in details.

Laudau-Ginzburg Theory for Liquid ^{3}He

The wave function of Cooper pair is

$$\psi = \sum_{m_s, m_l} \mathcal{A}_{m_s,m_l} \psi_{ms,m_l}, \quad (0.2)$$

where $m_s, m_l = +, 0, -1$ corresponds to the magnetic quantum number for spin and orbital angular momentum $M_S, M_L = 1, 0, -1$ respectively.

We can write the L-G functional in terms of a 3×3 matrix $A_{\alpha i}$, defined as $A_{\alpha i} = \sum_{m_s,m_l} \mathcal{A}_{m_s,m_l} \lambda_\alpha^{m_s} \lambda_i^{m_l}$, where $\lambda_\alpha^0 = \hat{z}_\alpha, \lambda_\alpha^\pm = (\hat{x}_{\alpha \pm i \hat{y}_\alpha})/\sqrt{2}$. Then The behavior near the critical temperature T_c is described by the quadratic term in the L-G functional $F_{LG} \supset (T - T_c) A^*_{\alpha i} A_{\alpha i}$, and the equilibrium order parameter $A^0_{\alpha i}$ can be found by solving G-L equations

$$\frac{\delta F^{LG}_{\text{bulk}}[A_{\alpha i}]}{\delta A_{\alpha i}} = 0, \quad (0.3)$$

where $F^{GL}_{\text{bulk}} \subset F_{LG}$ is the bulk condensation energy term in the L-G free energy functional[4].

By this method, we can find the order parameter for B and A phase:

- When the pressure below the so called policritical point, the solution of the L-G equation is

$$A^0_{\alpha i} = \Delta_B(T, P)\delta_{\alpha i}. \quad (0.4)$$

In terms of the amplitude, we have

$$\mathcal{A}_{+,-} = \mathcal{A}_{-,+} = \mathcal{A}_{0,0} = \Delta_B(T, P). \quad (0.5)$$

Therefore, the projection of the total angular momentum $m_j = m_s + m_l = 0$, and the state is isotropic, which indicates $J = 0$.

- When the pressure above the policritical point, we obtain the A phase solution [1]:

$$A^0_{\alpha i} = \Delta_A(T, P)\hat{z}_\alpha(\hat{x}_i + i\hat{y}_i). \quad (0.6)$$

In term of the amplitude, we have only one nonzero order parameter:

$$\mathcal{A}_{0,+} = \sqrt{2}\Delta_A(T, P), \quad (0.7)$$

from which we can tell the state has $m_s = 0$ and $m_l = 1$. The quantization axes for spin and orbital angular momentum are both along the same axis.

Manifold of the Degenerate State and Residue Symmetry

The transformation of the total group $g \in G$ can be described explicitly as

$$(gA^0)_{\alpha i} = e^{i\phi} R^S_{\alpha\beta} A_{\beta j} R^L_{ji}, \quad (0.8)$$

where R^S, R^L are the matrices of rotations $SO(3)_S, SO(3)_L$ in the orbital and spin spaces respectively, ϕ is the parameter of the global gauge transformation $U(1)$. Applying this to different state Eq.(0.4) and Eq.(0.6), we obtain[1]

$$\text{B phase:} \quad A^0_{\alpha i} = \Delta_B(T, P)e^{i\phi} R_{\alpha i} \quad (0.9)$$

$$\text{A phase:} \quad A^0_{\alpha i} = \Delta_A(T, P)\hat{s}_\alpha(\hat{e}^{(x)}_i + \hat{e}^{(y)}_i), \quad (0.10)$$

where $\hat{s}_\alpha$ is the spin axis and $\hat{e}^{(x)} \times \hat{e}^{(y)} = \hat{l}$ is the deribein. $\hat{l}$ is the orbital axis.

From these two general forms, we can obtain the manifolds and residue symmetries for the degenerate states:

- B phase: B phase is described by 1) ϕ parameter, and 2) the orthogonal real matrix $R_{\alpha i}$. The space corresponding to $U(1)$ is just S^1. As for $R_{\alpha i}$, notice it's from $R_{\alpha i} = R^S_{\alpha\beta}\delta_{\beta j}(R^L)^{-1}_{ij}$ and $\delta_{\beta j}$ is from the initial solution Eq.(0.4), $R_{\alpha i}$ describes the relative rotation of spin and orbital frames. When this apply to the initial state with $J = 0$, the total angular momentum of resulting state is no longer zero. Therefore, the manifold of the equilibrium degenerate states R_B is [1]

$$R_B = S^1 \times SO(3)_{rel}, \quad dim(R_B) = 4. \quad (0.11)$$

where S^1 is described by the phase ϕ and $SO(3)_{rel}$ is for the relative spin-to-orbit rotations $R_{\alpha i}$.

Because $R_{rel} = R^S(R^L)^{-1}$, it's invariant under a spin rotation R^S accompanied by an equal rotation R^L of the orbital space. Therefore, the residue symmetry is $H = SO(3)_J$, the group of combined rotations. As a result, the manifold R_B is just the coset space $R_B = G/H$.

- A phase: The general form of the solution Eq.(0.10) has the following symmetries:
  1. Spin rotation $SO(2)_S$ about axis $\hat{s}$;
  2. $U(1)_c$: $U(1)$ gauge transformation $e^{i\phi}$, accompanied by orbital rotation of the dreibein $\hat{e}^{(x)}, e^{(y)}$ and $\hat{l}$ about the axis $\hat{l}$ by the same angle ϕ, since under such a rotation $\hat{e}^{(x)}+ie^{(y)} \to e^{-i\phi}(\hat{e}^{(x)}+ie^{(y)})$, where the factor $e^{-i\phi}$ is cancelled out by the $U(1)$ transformation;
  3. A discrete combined symmetry Z_2^c, the spin rotation of $\hat{d}$ by angle π about a perpendicular axis, accompanied by gauge transformation $e^{i\pi}$. The former flips the sign which is further cancelled out by the latter $e^{i\pi}$.

  Therefore, the residue symmetry is $H = SO(2)_S \times U(1)_c \times Z_2^c$, and the manifold of the equilibrium degenerate state is [1]

$$R_A = G/H = S^1 \times SO(3)_{rel}/Z_2. \qquad (0.12)$$

  Be aware that here the $SO(3)_{rel}$ describes the rotations about the axis $\hat{l} = \hat{e}^{(x)} \times \hat{e}^{(y)}$ relative to gauge transformations, which is different from the one in B phase.

PHYSICAL RESULTS FROM THE BROKEN RELATIVE SYMMETRIES

Notice that both B and A phase have broken relative symmetries, summarized as follows:

- B phase: Relative spin-orbital rotation: Only the relative rotation of spin and orbital frame will change the degenerate state.
- A phase: Relative gauge-orbital rotation: Only the gauge transformation counted from the orbital rotations around $\hat{l}$ will change the degenerate state.

Such broken relative symmetries have very interesting physical consequence.

For B phase, the spin or orbital space are both isotropic. However, the broken spin-orbital rotational symmetry results in relative anisotropy of the magnetic and orbital properties in an isotropic liquid. If an anisotropy axis $\hat{n}_l$ of orbital properties is induced by external conditions, then an anisotropy axis relative to the orbital axis $\hat{n}_s = R_{\alpha i}\hat{n}_l$ of the magnetic properties will simultaneously appear.

For A phase, due to the breaking gauge-orbital rotational symmetry, all the gauge invariant quantities in A phase will obtain an anisotropy axis $\hat{l}$. They are invariant under rotations around such an axis $\hat{l}$.

SUMMARY

In this paper, we briefly introduced the symmetry breaking of ^{3}He and the resulting B and A phase. The new features, broken relative symmetries, generate many interesting physical properties of the superfluid ^{3}He.

There are also many other important and intriguing aspects in the study of superfluid ^{3}He. For example, in the topological perspective, topological defects are under extensive investigation; ^{3}He vortices also have lots of fascinating content to be studied. In addition, superfluid phases can also be studied with the help of gravity theory, which I found very attractive.

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