A Very Brief Review on $SL(2,\mathbb{R})$ and its Applications

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In this short note, we give a very^N brief overview of $SL(2,\mathbb{R})$ and its application. Dachuan said, for sufficiently large N, very^N fuse into the vacuum. But since I didn't figure out how large N should be, as always, I ended up with 3 and a half pages, diagram excluded.

INTRODUCTION

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is given by

$$[\Lambda_0,\Lambda_1]=\Lambda_2,\quad [\Lambda_0,\Lambda_2]=-\Lambda_1,\quad [\Lambda_1,\Lambda_2]=-\Lambda_0. \tag{0.1}$$

The simply connected Lie group with the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is denoted by $\widetilde{SL}(2,\mathbb{R})$ and its has the center

$$\mathcal{Z}=\{e^{2\pi n\Lambda_0}: n\in\mathbb{Z}\}. \tag{0.2}$$

The Casimir operator is given by

$$Q=\Lambda_0^2-\Lambda_1^2-\Lambda_2^2. \tag{0.3}$$

REPRESENTATIONS OF $\mathfrak{sl}(2,\mathbb{R})$

Here we list all the unitary irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$. The irreducible representations are labelled by two quantum numbers $q=\lambda(1-\lambda)\in\mathbb{R}$, the eigenvalue of the Casimir operator Q (where $\lambda\in\mathbb{R}$ or $\lambda\in\frac{1}{2}+i\mathbb{R}$), and $e^{-2\pi i\mu}$, the eigenvalue of the center element $e^{2\pi\Lambda_0}$ (where $\mu\in\mathbb{R}/\mathbb{Z}$). Here are the following irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$:

- Trivial representation I: $q=\mu=0$.
- Discrete series with lowest weight $\mathcal{D}^+_\lambda$: $\lambda>0,\mu=\lambda$.

$$\mathcal{D}^+_\lambda=\{|\lambda;m\rangle : m=\lambda,\lambda+1,\lambda+2,\cdots\}. \tag{0.4}$$

- Discrete series with highest weight $\mathcal{D}^-_\lambda$: $\lambda>0,\mu=-\lambda$.

$$\mathcal{D}^-_\lambda=\{|\lambda;m\rangle : m=-\lambda,-\lambda-1,-\lambda-2,\cdots\}. \tag{0.5}$$

- Principal series $\mathcal{C}^\mu_\lambda$: $\lambda\in\frac{1}{2}+i\mathbb{R}$, where we chosen $\mu\in(-1/2,1/2]$.

$$\mathcal{C}^\mu_\lambda=\{|\lambda,\mu;m\rangle : m=\mu,\mu\pm1,\mu\pm2,\cdots\}. \tag{0.6}$$

- Complementary series $\mathcal{E}^\mu_\lambda$: $|\mu|<\lambda<1/2$, and we have chosen $\mu\in(-1/2,1/2]$.

$$\mathcal{E}^\mu_\lambda=\{|\lambda,\mu;m\rangle : m=\mu,\mu\pm1,\mu\pm2,\cdots\}. \tag{0.7}$$

GLOBAL STRUCTURES

In the above section, we list all the unitary irreducible representation constructed from the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. This gives all the irreducible representation of the simply connected Lie group with this Lie algebra, which is denoted by $\widetilde{SL}(2,\mathbb{R})$. Lie groups with different global structures are acquired by taking quotient with respect to the center $\mathcal{Z}$ or a subset of $\mathcal{Z}$. Allowed irreps can be found by requiring the set we quotient out acting trivially on the irreps.

For instance, $PSL(2,\mathbb{R})=\widetilde{SL}(2,\mathbb{R})/\mathcal{Z}$. Therefore, $\mathcal{Z}$ must act trivially on the irreps of $PSL(2,\mathbb{R})$. Hence, only irreps with $e^{-2\pi i\mu}=1$ are allowed. As another example, $PSL(2,\mathbb{R})=SL(2,\mathbb{R})/\{1,-1\}$. Hence, for $SL(2,\mathbb{R})$, the condition is relaxed to $e^{-2\pi i\mu}=\pm1$.

Finally, as a bonus, we describe the allowed irreps of $G_B=\frac{\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}}{\mathbb{Z}}$ where we quotient the diagonal $\mathbb{Z}$ generated by the action

$$(g,\theta)\mapsto(e^{2\pi\Lambda_0}g,\theta+B)\in\widetilde{SL}(2,\mathbb{R})\times\mathbb{R},\quad B\in\mathbb{R}. \tag{0.8}$$

Before taking quotient, the irreps of $\widetilde{SL}(2,\mathbb{R})\times\mathbb{R}$ are irreps of $\widetilde{SL}(2,\mathbb{R})$ tensor product the irreps of $\mathbb{R}$. The irreps of $\mathbb{R}$ are given by $|k\rangle$ where $k\in\mathbb{R}$, and $\theta\in\mathbb{R}$ acts on $|k\rangle$ as $e^{ik\theta}|k\rangle$. Notice that there is no charge quantization on k because the group is $\mathbb{R}$ rather than $U(1)$. To quotient the diagonal $\mathbb{Z}$ means the irreps must be invariant under the action $(e^{2\pi\Lambda_0},B)$, which requires

$$e^{-2\pi i\mu}e^{iBk}=1, \tag{0.9}$$

that is,

$$\mu-\frac{Bk}{2\pi}\in\mathbb{Z}. \tag{0.10}$$

As we will see, an application of this construction is when we want to sum over irreps of $\widetilde{SL}(2,\mathbb{R})$ with fixed $e^{-2\pi i\mu}$, which is in the same spirit as one can gauge $\mathbb{Z}_N$ center symmetry out of $SU(N)$ by first extending $SU(N)$ to $U(N)=\frac{SU(N)\times U(1)}{\mathbb{Z}_N}$ in [5].

FOURIER ANALYSIS (OR HOW TO FIND WAVEFUNCTIONS)

To motivate this section, let's consider the a 1-dim particle in QM. A complete basis of wave function is given by

$$\frac{1}{\sqrt{2\pi}}e^{ipx}, \tag{0.11}$$

and any other wave function can be written as a linear superposition of the above. Notice that if we view the position space $\mathbb{R}$ as a group, then this is precisely the representation matrix (1-by-1, of course) of the charge p irrep of $\mathbb{R}$ up to normalization. This idea naturally generalizes to particles moving on Lie groups. Specifically, consider the action

$$S = \int dt\, \mathrm{Tr}(\partial_t g)(\partial_t g^{-1}), \quad g \in G, \tag{0.12}$$

then a basis of wave functions are given by the *matrix element functions* $U^j_{\alpha,k}$:

$$U^j_{\alpha,k} = \langle j|U_\alpha(g)|k\rangle, \quad (\overline{U}^j_{\alpha,k}(g) = U^j_{\alpha,k}(g^{-1}) = \langle k|U_\alpha(g)|j\rangle^*), \tag{0.13}$$

where α labels the irreps of G. Notice that since each matrix element function $U^j_{\alpha,k}$ carries two indices k, j, it is naturally to define two actions (i.e., the left action and the right action) of G on $U^j_{\alpha,k}$:

$$L(h) \cdot \overline{U}^k_{\alpha,j} = \overline{U}^k_{\alpha,n} U^n_{\alpha,j}(h), \tag{0.14}$$

$$R(h) \cdot \overline{U}^k_{\alpha,j} = \overline{U}^k_{\alpha,n}(h) \overline{U}^n_{\alpha,j}. \tag{0.15}$$

If you wonder why left action acts from the right and vice versa, this is because we are working with $\overline{U}$ rather than U. Anyway, this is consistent with the fact that there are two conserved currents in the above Lagrangian:

$$j_1 = g^{-1}\dot{g}, \quad j_2 = \dot{g}g^{-1}. \tag{0.16}$$

However, there is a caveat. If G is compact, then $U^j_{\alpha,k}(g)$ is obviously normalizable; however, if G is noncompact, it is often $U^j_{\alpha,k}(g)$ is not normalizable. For instance, for $G = \mathbb{R}$, $U_\alpha(\theta) = \frac{1}{\sqrt{2\pi}}e^{i\alpha\theta}$ is clearly not normalizable. But it is at least δ-normalizable, in the sense that,

$$\langle U_\alpha|U_\beta\rangle \equiv \int_{-\infty}^{\infty} d\theta\, U_\alpha(\theta)\overline{U}_\beta(\theta) = 2\pi\delta(\alpha-\beta). \tag{0.17}$$

However, some of the wave functions are not even δ-normalizable, therefore should be excluded. For $G = \widetilde{SL}(2,\mathbb{R})$, this is the complementary series $\mathcal{E}^\mu_\lambda$ and the discrete series $\mathcal{D}^\pm_\lambda$ with $\lambda < \frac{1}{2}$.

Similar to the $U(1)$ case, these δ-normalizable wave functions satisfies the following orthogonality conditions. For discrete series $D^\pm_\lambda$ with $\lambda > \frac{1}{2}$, we have

$$\langle \overline{U}^{\pm(\lambda+k)}_{\lambda,\pm(\lambda+j)}|\overline{U}^{\pm(\lambda'+k')}_{\lambda',\pm(\lambda'+j')}\rangle = \frac{8\pi^2}{2\lambda-1}\delta(\lambda-\lambda')\delta_{jj'}\delta_{kk'}, \tag{0.18}$$

where $\lambda, \lambda' > 1/2,\ j, k, j', k' \in \mathbb{Z}_{\geq 0}$. For principal series C^μ_q, we have

$$\begin{aligned}&\langle \overline{U}^{\mu+k}_{\frac{1}{2}+is,\mu+j}|\overline{U}^{\mu'+k'}_{\frac{1}{2}+is',\mu'+j'}\rangle \\ =&4\pi^2\frac{\cosh(2\pi s)+\cos(2\pi\mu)}{s\sinh(2\pi s)}\delta(s-s')\delta(\mu-\mu')\delta_{jj'}\delta_{kk'}.\end{aligned} \tag{0.19}$$

Here, we stop to introduce the concept of *Plancherel measure*, which is the analog of the normalization of the wave function $U_k = e^{ikx}$:

$$\langle U_k|U_{k'}\rangle = 2\pi\delta(k-k'). \tag{0.20}$$

In this case, the identity operator I is given by

$$I = (2\pi)^{-1}\int |\overline{U}_k\rangle\langle\overline{U}_k|dk. \tag{0.21}$$

Hence the Plancherel measure is $(2\pi)^{-1}dk$, and is computed from $\langle U_k|U_{k'}\rangle$. In general, consider

$$\langle \overline{U}^k_{\alpha,j}|\overline{U}^l_{\beta,m}\rangle = C_\alpha\delta(\alpha-\beta)\delta_{jm}\delta^{kl}, \tag{0.22}$$

then $C^{-1}_\alpha d\alpha$ is the Plancherel measure. For discrete series $\mathcal{D}^\pm_\lambda$, the Plancherel measure is

$$(2\pi)^{-2}(\lambda-\frac{1}{2})d\lambda, \quad \lambda > \frac{1}{2}. \tag{0.23}$$

For principal series $\mathcal{C}^{\mu=\frac{1}{2}+is}_\lambda$, the Plancherel measure is

$$(2\pi)^{-2}\frac{s\sinh(2\pi s)}{\cosh(2\pi s)+\cos(2\pi\mu)}dsd\mu. \tag{0.24}$$

APPLICATION I: FIRST ORDER FORMALISM OF JT GRAVITY

In this section, we briefly show how to see the partition function of JT gravity formulated using $SL(2,\mathbb{R})$ gauge theory on a disk D^2 is the same as the partition function of Schwarzian theory on the boundary $\partial D^2 = S^1$ following [2].

The Schwarzian theory is given by the action

$$S_{\mathrm{Schw}}[f] = -C\int_0^\beta du\left\{\tan\frac{\pi f}{\beta}\right\}, \quad \{F,u\} \equiv \frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2, \tag{0.25}$$

and the partition function over S^1 is given by

$$S_{\mathrm{Schwarzian}}(\beta) \propto \int_0^\infty ds s\sinh(2\pi s)e^{-\frac{\beta}{2C}s^2}. \tag{0.26}$$

For compact group, the partition function of 2d YM theory on D^2 is given by (this should be reviewed in Zipei's paper)

$$Z(g, e\beta) = \sum_R \dim R\chi_R(g)e^{-\frac{e\beta C_2(R)}{4N}}, \tag{0.27}$$

where $\text{tr}(\Lambda^i\Lambda^j) = N\delta^{ij}$, g is the holonomy of G along the boundary and $e\beta$ is the size of the boundary S^1. For non-compact group, the partition function then becomes

$$Z(g, e\beta) = \int dR\rho(R)\chi_R(g)e^{-\frac{e\beta C_2(R)}{4N}}, \tag{0.28}$$

and $\text{tr}(\Lambda^i\Lambda^j) = N\eta^{ij}$.

For this, we first extend the gauge group to G_B defined in the previous section and add to the action

$$\Delta S_E = -i\int_\Sigma \phi^{\mathbb{R}}F^{\mathbb{R}} + i\oint_{\partial\Sigma}\phi^{\mathbb{R}}A^{\mathbb{R}}. \tag{0.29}$$

The holonomy along the boundary is given by $\tilde{g} = \oint_{\partial\Sigma} A^i\Lambda_i \in \widetilde{SL}(2,\mathbb{R})$ and $\theta = \oint_{\partial\Sigma} A^{\mathbb{R}}$, as

$$Z_{k_0}(\tilde{g}, e\beta) = \int d\theta Z((\tilde{g},\theta), e\beta)e^{-ik_0\theta}, \tag{0.30}$$

where we've chosen $\phi^{\mathbb{R}}|_{\partial\Sigma} = k_0$, and this allows us to isolate the contribution of the $\mathbb{R}$ labelled by k_0, and thus fixes $e^{2\pi i\mu}$ with $\mu \in \frac{Bk_0}{2\pi} + \mathbb{Z}$. Then, we send $\mu \to i\infty$, or equivalently $kB \to \infty$,

$$G = G_B \quad \text{with} \quad B \to \infty, \quad \phi^{\mathbb{R}}|_{\partial\Sigma} = k_0 = -i. \tag{0.31}$$

For fixed G_B holonomy $(g, e\beta)$, the partition function is given by

$$\begin{aligned} Z(g, e\beta) \propto &\int_{-\infty}^{\infty} dk \int_0^\infty ds \frac{s\sinh(2\pi s)}{\cosh(2\pi s) + \cos(Bk)} \\ &\times \chi_{(s,\mu=\frac{Bk}{2\pi},k)}(g)e^{-e\beta s^2/2} + (\text{discrete series}), \end{aligned} \tag{0.32}$$

and after (0.30) and taking $k_0 = -i$ and $B \to \infty$, the leading order contribution for $\tilde{g} = 1$ is given by

$$Z_{k_0}(1, e\beta) \propto C\int_0^\infty ds s\sinh(2\pi s)e^{-e\beta s^2/2} + O(e^{-B}). \tag{0.33}$$

Thus we recover the partition function (0.26) of the Schwarzian theory.

APPLICATION II: STRING IN AdS_3

In this section, we briefly review the spectrum of strings in AdS_3 and its relation to the representation theory of $\widetilde{SL}(2,\mathbb{R})$ following [3].

Notice that the group manifold of $\widetilde{SL}(2,\mathbb{R})$ is the AdS_3 with non-compact time direction. Hence string theory on AdS_3 is made of $\widetilde{SL}(2,\mathbb{R})$ WZW model tensor product some internal CFT. This internal CFT usually arise from the string compactification. The action of the $\widetilde{SL}(2,\mathbb{R})$ WZW model is given by

$$S = \frac{k}{8\pi\alpha'}\int d^2\sigma \text{Tr}\Big(g^{-1}g\partial_\mu g g^{-1}\partial^\mu g\Big) + k\Gamma_{WZ}. \tag{0.34}$$

The theory has a set of conserved left moving and right moving currents $J^a_R(x^+)$ and $J^a_L(x^-)$, corresponding to the $\widetilde{SL}(2,\mathbb{R}) \times \widetilde{SL}(2,\mathbb{R})$ symmetries of the Lagrangian.

Before study the theory quantum mechanically, a little discussion classical solutions might be helpful. The theory has a spectral flow symmetry, parameterized by $w \in \mathbb{Z}$, which allows one to generate new solutions from the known one. Under the spectral flow, the currents transform as

$$J^3_{R/L} = \tilde{J}^3_{R/L} - \frac{k}{2}w, \quad J^\pm_{R/L} = \tilde{J}^\pm_{R/L}e^{\pm iwx^\pm}, \tag{0.35}$$

and in terms of the Fourier modes

$$J^3_n = \tilde{J}^3_n - \frac{k}{2}w\delta_{n,0}, \quad J^\pm_n = \tilde{J}^\pm_{n\pm w}. \tag{0.36}$$

As one may recognize, this is the spectral flow in CFT.

There are three classes of solutions we are interested in:

- *Geodesic*: For these solutions, the string collapsed to a point, hence the string world sheet shrinks into a world line in AdS_3.
- *Short Strings*: These solutions are acquired from the spectral flow of the timelike geodesics. They describe a string expand and contract repeatedly.
- *Long Strings*: These solutions are acquired from the spectral flow of the spacelike geodesic. They describe a string, which starts its life as a circular string of an infinite radius on the boundary of the AdS_3 at $t = -\infty$, collapses into the bulk and then expand to an infinite circle on the boundary again. The reason for such solution to have finite energy is the NS-NS B field tends to extend the string, which balance the contraction force of the string tension.

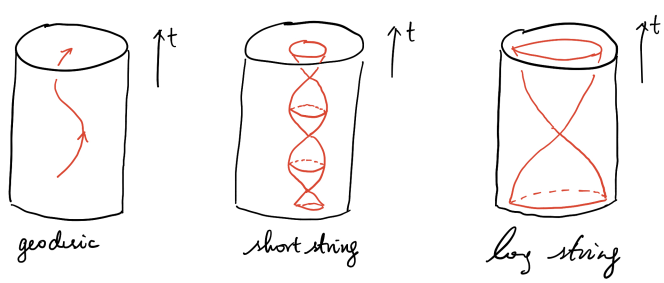

FIG. 1: A terrible drawing to illustrate the three types of classical solutions mentioned above.

We pause here to explain the two puzzled before [3]. People used to believe the spectrum only contains the representations of $\widehat{SL}_k(2,\mathbb{R})$ whose zero modes are the discrete series with $0 < \lambda < \frac{k}{2}$, whose L_0 is bounded below. This raises the following puzzles. First, there is an upper bound on the mass of the string states in AdS_3, which implies the energy contribution from the internal CFT can not be too high. That is, if the compact dimensions contain a circle, then there will be a cut-off on the winding numbers, which seems quite arbitrary. Second, among those representations, there is no one correspond to the long string solutions.

In the quantum theory, the $\widetilde{SL}(2,\mathbb{R})$ extends into Kac-Moody algebra $\widehat{SL}_k(2,\mathbb{R})$. The full representation is acquired by acting the raising operators (creating string oscillation) on the zero modes, which are representations of $\widetilde{SL}(2,\mathbb{R})$. In the low energy limit, the strings will become particles and only those zero modes will survive. Hence the WZW model will reduce to the quantum mechanics over $AdS_3 = \widetilde{SL}(2,\mathbb{R})$. From previous discussion, we learn we must keep all the (δ-)normalizable representations, including the principal continuous series $\mathcal{C}^\mu_\lambda$ and the discrete series $\mathcal{D}^\pm_\lambda$ with $\lambda > \frac{1}{2}$ and we must include both. We will denote the corresponding representations of $\widehat{SL}_k(2,\mathbb{R})$ as $\widehat{\mathcal{C}}^\mu_\lambda$ and $\widehat{\mathcal{D}}^\pm_\lambda$ respectively.

However, there are still two extra complexities. First, the representation of $\widehat{SL}_k(2,\mathbb{R})$ will contain negative norm states, thus must be removed from the spectrum by imposing the Virasoro constraint:

$$(L_n + \mathcal{L}_n - \delta_{n,0})|phsical\rangle = 0, \quad n \geq 0, \tag{0.37}$$

where L_n is the Virasoro generators of the $\widehat{SL}_k(2,\mathbb{R})$ theory and $\mathcal{L}_n$ is the Virasoro generators of the internal CFT. Those representations still include negative norm states after imposing the Virasoro constraints must not be included in the theory.

Second, the $\widehat{SL}_k(2,\mathbb{R})$ has the spectral flow 0.36, under which

$$\tilde{L}_n = L_n + wJ^3_n - \frac{k}{4}w^2\delta_{n,0}. \tag{0.38}$$

Unlike the spectral flow for $U(1)$ symmetry in the case of compact boson, this does lead to new representations which generically are not isomorphic to the $\widehat{\mathcal{C}}^\mu_\lambda$ and $\widehat{\mathcal{D}}^\pm_\lambda$. We will denote them as $\widehat{\mathcal{C}}^{\mu,w}_\lambda$ and $\widehat{\mathcal{D}}^{\pm,w}_\lambda$. Hence, we must keep those new representations which satisfy unitarity after imposing the Virasoro constraint.

After some tedious algebra which you can find in the appendix of [3], one finds the spectrum of the $\widetilde{SL}_k(2,\mathbb{R})$ WZW model contains two types of representations. The spectral flow of the principal series $\widehat{\mathcal{C}}^{\mu,w}_{\frac{1}{2}+is,L} \otimes \widehat{\mathcal{C}}^{\mu,w}_{\frac{1}{2}+is,R}$ and the spectral flow of the discrete series $\widehat{\mathcal{D}}^{+,w}_{\lambda,L} \otimes \widehat{\mathcal{D}}^{+,w}_{\lambda,R}$ with $\frac{1}{2} < j < \frac{k-1}{2}$. For string theory, we tensor the above with the representations of the internal CFT and impose the Virasoro constraints.

Two comments:

- We have to include two copies of the same representations for the left-mover and the right-mover because there are two copies of the Virasoro. If you wonder why we can't have something like $\widehat{\mathcal{C}} \otimes \widehat{\mathcal{D}}$, this is because, in the low energy limit where strings become particles, the wave functions are of the same irreps under the two copies $\widetilde{SL}(2,\mathbb{R})$, see (0.14).
- And if you wonder we do not include $\widehat{\mathcal{D}}^-$, this is because $\widehat{\mathcal{D}}^-$ is actually related to $\widehat{\mathcal{D}}^+$ under the spectral flow.

How does this spectrum helps solve the two puzzles we mentioned previously? First, the $\widehat{\mathcal{C}}^{\mu,w}_{\frac{1}{2}+is,L} \otimes \widehat{\mathcal{C}}^{\mu,w}_{\frac{1}{2}+is,R}$ would correspond to the long strings in the classical limit. The appearance of the principal representation also explains what happens if we try to push the internal energy h very high, the discrete representation which seems to violate the unitarity bound would decay, and a continuous representation would appear and save the day.

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