Tensor Category Theory and Anyon Quantum Computation

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Abstract

We discuss the fusion and braiding of anyons, where different fusion channels form a Hilbert space that can be used for quantum computing. These operations can be nicely formulated using tensor category theory. They are highly-constrained, which means most of the properties can be derived only from some simple fusion rules, together with those constraints. These operations can then be used to approximate logic gates that act on the Hilbert space of fusion channels.

1 Introduction

In three-dimensional systems, particles are either bosons or fermions. Exchanging them would at most produce a sign, and a composite object would either be a boson or a fermion. For example, a composite object of two fermions is always a boson. In contrast, there can be more exotic species of particles in two-dimensional systems, when exchanged would give more than a sign, such as a specific phase factor $e^{i\theta}$. We can label different species by their exchange properties. Moreover, combining the same set of constituents could result in objects of different species. For example, in a system that we will discuss later, called Fibonacci anyons, there are two species of particles, 1 and τ. The object from combining two τ particles has two possibilities, instead of one: it could be a 1 particle or a τ particle. If this two-τ object is combined with a third or more τ particles, there would be even more possibilities.

It turns out that we can use these different possibilities to represent $|0\rangle$ and $|1\rangle$ qubits for quantum computing, and exchanging some of the anyon constituents can serve as linear operators on the qubit. Armed with qubits and linear operators, or logic gates, we can perform quantum computation. To do this in a precise manner, we need to understand how those different possibilities transform among each other when anyon constituents are exchanged. This is where tensor category theory comes in, it accurately captures the algebraic property of anyon exchanges.

In this article, we will first discuss the fusion (combining them) and braiding (exchanging) of anyons in Sec.2, which are dictated by a set of rules. Those rules must satisfy some constraints that arise from physical grounds. This motivates the definition of tensor category theory in Sec. 3, a mathematical construct that satisfies those constraints. Then in Sec. 4 we see how the tensor category theory facilitates quantum computation using anyons.

2 Fusion and Braiding of Anyons

Consider a sytem with several species of anyons, labeld a, b, c, $\cdots$, one of which, labeled 1, would be the trivial species, kind of like a boson in 3d. Combining the trivial particle with any other particle does not change its statistical property, also similar to bosons.

If we put two anyons a and b in a box, and observe the overall statistical behavior of the box, we get the fusion rule

$$a \times b = \sum_d N_{ab}^d d\,, \tag{1}$$

with potentially multiple outcomes d, and it's possible for $N_{ab}^d > 1$, meaning that there are multiple ways a and b could fuse to d. We could fuse this with a third anyon d,

$$(a \times b) \times c = \sum_d N_{ab}^d \,(d \times c) = \sum_e \sum_d N_{ab}^d N_{dc}^e e\,, \tag{2}$$

which enumerates the several outcomes e of fusing a, b and c.

2.1 Hilbert space from fusion channels

Instead of looking at all possible outcomes, we could focus on only a specific outcome of fusion. For example, consider the cases where $a \times b$ fuse to x,

$$a \times b \to x\,, \tag{3}$$

where there would be N^x_{ab} ways to do this. Thus we can imagine that these N^x_{ab} ways form a Hilbert space with states

$$|a, b \to x; \mu\rangle\ , \tag{4}$$

where μ labels the different ways to fuse to x. The dimension of this Hilbert space would be N^x_{ab}. In some common systems N is at most one, so these Hilbert spaces are one dimensional and we may drop the label μ,

$$|a, b \to x\rangle\ . \tag{5}$$

Unfortunately, with only one state, we cannot perform quantum computation.

Similarly, for the fusion of three anyons

$$|a, b, c \to y; \nu\rangle \tag{6}$$

also form a Hilbert space of dimension $\sum_d N^d_{ab} N^e_{dc}$, which is in general bigger than one even if $N^x_{ab} \le 1$. There is a natural way to label the states in the Hilbert space: the intermediate step d from fusing a and b, so the states are

$$|d\rangle = |a, b \to d\rangle\, |d, c \to y\rangle\ . \tag{7}$$

With multiple states in the Hilbert space, we could potentially perform quantum computation by finding linear operators that act upon those states.

Alternatively, we can also look at these as splitting channels, where e can be split into a, b, c in various ways.

2.2 Operators on the Hilbert space

Now let's find operators to act on the Hilbert space, which will need to change the intermediate state d.

2.2.1 Exchanging a and b, R matrix

Consider exchanging a and b, as shown in Fig. 1. . Since in the end we still have a and b fuse together, this should still produce d and introduce only a phase factor R^d_{ab},

$$\begin{aligned}\mathbf{R}\, |a, b \to d\rangle\, |d, c \to y\rangle \\ = R^d_{ab}\, |a, b \to d\rangle\, |d, c \to y\rangle\ .\end{aligned} \tag{8}$$

This does not transform it into other basis.

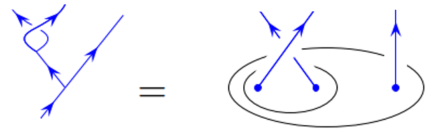

Figure 1: The **R** matrix operation.

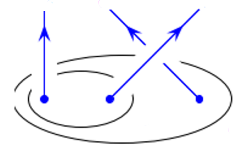

Figure 2: The **B** matrix operation.

2.2.2 Exchanging b and c, B matrix

If instead we exchange b and c, as in Fig. 2 then since these two are not in a definite fusion channel, the phase R_{ab} cannot be applied directly. We should first do a change of basis, so that b and c fuse first.

Physically, fusing a, b and c together to get e should not depend on the order of fusion, so $(a \times b) \times c \to e$ and $a \times (b \times c) \to e$ should represent the same Hilbert space, and the basis vectors should be related by a unitary transformation,

$$\begin{aligned}&|a, b \to x\rangle\, |x, c \to e\rangle \\ &\quad = \sum_y \left(F^e_{abc}\right)^x_y |b, c \to y\rangle\, |a, y \to e\rangle\ .\end{aligned} \tag{9}$$

Then, exchanging b and c would amount to simply a phase R^y_{ab} to each basis on the right, so

$$\begin{aligned}&\mathbf{B}\, |a, b \to x\rangle\, |x, c \to e\rangle \\ &= \sum_y R^y_{bc} \left(F^e_{abc}\right)^x_y |b, c \to y\rangle\, |a, y \to e\rangle\ .\end{aligned} \tag{10}$$

We then want to express this in terms of the original basis, so

$$\begin{aligned}&\mathbf{B}\, |a, b \to x\rangle\, |x, c \to e\rangle \\ &= \sum_y \left(F^e_{abc}\right)^x_y R^y_{bc} \left[\left(F^e_{abc}\right)^{-1}\right]^y_z |a, b \to z\rangle\, |z, c \to e\rangle\ ,\end{aligned} \tag{11}$$

or roughly

$$\mathbf{B} = \mathbf{F}^{-1}\mathbf{R}\mathbf{F}\,, \tag{12}$$

which is a valid operator that acts on the Hilbert space.

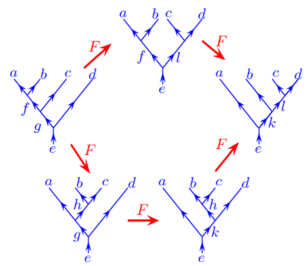

Figure 3: Relation between **F** matrices.

2.3 What we need for a math theory

Up to now, we see that anyons has several different types of labels, and we can form composites from several anyons. There are operations we can do on the composites, such as exchanging them or altering the order of fusing multiple anyons. We also want to compute the **F** matrices from the fusion rules N_{ab}^{c}. These actually follow some constraints. For example, we may fuse four anyons in various orders, and $((a\times b)\times c)\times d$ and $a\times(b\times(c\times d))$ can be related in two different ways,

$$((a\times b)\times c)\times d=(a\times b)\times(c\times d)=a\times(b\times(c\times d)) \tag{13}$$

and

$$\begin{aligned}((a\times b)\times c)\times d=&(a\times(b\times c))\times d\\ =&a\times((b\times c)\times d)\\ =&a\times(b\times(c\times d))\end{aligned} \tag{14}$$

have to be the same, where each alternative association order is related by an **F** matrix. An illustration is shown in Fig. 3. This puts a constraint on the **F** matrices, and is called the pentagon relation because of the shape of Fig. 3. It turns out tensor category is suitable to describe these properties of anyons.

3 Modular Tensor Category Theory

We now define a category theory that suitably describes anyon operations. A category **C** consists of a collection of objects $A\in|\mathbf{C}|$ and a set of morphisms $f\in\hom(A,B)$ that are basically mapping between A and B,

$$A\xrightarrow{f}B\,, \tag{15}$$

which contains identity and can be composed, $g\circ f$. In our anyon case, the objects would be an anyon or a set of anyon, while the morphisms would be fusion, splitting, braiding or change of associativity order (the matrix $\boldsymbol{F}$), which relates a set of anyons either to itself or to another set of anyons. In the following, we will define morphisms that represent the operations we've met in the previous section.

We further define a "tensor" operation

$$\otimes:\mathbf{C}\times\mathbf{C}\to\mathbf{C}\,, \tag{16}$$

which is a "functor", that maps one category to the other (taking objects to objects and morphisms to morphisms, while preserving properties such as morphism composition). For

$$A\otimes B\to C\,, \tag{17}$$

the morphisms act as you would expect,

$$(f\otimes g)\,(A\otimes B)=f\,(A)\otimes f\,(B)\ . \tag{18}$$

This represents combining anyon sets.

In addition, the category theory for anyon comes with natural isomorphisms (an ivertible morphism with some good properties)

$$\begin{aligned}\alpha_{A,B,C}:&(A\otimes B)\otimes C\to A\otimes(B\otimes C)\\ \lambda_A:&1\otimes A\to A\\ \rho_A:&A\otimes 1\to A\\ \sigma_{A,B}:&A\otimes B\to B\otimes A\end{aligned} \tag{19}$$

Basically, α is the change of associativity order, λ and ρ describe fusing with a trivial particle 1, and σ describes braiding (exchange). And of course, among the objects there are "simple objects" that represent single anyons, a, b, c, $\cdots$, and the rules

$$a\otimes b\cong\bigoplus N_{ab}^{c}c\,. \tag{20}$$

These would be our fusion rules.

Up to now, it is stated that the various morphisms represent the familiar anyon operations. For this to be true, they have to satisfy the constraints such as Fig. 3. Thus, we require that α and ρ, λ satisfy the relations in Fig. 4, which is consistent with associativity order change and the fact that fusing with a trivial partile is like doing nothing. Also, the braiding morphism σ satisfy the hexagon relation in Fig. 5 which follows from the fact that braiding with the trivial particle does nothing, and different ways to achieve braiding and changing associativity order should commute, as long as the end product is the same.

Also, the notion of the Hilbert space of different fusion channels corresponds to the space of morphisms.

$$\begin{tikzcd}
 & ((A\otimes B)\otimes C)\otimes D \arrow[dl, "\alpha_{A,B;C}\otimes 1_D"'] \arrow[dr, "\alpha_{A\otimes B,C;D}"] & \\
(A\otimes(B\otimes C))\otimes D \arrow[d, "\alpha_{A,B\otimes C;D}"'] & & (A\otimes B)\otimes(C\otimes D) \arrow[d, "\alpha_{A,B;C\otimes D}"] \\
A\otimes((B\otimes C)\otimes D) \arrow[rr, "1_A\otimes\alpha_{B,C;D}"'] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd}$$

(a)

$$\begin{tikzcd}
(A\otimes \mathbf{1})\otimes B \arrow[rr, "\alpha_{A,\mathbf{1};B}"] \arrow[dr, "\rho_A\otimes 1_B"'] & & A\otimes(\mathbf{1}\otimes B) \arrow[dl, "1_A\otimes\lambda_B"] \\
 & A\otimes B &
\end{tikzcd}$$

(b)

Figure 4: Pentagon and triangle relations for category theory.

$$\begin{tikzcd}
 & A\otimes(B\otimes C) \arrow[r, "\sigma_{A,B\otimes C}"] & (B\otimes C)\otimes A \arrow[dr, "\alpha_{B,C;A}"] & \\
(A\otimes B)\otimes C \arrow[ur, "\alpha_{A,B;C}"] \arrow[dr, "\sigma_{A,B}\otimes 1_C"'] & & & B\otimes(C\otimes A) \\
 & (B\otimes A)\otimes C \arrow[r, "\alpha_{A,B;C}"'] & B\otimes(A\otimes C) \arrow[ur, "1_B\otimes\sigma_{A,C}"'] &
\end{tikzcd}$$

(a)

$$\begin{tikzcd}
A\otimes\mathbf{1} \arrow[r, "\lambda_A"] \arrow[d, "\tau_{\mathbf{1},A}"'] & A \\
\mathbf{1}\otimes A \arrow[ur, "\rho_A"'] &
\end{tikzcd}$$

(b)

Figure 5: Hexagon relation for $\sigma_{A,B}$.

For example, the space of (4) for fusing two anyons is

$$\mathrm{Hom}\,(a\otimes b, c)\ , \tag{21}$$

the space of all morphisms between $a\otimes b$ and c, while the space for fusing three anyons would analogously be

$$\mathrm{Hom}\,(a\otimes b\otimes c, e)\ . \tag{22}$$

These can be determined from the fusion rules N_{ab}^c.

3.1 Computing the F matrix

It turns out that once the fusion rules N_{ab}^c are specified, all other matrices, **F**, **B** and **R**, (which are needed to perform quantum computation) follows from the constraints that the category theory, where the**F** matrix is the matrix form of the α morphism, and the **R** matrix is from the σ morphism. For example, from the pentagon relation in Fig. 4, one finds that

$$\left[F_e^{fcd}\right]_{gl}\left[F_e^{ab\ell}\right]_{fk}=\sum_h\left[F_g^{abc}\right]_{fh}\left[F_e^{ahd}\right]_{gk}\left[F_k^{bcd}\right]_{hl}\ . \tag{23}$$

Once this is done, we can turn to the hexagon relation in Fig. 5, which relates the **R** matrices,

$$R_x^{AC}\left(F_D^{BAC}\right)_{xy}R_y^{AB}=\sum_z\left(F_D^{BCA}\right)_{zy}R_D^{Az}\left(F_D^{ABC}\right)_{yx}\ , \tag{24}$$

one may solve **R** as well. The **B** matrices then follow.

4 Quantum Computation Using Anyons

Here we present a specific example, Fibonacci anyons, where there are two types of anyons, 1 and τ, and the fusion rules N_{ab}^c are

$$\begin{aligned}1\otimes 1 &=1\\ 1\otimes\tau &=\tau\\ \tau\otimes\tau &=1+\tau\,.\end{aligned} \tag{25}$$

Now consider fusing three τ particles, $(\tau\otimes\tau)\otimes\tau$. There are three posibilities:

$$\begin{aligned}(\tau\otimes\tau)\otimes\tau &\to(\tau)\otimes\tau\to 1\\ (\tau\otimes\tau)\otimes\tau &\to(\tau)\otimes\tau\to\tau\\ (\tau\otimes\tau)\otimes\tau &\to(1)\otimes\tau\to\tau\,,\end{aligned} \tag{26}$$

so we see that there are two different ways that the end result is τ, thus

$$\mathrm{Hom}\,(\tau,\tau\otimes\tau\otimes\tau) \tag{27}$$

is 2-dimensional, and the two different states

$$\begin{aligned}(\tau\otimes\tau)\otimes\tau &\to(\tau)\otimes\tau\to\tau\\ (\tau\otimes\tau)\otimes\tau &\to(1)\otimes\tau\to\tau\end{aligned} \tag{28}$$

may be treated as qubits $|0\rangle$ and $|1\rangle$. Graphically these can be represented as in Fig. 6. To form lin-

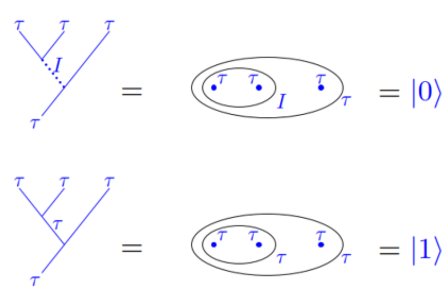

Figure 6: The two fusion channels of $\tau\otimes\tau\otimes\tau\to\tau$ that serve as qubits.

ear operators on them, we consider braiding operations such as in Fig. 2, which involves the **F** matrices $[F_\tau^{\tau\tau\tau}]_{ab}$, where $a,b=1,\tau$, and the **R** matrices

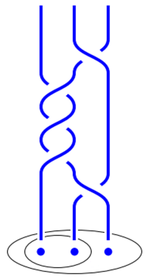

Figure 7: Sequence of braidings.

$[R_x^{\tau\tau}]_{ab}$. Utilizing (23) and (24), it can be solved so that

$$F = \begin{bmatrix} \phi^{-1} & \sqrt{\phi^{-1}} \\ \sqrt{\phi^{-1}} & \phi^{-1} \end{bmatrix}$$
$$R = \begin{bmatrix} e^{-4i\pi/5} & 0 \\ 0 & -e^{-2i\pi/5} \end{bmatrix}, \tag{29}$$

where ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$. Thus, the **B** matrix follows too,

$$B = F^{-1} R F \,, \tag{30}$$

which can be readily obtained but kind of ugly.

Now we have the explicit forms of those matrices. The nice thing about them is that physically (at least at thought experiment level) they are readily realized as braiding among the three particles. For example, the operation in Fig. 7 would result in a concatination of those matrices. However one may wish they have a nicer one that performs a logical operation, such as this one,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{31}$$

or something similar. It turns out that these can be approximated by a long sequence of braiding operations, such as in Fig. 8, where a longer sequence can provide a better approximation.

For a different anyon system, the fusion rules N would be different. However, one may still utilize the constraints in (23), (24) to solve for the **F**, **R** and **B**matrices to derive the operation of braiding on the Hilbert space of fusion channels.

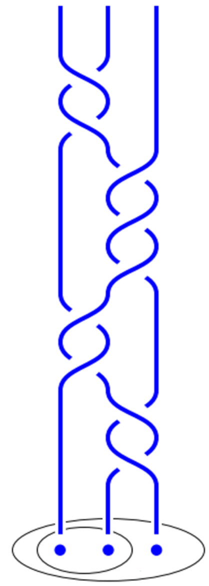

Figure 8: A sequence of braidings that approximate (31).

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