A Brief Introduction to Monstrous Moonshine

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INTRODUCTION

This paper is mainly about the Monstrous Moonshine, which describes a weird relation by the monster and the elliptic modular function. This relation was first observed by John Mckay and he proposed this conjecture in 1978. It was proven by Richard Borcherds in 1992 using the no-ghost theorem from string theory and the theory of vertex operator algebras and generalized Kac–Moody algebras [10]. The first half part is about some conceptions that appear in the moonshine; and the second half part is a very brief outline of its proof.

We know that finite simple groups have been completely classified. Every finite simple group belongs to countable infinite families or is one of the 26 sporadic groups. The symmetry described by sporadic groups are quite different, so it took people several decades to figure all of them out. The monster, is the largest sporadic simple group, having order

$$\begin{aligned} &808,017,424,794,512,875,886,459,904,\\ &961,710,757,005,754,368,000,000,000\\ &= 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17\\ &\cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \end{aligned}$$

It is almost impossible to elaborate the detail structure of the monster. For example, we are used to start from the multiplication table when studying entry-level groups, but this doesn't work for the monster. Although there are two matrices that generates the monster, their sizes are incredibly large – they have 196883×196883 entries. The reason why multiplying two elements is difficult is not the huge order of the monster (for example, A_{100} has much more elements than the monster but multiplying 2 elements of A_{100} is rather easy), but the lack of small representations. The monster has 194 irreducible representations (discovered by Fischer, D. Livingstone and M. P. Thorne[1]), the smallest one except for the trivial representation has dimension 196883.

In Galois Theory, there is a function $j(\tau)$, called the elliptic modular function. The Laurent expansion of $j(\tau)$ is,

$$j(\tau) = q^{-1} + 196884q + 21493760q^2 + ... \quad (q = e^{2\pi i\tau})$$

John McKay found that the coefficient of the first order term 196884=196883+1, where 196883 is the dimension of the smallest irreducible representation of the monster. The "monstrous moonshine" refers to this strange relation between the structure of the monster and the coefficients of modular functions.

This paragraph is about the modular function $j(\tau)$. The group $SL_2(\mathbf{Z})$ acts on the upper half plane $H = \{\tau \in \mathbf{C}|Im(\tau) > 0\}$ by, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \frac{a\tau+b}{c\tau+d}$.If we let $a = c = 1$ and $d = 0$, we get a symmetric operation $\begin{pmatrix} 1 & b \\ 1 & 0 \end{pmatrix}(\tau) = \tau + b, b \in \mathbf{Z}$, which keeps $j(\tau)$ invariant. As a result, $j(\tau)$ is periodic and can be written as a series in $q = e^{2\pi i\tau}$. Apparently, the symmetry of $SL_2(\mathbf{Z})$ is a very strong constraint, and in fact, such a constraint is so strong that $j(\tau)$ is unique up to normalization. Its exact form is:

$$j(\tau) = \frac{(1 + 240\sum_{n>0}\sigma_3(n)q^n)^3}{q\prod_{n>0}(1-q^n)^{24}}$$

where $\sigma_3(n) = \sum_{d|n} d^3$ is the sum of the cubes of the divisors of n [2]. In general, we can replace this $SL_2(\mathbf{Z})$ with other groups, and we will get more complicated modular functions, which are called Hauptmoduls (fortunately we are not going to discuss them here). Another way of thinking about j is that it is an isomorphism from the quotient space $H/SL_2(\mathbf{Z})$ to the complex plane.

John Thompson found that not only the coefficient of the first order term, but also the next a few coefficients of the elliptic modular function are related with the dimension of the irreducible representations of the monster by some simple relations. He conjectured that there should exist a graded representation $V = \sum_{n\in\mathbf{Z}} V_n$ of the monster, such that the dimension of V_n is the coefficient $c(n)$ of q^n in $j(\tau)$. With this graded representation, we can construct a series which is called McKay-Thompson series $T_g(\tau) = \sum_{n\in\mathbf{Z}} Trace(g|V_n)q^n$, whose coefficients are given by the traces of elements g of the monster on the representation V_n. The question is whether such a graded representation really exists, and if so, whether it is constructable. Both answers are yes, the representation was constructed by I. B. Frenkel, J. Lepowsky, and A. Meurman using vertex operators[3, 4]. There are some more information about vertex algebras[5].

Now we can state the "monstrous moonshine" again, in a more rigorous way.

Theorem 1: Suppose that $V = \sum_{n\in\mathbf{Z}} V_n$ is the infinite dimensional graded representation of the monster constructed by Frenkel, Lepowsky and Meurman. Then for any element g of the monster the Thompson series $T_g(\tau) = \sum_{n\in\mathbf{Z}} Trace(g|V_n)q^n$ is a Hauptmodul for a genus 0 subgroup of $SL_2(\mathbf{Z})$. V satisfies the main conjecture in Conway and Norton's paper [8].

The symmetry described by a group is usually indicated by the algebraic structure it preserves on a vector space. For example, orthogonal groups preserve the Kronecher symbol. The monster module constructed by Frenkel-Lepowsky-Meurman has a vertex operator algebra structure invariant under the action of the monster. The definition of vertex operators is given in the appendix A.

The above part is explaining what the moonshine is, with nothing about its proof. If one wants to prove the theorem above, explicitly calculating the Thompson series using the monster Lie algebra M is a feasible way. M is a $Z^2 = Z \oplus Z$ graded Lie algebra, whose piece of degree $(m,n) \in^2$ has dimension c(mn) when both m and n are non-zero. The monster Lie algebra is a generalized Kac-Moody algebra[6,7] (See definition of the generalized Kac-Moody algebra in Appendix B), and Kac-Moody algebras have a denominator formula which says that a product over positive roots is equal to a sum over the Weyl group. The denominator formula for the monster Lie algebra is:

$$j(\sigma) - j(\tau) = p^{-1} \prod_{m>0, n\in\mathbf{Z}} (1 - p^m q^n)^{c(mn)}$$

Where $p = e^{2\pi i\sigma}$ and $q = e^{2\pi i\tau}$[6]. There are similar identities with $j(\tau)$ replaced by the McKay-Thompson series of any element of the monster, which look like:

$$p^{-1}exp(-\sum_{i>0}\sum_{m>-,n\in\mathbf{Z}} Tr(g^i|V_m n)p^{mi}q^{ni}/i)$$
$$= \sum_{m\in\mathbf{Z}} Tr(g|V_m)p^m - \sum_{n\in\mathbf{Z}} Tr(g|V_n)p^n$$

These relations between the coefficients $Tr(g|V_n)$ of the Thompson series are strong enough to determine them from their first few coefficients. If we compare the coefficients of p^2 and p^4 of BHS, we find recursion formulas of $c_g(n) = Tr(g|V_n)$. They look like:

$$c_g(4k) = c_g(2k+1) + (c_g(k)^2 - c_{g^2}(k))/2$$
$$+\textstyle\sum_{1\le j<k} c_g(j)c_g(2k-j)$$

$$c_g(4k+1) = c_g(2k+3) - c_g(2)c_g(2k)$$
$$+(c_g(k)^2 + c_{g^2}(k))/2$$
$$+(c_g(k+1)^2 - c_{g^2}(k+1))/2$$
$$+\textstyle\sum_{1\le j<k} c_g(j)c_g(2k-j+2)$$

$$c_g(4k+2) = c_g(2k+2) + \textstyle\sum_{1\le j\le k} c_g(j)c_g(2k-j+1)$$

$$c_g(4k+3) = c_g(2k+4) - c_g(2)c_g(2k+1)$$
$$-(c_g(2k+1)^2 - c_{g^2}(2k+1))/2$$
$$+\textstyle\sum_{1\le j\le k+1} c_g(j)c_g(2k-j+3)$$
$$+\textstyle\sum_{1\le j\le k} c_{g^2}(j)c_g(4k-4j+2)$$
$$+\textstyle\sum_{1\le j\le 2k} (-1)^j c_g(j)c_g(4k-j+2)$$

where $c_g(n) = Tr(g|V_n)$, $c_{g^2}(n) = Tr(g^2|V_n)$. If $n = 4$ or $n > 5$, the coefficients $c_g(n)$ is completely determined by coefficients $c_g(i)$ and $c_{g^2}(i)$ for $1 \le i < n$. If we know the coefficient $c_g(n) = Tr(g|V_n)$ for $n = 1, 2, 3$ and 5, all the coefficients of the Thompson-Mckay series are determined. The coefficient $c_g(5)$ is not determined,because when you try to plug in $k = 1$ into the second equation, it gives $c_g(5) = c_g(5)$.

The interesting thing is, Hauptmoduls suggested by Conway and Norton[8] satisfy the same identities and have the same first five coefficients. How to verify this? It was proved by Koike[9] that the recursion functions for modular functions are the same as Thompson series. As for the first few coefficients, We can check them case by case. Although they are no quite easy to work out, fortunately we just need to know first five of them. As a result, the Thompson series $T_g(\tau) = \sum_{n\in\mathbf{Z}} Trace(g|V_n)q^n$is indeed a Hauptmodul, as is claimed by Theorem 1.

Reference

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Appendix A: Definition of vertex algebras

A vertex algebra over the real numbers is a vector space V over the real numbers with an infinite number of bilinear products, written $u_n v$ for $u, v \in V$, $n \in Z$, such that

(1) $u_n v = 0$ for n sufficiently large

(2) $\sum_{i\in Z} \binom{m}{i} (\mathrm{u}_{q+i}v)_{m+n-i}\omega$

$=\sum_{i\in Z}(-1)^i \binom{q}{i} (\mathrm{u}_{m+q-i}(v_{n+i}\omega)$

$-(-1)^q v_{n+q-i}(u_{m+i}\omega))$

for all u,v, and w in V and all integers m, n and q.

(3) There is an element $1 \in V$ such that $v_n 1 = 0$ if $n \geq 0$ and $v_1 1 = v$.

Appendix B: Definition of the generalized Kac-Moody algebra

A Lie algebra G is defined to be a generalized Kac-Moody algebra if it has an almost positive definite contravariant bilinear form, which means that G has the following three properties.

(1) G can be Z-graded as $G = \oplus_{i\in Z} G_i$and G_i is finite dimensional if $i \neq 0$

(2) G has an involution ω which maps G_i into G_{-i} and acts as 1 on G_0, so in particular G_0 is abelian.

(3) G has an invariant bilinear form (,) invariant under ω such that G_i and G_j are orthogonal if $i \neq j$, and such that $(g, \omega(g)) > 0$ if g is a nonzero homogeneous element of G of nonzero degree.