GROUP ACTIONS ON MULTISYMPLECTIC MANIFOLDS

BRIAN TRAN

Abstract. Analogous to the Poisson bracket on symplectic manifolds, the observables on multisymplectic manifolds can be equipped with an algebra structure. We investigate these algebraic structures; paying particular attention to the conserved quantities arising from group actions on a multisymplectic manifold.

Introduction

In this paper, we investigate the algebraic structures associated to multisymplectic manifolds. Keeping the prototypical example of classical field theory in our mind, this algebraic structure provides a covariant generalization of the Poisson bracket which one usually employs in the quantization of a symplectic system. For field theory, there are two immediate issues in the symplectic formulation: first, it requires that one introduce an explicit foliation of spacetime and hence break manifest covariance; furthermore, the resulting symplectic phase space is infinite-dimensional, which introduces many functional analytic issues. By studying the algebraic structure of the multisymplectic formulation of classical field theories (which is both covariant and finite-dimensional, see Appendix B for a primer), the hope is that one can rigorously formulate a quantization theory for field theories; this is still an active area of research but there has been progress for specific types of manifolds, e.g. the multisymplectic quantization of hyperkähler manifolds [1].

In this paper, we put the major ideas and concepts in the main body of the paper, and leave specific examples (appendix B) and proofs (appendix C) to the appendices.

Multisymplectic Geometry and the Algebra of Observables

In this section, we introduce the basics of multisymplectic geometry and the algebra of observables. We say a manifold M with a closed $n+1$ form $\omega \in \Omega^{n+1}(M)$ is a pre-n-plectic manifold, where $1 \leq n \leq \dim(M) - 1$. Furthermore, if ω is non-degenerate, in the sense that the map $v \mapsto i_v\omega$ is injective, then we call the system (M, ω) $n-$plectic. The simplest examples are $n = 1$ which is a symplectic manifold and $n = \dim(M) - 1$ which is an oriented manifold (where ω is the volume form). In the context of classical field theory (see Appendix B), the relevant multisymplectic form is defined over the restricted dual jet bundle of a fiber bundle; one can think of the multisymplectic form as encoding a symplectic form in each spacetime direction; upon introducing an explicit foliation of spacetime, the multisymplectic form transgresses into a symplectic form.

Given an $n-$plectic manifold (M, ω), we say that an $(n-1)-$form H is Hamiltonian if there exists a vector field $X_H \in \mathfrak{X}(M)$, called the associated Hamiltonian vector field, such that $dH = -i_{X_H}\omega$. We denote the space of such Hamiltonian forms $\Omega^{n-1}_{Ham}(M) \subset \Omega^{n-1}(M)$.

Now, consider the following Lie $\infty-$algebra (which is a graded generalizaton of a Lie algebra; although I won't state the precise definition here, it is just a graded algebra with a graded analog of the Jacobi identity),

$$L_\infty(M, \omega) = (L = \oplus_{i=0}^{n-1} L_i, \{[\cdot, \cdots, \cdot]_{k=1}^{n+1}\}),$$

where $L_0 \equiv \Omega^{n-1}_{Ham}(M), L_i \equiv \Omega^{n-1-i}(M)$ for $1 \leq i \leq n-1$ (an element of L_α is said to have degree α), and the $k-$ary brackets are defined as follows. $[\cdots]_k : L^{\otimes k} \to L$; $[\alpha]_1 = d\alpha$ if $\deg \alpha > 0$, and $[\alpha]_1 = 0$ if $\deg \alpha = 0$. For $k > 1$,

$$[\alpha_1, \ldots, \alpha_k]_k = -(-1)^{k(k+1)/2} i_{X_{\alpha_1}} \cdots i_{X_{\alpha_k}} \omega$$

if $\deg(\alpha_1 \otimes \cdots \otimes \alpha_k) = 0$ and 0 otherwise. This algebra is known as the Poisson Bracket Lie $n-$algebra and serves as the algebra of observables in the multisymplectic setting.

A particularly important subalgebra is $(L_0, [\cdot, \cdot]_2)$, $[\alpha, \beta]_2 = i_{X_\alpha} i_{X_\beta}\omega$. $[\alpha, \beta]_2$ is clearly an $n-1$ form by definition. In fact, it is also a Hamiltonian form; its associated Hamiltonian vector field is just $[X_\alpha, X_\beta]$ (which follows routinely from Cartan's magic formula and closedness of ω, see Appendix C). Thus, we have an algebra morphism from the Lie algebra of vector fields equipped with the Jacobi Lie bracket to $(L_0, [\cdot, \cdot]_2)$. Note that $[\cdot, \cdot]_2$ is just the multisymplectic analog of the Poisson bracket (see Appendix B); in the case $n = 1$ (i.e. symplectic geometry), this is precisely the Poisson bracket.

Conserved Quantities

Next, we state the algebraic properties of conserved quantities. Conserved quantities are of course fundamental in a physical system; in the setting of quantization, they provide operators which commute with the

Department of Mathematics, University of California at San Diego, La Jolla, CA 92093.

Hamiltonian governing the system. Even in the classical context, they are useful as they provide a way to effectively reduce the dimensionality of the system (a process known as symplectic reduction in mechanics or more generally, multisymplectic reduction).

We say a form $\alpha \in \Omega(M) \equiv \oplus_k \Omega^k(M)$ is strictly (resp. locally, globally) conserved in the direction $v \in \mathfrak{X}(M)$ if $\pounds_v \alpha = 0$ (resp. $\pounds_v \alpha$ is closed, $\pounds_v \alpha$ is exact), where $\pounds$ denotes the Lie derivative. Of the three notions of conserved, the notion of globally conserved is usually emphasized in physics, where we say that a symmetry of a Hamiltonian/Lagrangian density is a transformation which leaves the density invariant up to a total derivative. Subsequently, for brevity, we will focus on globally conserved quantities, although similar statements can be made in the local and strictly conserved cases. Note that $\alpha \in \Omega^{n-1}_{Ham}(M)$ is globally conserved by X_H (i.e. $\pounds_{X_H} \alpha$ is exact) if and only if $\pounds_{X_\alpha} H$ is exact (see Appendix C).

The algebraic structure of the conserved quantities arises from the fact that they form subalgebras of $L_\infty(M, \omega)$. In particular, if $C(v)$ denotes the set of globally conserved forms in the direction v, where v is also a multisymplectic symmetry (i.e. $\pounds_v \omega = 0$), then $L_\infty(M, \omega) \cap C(v)$ is an $L_\infty-$subalgebra of $L_\infty(M, \omega)$ (and similarly for strict and locally conserved quantities). For a proof, see Appendix C.

Homotopy Co-momentum Maps for Multisymplectic Group Actions

Since the algebra of conserved quantities is fundamental to the structure of a multisymplectic system, we want to be able to construct such conserved quantities. One way to do this is by finding homotopy co-momentum maps. These are the multisymplectic generalizaton of momentum maps in symplectic geometry, which translate symmetries into conserved quantities (e.g. the momentum map for rotational symmetry is angular momentum); for a primer on momentum maps, see Appendix A (if you're unfamiliar with momentum maps, I recommend reading this before our discussion of homotopy co-momentum maps).

Consider the following $L_\infty-$algebra:

$$L_\infty(\mathfrak{g}) \equiv (\oplus_{k=1}^n \Lambda^k \mathfrak{g}, [\cdot,\cdot])$$

where the bracket is extended from the usual bracket on one copy of $\mathfrak{g}$:

$$[\xi_1 \wedge \cdots \wedge \xi_k, \eta_1 \wedge \cdots \wedge \eta_k] \equiv [\xi_1, \eta_1] \wedge \cdots \wedge [\xi_k, \eta_k].$$

Define the homology differential $\partial_k(\xi_1 \wedge \cdots \wedge \xi_k) = \sum_{1 \leq i < j \leq k} (-1)^{i+j} [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \hat{\xi}_i \wedge \cdots \wedge \hat{\xi}_j \wedge \cdots \wedge \xi_k$. The skew symmetry and Jacobi identity of the bracket implies that $\partial_{k-1} \circ \partial_k = 0$ and so the above forms a chain complex. Its homology will allow us to describe the conserved quantities. Now, suppose we have a $G-$action on M which is multisymplectic, i.e. the $G-$action preserves the multisymplectic form: $g^*\omega = \omega$ or infinitesimally, $\pounds_\xi \omega = 0$. We define a homotopy co-momentum map as an $L_\infty-$morphism $(f) : L_\infty(\mathfrak{g}) \to L_\infty(M, \omega)$, where (f) denotes a collection of maps $f_i : \Lambda^i \mathfrak{g} \to L_{i-1}$, and in addition, for $\xi \in \Lambda^1 \mathfrak{g} = \mathfrak{g}$, $df_1(\xi) = -i_\xi \omega$, where the vector field on the RHS is from the infinitesimal action on M. Thus, a homotopy co-momentum map is an $L_\infty-$morphism of the above algebras, with the additional condition that $f_1 : \mathfrak{g} \to L_0 = \Omega^{n-1}_{Ham}(M)$ is a co-momentum map in the usual sense as described in Appendix A; as we will see, this will single out a stronger conservation property that f_1 has than the other f_k. We will provide an example of such an f_1 in the context of field theory in Appendix B; unsurprisingly, it is the usual Noether current. Using the algebras of both and the homology operator, the condition that (f) preserves the algebras is equivalent to $-f_{k-1}(\partial\xi) = df_k(\xi) + (-1)^{k(k+1)/2} i_{\xi_1} \ldots i_{\xi_k} \omega$, where $\xi = \xi_1 \wedge \cdots \wedge \xi_k \in \Lambda^k \mathfrak{g}$.

As we noted, the space $\oplus \Lambda^k \mathfrak{g}$ forms a chain complex, so we define its relevant homology spaces: the cycles $Z_k(\mathfrak{g}) = \ker \partial_k$, the boundaries $B_k(\mathfrak{g}) = \text{im}\partial_{k+1}$, and the $k-$th homology $H_k(\mathfrak{g}) = Z_k(\mathfrak{g})/B_k(\mathfrak{g})$. Now, suppose the group action globally conserves the Hamiltonian $H \in \Omega^{n-1}_{Ham}(M)$ of our system (the system is governed the vector field X_H of $dH = -i_{X_H}\omega$), then $f_k(\xi)$ is locally conserved by X_H for any $\xi \in Z_k(\mathfrak{g})$ and $f_k(\eta)$ is globally conserved by X_H for any $\eta \in B_k(\mathfrak{g})$ (For a proof of the result, see Appendix C). Note the contrast to the result for the co-momentum map in symplectic geometry (see Appendix A), where an $H-$preserving symplectic action preserves $\mu^*(\eta)$ for any $\eta \in \mathfrak{g}$; this should not be too surprising, as the conserved quantities in the multisymplectic case contain much more content than the class of conserved functions; they contain the whole class of $k-$form Noether currents (k from 0 to $n-1$). However, while the f_k have weaker conservation properties than the usual co-momentum map in symplectic geometry (that is, $f_k(\xi)$ needs ξ to be a cycle (resp. boundary) to be locally (resp. globally) conserved), f_1 has the stronger conservation property that $f_1(\xi)$ is globally conserved for any $\xi \in \mathfrak{g}$, since we explicitly required that $df_1(\xi) = -i_\xi \omega$. The proof is identical to the proof of the conservation of the (co-)momentum map in symplectic geometry, except there we assumed that the group action strictly conserved H whereas here we assume it globally conserves H (if we instead assumed the multisymplectic action strictly conserved H, then $f_1(\xi)$ is strictly conserved for any $\xi \in \mathfrak{g}$). Thus, the subalgebra of the degree 1 homotopy co-momentum maps contained in $(L_0 = \Omega^{n-1}_{Ham}(M), [\cdot,\cdot]_2) \subset L_\infty(M,\omega)$ forms a special algebra of conserved currents. This is the algebra of Noether currents as we usually think of it (again, see Appendix B).

Conclusion

We investigated the algebraic structures associated to multisymplectic manifolds, which generalize the usual Poisson algebra of functions in symplectic geometry to an algebra on $k-$forms. It also extends the usual conservation of co-momentum maps in symplectic geometry, modulo conditions concerning the homology spaces Z_k and B_k of the Lie algebra, but reproduces the stronger conservation property for the special subalgebra of Noether currents. This algebra and its subalgebras on multisymplectic manifolds are at the heart of recent attempts to quantize multisymplectic systems and is an active area of research. Another interesting question is the generalization of integrable systems in symplectic geometry: in symplectic geometry, if one has a complete set of commuting observables, one has a completely integrable system and the dynamics are solved (Louville-Arnold theorem). The multisymplectic generalization is much more intricate, as the theory of completely integrable PDE is less understood (but see [6] for an excellent framework of pluri-Lagrangian systems for approaching this problem).

(Note: I recommend reading appendix B for putting our discussion into the context of Hamiltonian field theory.)

Appendix A. Momentum Maps in Symplectic Geometry

Let (M,ω) be a symplectic manifold and suppose we have a Lie group G (with Lie algebra $\mathfrak{g}$) acting on (M,ω) by symplectomorphisms, i.e. $g^*\omega = \omega$ for every $g \in G$; infinitesimally, $\pounds_{\xi_M}\omega = 0$ (see below for the definition of the vector field ξ_M). We say that this action is Hamiltonian if it admits a momentum map $\mu : M \to \mathfrak{g}^*$ (here $\mathfrak{g}^*$ is the linear dual of $\mathfrak{g}$, with duality pairing which we denote $\langle\cdot,\cdot\rangle$), i.e. μ satisfies

$$d\langle \mu, \xi\rangle = i_{\xi_M}\omega,\ \forall \xi \in \mathfrak{g},$$

where ξ_M is the infinitesimal generator of the group action on M, i.e. it is a vector field on M, defined by $(\xi_M)_p \equiv \frac{d}{dt}|_0 \exp(t\xi)p$. One also usually requires that the momentum map is Ad*-equivariant; $\mu \circ g = \mathrm{Ad}^*_g \circ \mu$ (this is so that the pre-image of the momentum map is stable under the group action and thus one can perform symplectic reduction).

Aside from symplectic reduction, the significance of a momentum map is the following: if the group action admits a momentum map and it preserves a Hamiltonian H (in the sense that $g^*H = H$ or equivalently $\pounds_{\xi_M}H = 0$), then the momentum map is preserved under the time evolution of H. The proof is as follows: Compute

$$\begin{aligned} 0 &= \pounds_{\xi_M}H = i_{\xi_M}dH = -i_{\xi_M}i_{X_H}\omega = i_{X_H}i_{\xi_M}\omega \\ &= i_{X_H}d\langle\mu,\xi\rangle = \pounds_{X_H}d\langle\mu,\xi\rangle = \frac{d}{dt}\langle\mu,\xi\rangle, \end{aligned}$$

(where the d/dt is relative to the time flow generated by X_H). Thus, if we have a group action which admits a momentum map and is also a symmetry of the Hamiltonian which evolves our system, then that momentum map is conserved. This is of course Noether's theorem stated in the language of symplectic geometry.

Now, we define a dual notion to momentum map, the co-momentum map (these are equivalent for connected Lie groups). We say that the group action admits a co-momentum map if there exists a map $\mu^* : \mathfrak{g} \to C^\infty(M)$ such that $d\mu^*(\xi) = i_{\xi_M}\omega$ and μ^* is a Lie algebra homomorphism from the Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ to the algebra of observables $(C^\infty(M), \{\cdot,\cdot\})$, i.e. $\mu^*[\xi,\eta] = \{\mu^*(\xi), \mu^*(\eta)\}$. The comomentum map has the same conservative property as the momentum map: if the group action admits a comomentum map and additionally preserves some Hamiltonian H, then the comomentum map is preserved under the time evolution of H.

The comomentum map is the object that generalizes in the case of multisymplectic geometry; in particular, by replacing $(C^\infty(M), \{\cdot,\cdot\})$ with $L_\infty(M,\omega)$ and Lie algebra homomorphism to L_∞ morphism.

Appendix B. Multisymplectic Hamiltonian Field Theory

To keep the exposition simple, we will not dive in to the details of jet bundle geometry too thoroughly (but see [3] and then [2] for great introductions to this theory). Instead, to keep things simple, we will work in coordinates.

Let X be an $(n+1)-$dimensional spacetime and $Y \to X$ be a fiber bundle over spacetime. Intuitively, this is the space where our fields live; more precisely, a field is a section of this fiber bundle (if you're not familiar with fiber bundles, think of it locally as a Cartesian product $X \times Q$; the fields then map X into Q, e.g. $Q = \mathbb{R}$ is the case of a real scalar field). This space is coordinatized by the coordinates of the spacetime x^μ and the coordinates of the fiber ϕ^a, so our coordinates are (x^μ, ϕ^a). The relevant multisymplectic manifold in the Hamiltonian setting is the restricted dual jet bundle associated to the fiber bundle; $M \equiv \widetilde{(J^1Y)^*}$. The coordinates on this space are (x^μ, ϕ^a, p^μ_a). The p^μ_a are thought of as the multi-momenta: we have a momentum coordinate for each spacetime coordinate and fiber coordinate. These momenta arise from the covariant Legendre transform,

$$p^\mu_a = \frac{\partial L}{\partial(\partial_\mu \phi^a)},$$

which passses from the Lagrangian picture (involving the first order derivatives $\partial_\mu \phi^a$) to the (covariant!) Hamiltonian picture (involving the multi-momenta p^μ_a). Note that in order for the coordinates p^μ_a to be non-degenerate i.e. good coordinates, the Lagrangian should be hyperregular; otherwise, we should be working on some constraint submanifold of the restricted dual jet bundle (more precisely, on the image of the Legendre transform).

We define the Cartan $(n+1)-$form

$$\theta = -p^\mu_a \wedge d\phi^a \wedge d^n x_\mu,$$

where $d^n x_\mu \equiv i_{\partial_\mu} d^{n+1}x$. The multisymplectic $(n+2)-$form is then defined as

$$\omega = -d\theta = dp^\mu_a \wedge d\phi^a \wedge d^n x_\mu;$$

then (assuming the Lagrangian is hyperregular), (M, ω) is an $(n+1)-$plectic manifold (note ω is not a form on X but a form on M, which is an $(n+1+\dim(Q) + (n+1)\dim(Q))-$dimensional manifold). Think of ω as the spacetime generalization of the symplectic form in mechanics, $dp_a \wedge dq^a$; in fact, the multisymplectic form ω encodes such a symplectic form in each spacetime direction, so is a covariant generalization.

Now, let's get a feel for what the governing equations $dH = -i_{X_H}\omega$ look like. Let's make a choice for $H \in \Omega^{n-1}_{Ham}(M)$; let's say $H = \mathcal{H}(\phi^a, p^a_\mu) d^n x_0$, where $\mathcal{H} \in C^\infty(M)$. In coordinates, a vector field on M (and in particular, X_H) can be written

$$X_H = \dot{\phi}^a \frac{\partial}{\partial \phi^a} + \dot{p}^\mu_a \frac{\partial}{\partial p^\mu_a},$$

where $\dot{x}$ denotes the time derivative of x along the flow of X_H. Now, compute $dH = -i_{X_H}\omega$ in coordinates.

$$dH = \frac{\partial \mathcal{H}}{\partial \phi^a} d\phi^a \wedge d^n x_0 + \frac{\partial \mathcal{H}}{\partial p^\mu_a} dp^\mu_a \wedge d^n x_0$$

and

$$i_{X_H}\omega = \dot{p}^\mu_a d\phi^a \wedge d^n x_\mu - \dot{\phi}^a dp^\mu_a \wedge d^n x_\mu.$$

Comparing both sides of $dH = -i_{X_H}\omega$, we get

$$\dot{p}^0_a = -\frac{\partial \mathcal{H}}{\partial \phi^a}, \ \dot{\phi}^a = \frac{\partial \mathcal{H}}{\partial p^0_a}.$$

These are the usual time evolution Hamilton's equations that one sees in Hamiltonian field theory. Why was time singled out here (i.e. the 0 component of momenta)? This is because we chose $H = \mathcal{H} d^n x_0$. The key point is that we could've chosen a different slicing of the form and produced a different set of equations corresponding to the evolution of the system in the direction of the slicing, i.e. this construction is covariant. The most general form of these equations is known as the De Donder-Weyl equations,

$$\partial_\mu p^\mu_a = -\frac{\partial \mathcal{H}}{\partial \phi^a}, \ \partial_\mu \phi^a = \frac{\partial \mathcal{H}}{\partial p^\mu_a}.$$

The first equation is just the Legendre transform of the perhaps more familiar Euler-Lagrange equations

$$\partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi^a)} - \frac{\partial L}{\partial \phi^a} = 0,$$

and the second equation is just the inverse Legendre transform.

Recall our discussion of $L_\infty(M, \omega)$ in the abstract setting and in particular the subalgebra $(L_0, [\cdot,\cdot]_2)$. Let's see how this subalgebra looks in this setting with the above H and a similar $F = \mathcal{F} d^n x_0$. Compute

$$[H, F]_2 = i_{X_H} i_{X_F}\omega = \Big(\frac{\partial \mathcal{H}}{\partial p^0_a}\frac{\partial \mathcal{F}}{\partial \phi^a} - \frac{\partial \mathcal{F}}{\partial p^0_a}\frac{\partial \mathcal{H}}{\partial \phi^a}\Big) d^n x_0.$$

This is just the usual Poisson bracket of the functions $\mathcal{H}, \mathcal{F}$ (multiplied by $d^n x_0$) and so clearly $(L_0, [\cdot,\cdot]_2)$ forms a Lie algebra contained inside the $L_\infty-$algebra. Of course, as noted before, $[\cdot,\cdot]_2$ is more general (covariant), since we did not have to slice our forms as we did.

Now, as promised, let's construct the degree 1 homotopy co-momentum map f_1. Suppose we have a group action on M that preserves θ, $g^*\theta = \theta$ which implies (since pullbacks commute with exterior derivatives) that the action is multisymplectic, $g^*\omega = \omega$. Infinitesimally, we have $\pounds_{\xi_M}\theta = 0, \pounds_{\xi_M}\omega = 0$. Note that usually, one considers actions on the fields i.e. sections of the fiber bundle space $Y \to X$ and lifts these actions to the restricted dual jet bundle M; these types of actions automatically preserve θ (this extends the usual result in symplectic geometry: cotangent lifted actions preserve the tautological one-form on the cotangent bundle). Let ξ_M denote the infinitesimal generator vector field on M associated with $\xi \in \mathfrak{g}$. I claim

$$f_1(\xi) \equiv -i_{\xi_M}\theta$$

is a co-momentum map, i.e. that $df_1(\xi) = -i_{\xi_M}\omega$. To see this, one uses Cartan's magic formula $\pounds_v = di_v + i_v d$. Compute

$$df_1(\xi) = -di_{\xi_M}\theta = i_{\xi_M}d\theta - \underbrace{\pounds_{\xi_M}\theta}_{=0} = -i_{\xi_M}(-d\theta) = -i_{\xi_M}\omega,$$

which verifies that f_1 is a co-momentum map. If you look at the coordinate expression of θ (remember that p^μ_a is the image of the Legendre transform

$\partial L/\partial(\partial_\mu \phi^a)$, you can convince yourself that f_1 is the usual field-theoretic Noether current. If this group action also preserves the H of our theory, this quantity is conserved in the (strong homology-independent) sense described in the main body of the paper.

Appendix C. Some proofs

In this part of the Appendix, we'll prove some of the claimed results that we left out of the main body of the paper for brevity. The main tools are just Cartan's magic formula and the Lie homotopy formula, which we will state below.

Claim 1: The first claim we made is that given two Hamiltonian vector fields X_α and X_β corresponding to $\alpha, \beta \in \Omega^{n_1}_{Ham}(M)$ respectively, $[X_\alpha, X_\beta]$ is the Hamiltonian vector field corresponding to $[\alpha, \beta]_2$, i.e. that we have a Lie algebra morphism.

Proof. All we need are Cartan's magic formula $\pounds_v = i_v d + d i_v$, the Lie homotopy formula $i_{[u,v]} = \pounds_u i_v - i_v \pounds_u$, and the fact that ω is closed by definition. By definition, we know $i_{X_\alpha}\omega = -d\alpha, i_{X_\beta}\omega = -d\beta, [\alpha, \beta]_2 = i_{X_\alpha} i_{X_\beta}\omega$. We want to show that

$$i_{[X_\alpha, X_\beta]}\omega = -d[\alpha, \beta]_2.$$

Starting from the LHS, compute

$$\begin{aligned} i_{[X_\alpha, X_\beta]}\omega &= \pounds_{X_\alpha} i_{X_\beta}\omega - i_{X_\beta}\pounds_{X_\alpha}\omega \\ &= -\pounds_{X_\alpha} i_{X_\beta}\omega - i_{X_\beta} i_{X_\alpha} \underbrace{d\omega}_{=0} - i_{X_\beta} d i_{X_\alpha}\omega \\ &= -\pounds_{X_\alpha} i_{X_\beta}\omega + i_{X_\beta} \underbrace{dd\alpha}_{=0} \\ &= -d i_{X_\alpha} i_{X_\beta}\omega - i_{X_\alpha} d i_{X_\beta}\omega \\ &= -d i_{X_\alpha} i_{X_\beta}\omega + i_{X_\alpha} \underbrace{dd\beta}_{=0} \\ &= -d[\alpha, \beta]_2. \end{aligned}$$

□

Claim 2: Next, we claimed that $\pounds_{X_H}\alpha$ is exact if and only if $\pounds_{X_\alpha} H$ is exact. $H, \alpha \in \Omega^{n-1}_{Ham}(M)$ were arbitrary, so we only need to show one direction.

Proof. Suppose $\pounds_{X_H}\alpha$ is exact, i.e. equals $d\beta$ for some $\beta \in \Omega^{n-2}$. Then, using Cartan's magic formula, we have

$$\begin{aligned} d\beta = \pounds_{X_\alpha} H &= i_{X_\alpha} dH + d(i_{X_\alpha} H) \\ &= -i_{X_\alpha} i_{X_H}\omega + d(i_{X_\alpha} H) \\ &= i_{X_H} i_{X_\alpha}\omega + d(i_{X_\alpha} H) \\ &= -i_{X_H} d\alpha + d(i_{X_\alpha} H) \\ &= -\pounds_{X_H} d\alpha + d(i_{X_H}\alpha) + d(i_{X_\alpha} H), \end{aligned}$$

which shows that

$$\pounds_{X_H} d\alpha = d\Big(-\beta + i_{X_H}\alpha + i_{X_\alpha} H \Big)$$

is exact. □

Claim 3: Next, we claimed that for a multisymplectic vector field, i.e. v such that $\pounds_v \omega = 0$, $L_\infty(M, \omega) \cap C(v)$ is an $L_\infty-$subalgebra of $L_\infty(M, \omega)$, where $C(v)$ denotes globally conserved forms in the direction v.

Proof. We want to show that the bracket of $v-$globally conserved quantities is again $v-$globally conserved. In fact, we can prove the stronger statement: the bracket of $v-$locally conserved quantities is $v-$strictly conserved. Thus, we want to show

$$\pounds_v([\alpha_1, \ldots, \alpha_k]_k) = 0$$

for any $\alpha_i \in L$. The case $k = 1$ (where $[\alpha]_1 = d\alpha$ for deg $\alpha \neq 0$) is analogous to the previous proofs, so I'll leave that up to you. For the case $k > 1$, recall the $k-$ary bracket $[\ldots]_k$ is only non-trivial when α_i are all degree 0 i.e. in $\Omega^{n-1}_{Ham}(M)$. This means they each admit a Hamiltonian vector field X_{α_i}. Then,

$$\pounds_v([\alpha_1, \ldots, \alpha_k]_k) = (-1)^{k(k+1)/2} \pounds_v i_{X_{\alpha_1}} \cdots i_{X_{\alpha_k}}\omega.$$

Recall the Lie homotopy formula $i_{[u,v]} = \pounds_u i_v - i_v \pounds_u$. Now, we claim that $\pounds_v$ commutes with $i_{X_{\alpha_i}}$, when acting on ω. To see this, using the Lie homotopy formula, we have

$$\begin{aligned} i_{[v, X_{\alpha_i}]}\omega &= \pounds_v i_{X_{\alpha_i}}\omega - i_{X_{\alpha_i}} \underbrace{\pounds_v\omega}_{=0} \\ &= -\pounds_v d\alpha_i = -d(\pounds_v \alpha_i) = 0, \end{aligned}$$

where in the last line, we used that the Lie derivative commutes with the exterior derivative and that α_i is $v-$locally conserved (recall we weakened our assumptions), so acting on it by the exterior derivative gives zero. Thus, $\pounds_v$ commutes with $i_{X_{\alpha_i}}$ when acting on ω, so in the above expression,

$$\pounds_v([\alpha_1, \ldots, \alpha_k]_k) = (-1)^{k(k+1)/2} \pounds_v i_{X_{\alpha_1}} \cdots i_{X_{\alpha_k}}\omega,$$

we can commute $\pounds_v$ to the right, the extra commutator terms $i_{[v, X_{\alpha_i}]}$ can be moved to the right by total anti-symmetry of ω with contractions so they are annihilated. We are left with

$$(-1)^{k(k+1)/2} i_{X_{\alpha_1}} \cdots i_{X_{\alpha_k}} \underbrace{\pounds_v\omega}_{=0}.$$

□

Claim 4: Finally, we claimed that the homotopy comomentum map satisfies: $f_k(\xi)$ is locally conserved by X_H for any $\xi \in Z_k(\mathfrak{g})$, where H is globally conserved by the group action. Similarly, $f_k(\eta)$ is globally conserved by X_H for any $\eta \in B_k(\mathfrak{g})$.

***Proof*.** Suppose as above; $\xi = \xi_1 \wedge \cdots \wedge \xi_k \in Z_k(\mathfrak{g}), \eta = \eta_1 \wedge \cdots \wedge \eta_k \in B_k(\mathfrak{g})$, and H globally conserved by the group action. From the definition of (f), we have

$$-f_{k-1}(\partial\xi) = df_k(\xi) + (-1)^{k(k+1)/2} i_{\xi_1} \dots i_{\xi_k} \omega.$$

Instead of writing $(-1)^{k(k+1)/2}$, I'll just put $\pm$, since it will be zero anyway. We want to show that $\mathcal{L}_{X_H} f_k(\xi)$ is closed, i.e. $d\mathcal{L}_{X_H} f_k(\xi) = 0$. Since the exterior derivative commutes the Lie derivative, this is

$$\mathcal{L}_{X_H} df_k(\xi) = -\mathcal{L}_{X_H} f_{k-1}(\underbrace{\partial\xi}_{=0}) \pm \mathcal{L}_{X_H} i_{\xi_1} \cdots i_{\xi_k} \omega,$$

where we used that $\xi \in Z_k$ so $\partial\xi = 0$ (and (f) is an algebra morphism which implies $f_{k-1}(0) = 0$). Now, since the group action preserves H, we have $g^* H = H$. Since pullbacks commute with exterior derivatives, this gives $g^*(-i_{X_H}\omega) = g^* dH = dH = -i_{X_H}\omega$. Infinitesimally, this says $\mathcal{L}_\xi i_{X_H}\omega = 0$. By the Lie homotopy formula, we can write this as $i_{X_H}\mathcal{L}_\xi + i_{[X_H,\xi]} = 0$. Now, note $i_{[X_H,\xi]}\omega = 0$ (just use the definitions as we've been doing, since ξ preserves ω and H). Thus, we can perform the same maneuver as in the proof of Claim 3 to move $\mathcal{L}_\xi$ to the right where it annihilates ω, since the action is multisymplectic, $\mathcal{L}_\xi \omega = 0$.

The case of $f_k(\eta)$ globally conserved by X_H for $\eta \in B_k(\mathfrak{g})$ follows similarly. □

References

[1] T. Barron and B. Serajelahi. *Berezin-Toeplitz Quantization, Hyperkähler Manifolds, AND Multisympletic Manifolds*. Glasgow Mathematical Journal, 59(1), 167-187 (2017).

[2] M. de León, P. D. Prieto-Martínez, N. Román-Roy, and S. Vilariño. *Hamilton-Jacobi theory in multisymplectic classical field theories*. Journal of Mathematical Physics 58(9), (2017).

[3] M. Gotay, J. Isenberg, J. Marsden, and R. Montgomerey. *Momentum Maps and Classical Relativistic Fields. Part I: Covariant Field Theory*, preprint, 1998, arXiv:physics/9801019.

[4] X. Grácia, J. de Lucas, M. C. Muñoz, and S. Vilariño. *Multisymplectic structures and invariant tensors for Lie systems*. J. Phys. A: Math. Theor. 52 215201 (2019).

[5] J. Marsden and S. Shkoller. *Multisymplectic geometry, covariant Hamiltonians, and water waves*, Math Proc Cambridge Philos Soc 125(3), 553-575 (1999).

[6] M. Petrera and Y. Suris. *Variational symmetries and pluri-Lagrangian systems in classical mechanics*. Journal of Nonlinear Mathematical Physics 24, 121-145 (2017).

[7] L. Ryvkin, T. Wurzbacher, and M. Zambon. *Conserved Quantities on Multisymplectic Manifolds*. Journal of the Australian Mathematical Society, 108(1), 120-144 (2020).