Strong coupling expansion of lattice gauge theory using characters

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In this paper, we will talk about the strong coupling expansion, especially in the case of the pure SU(N) lattice gauge theory with Wilson's action.

INTRODUCTION

Perturbation theory can be applied to gauge theory in the continuum as well as on a lattice. We can use perturbative calculations in discussing the continuum limit of lattice gauge theory. However, several interesting aspects are inaccessible to a perturbative treatment. So it requires non-perturbative methods. Strong-coupling expansion is one of these which amount to expansions in powers of the inverse coupling.

STRONG-COUPLING GRAPHS

In the following we consider the pure SU(N) lattice gauge theory with Wilson's action:

$$S = \sum_p S_p(U_p) \tag{0.1}$$

$$S_p(U_p) = -\frac{\beta}{2N}\{\mathrm{Tr}\,U + \mathrm{Tr}\,U^\dagger - 2N\} \tag{0.2}$$

Here, p is the smallest closed loops on the lattice, *plaquettes* with four links. Actually we can define gauge invariant actions in other ways, but Wilson's action appears to be the simplest one. Consider a system with high temperature, i.e. $\beta \to 0$, we know that $e^{-S_p(U)} \to 1$. We can write

$$\begin{aligned} e^{-S} &= \prod_p e^{-S_p(U)} \\ &= \prod_p (1 + f_p) \end{aligned} \tag{0.3}$$

where f_p is a correction term, which vanishes when β goes to zero. Using the character expansion and factoring out the trivial character, we can write

$$e^{-S_p(U_p)} = c_0(\beta)\{1 + \sum_{r\neq 0} d_r a_r(\beta)\chi_r(U_p)\} \tag{0.4}$$

d_r is the dimension of the representation r, a_r is the coefficient. We obtain

$$e^{-S} = c_0(\beta)^{6\Omega} \prod_p \{1 + \sum_{r\neq 0} d_r a_r(\beta)\chi_r(U_p)\} \tag{0.5}$$

where Ω is the lattice volume. We can expand this product and get a sum of terms. We will define a graph $\mathcal{G}$ for each term. First, we notice that these terms have the form

$$d_{r1}a_{r1}\chi_{r1}(U_{p1}) \cdot d_{r2}a_{r2}\chi_{r2}(U_{p2}) \ldots \tag{0.6}$$

The graph $\mathcal{G}$ maps each plaquette p to a representation r_p according to the form of these terms. If some plaquettes don't occur in this term, they are associated to the trivial representation $r = 0$. Different plaquettes can be associated to the same representation r. Now we can write the partition function

$$\begin{aligned} Z &= \int \prod_b dU(b) e^{-S} \\ &= c_0^{6\Omega} \sum_{\mathcal{G}} \Phi(\mathcal{G}) \end{aligned} \tag{0.7}$$

where

$$\Phi(\mathcal{G}) = \int \prod_b dU(b) \prod_{p\in\mathcal{G}} d_{r_p} a_{r_p} \chi_{r_p}(U_p) \tag{0.8}$$

$\Phi(\mathcal{G})$ is called the contribution of the graph $\mathcal{G}$.
The simplest example is a cube, with six plaquettes. All of these plaquettes have the fundamental representation f of SU(N). We can write the $\Phi(\mathcal{G})$

$$\Phi(\mathcal{G}) = a_f(\beta)^6 \int \prod dU(b) \prod_{i=1}^{6} [d_f \mathrm{Tr}(U_{p_i})] \tag{0.9}$$

According to the integration rule that

$$\int dU\, \mathrm{Tr}(UV_1)\, \mathrm{Tr}(U^{-1}V_2) = \mathrm{Tr}(V_1 V_2) \tag{0.10}$$

That means the integration is equivalent to the value on the surface. So the contribution of the cube is equivalent to that of one plaquette.

$$\int \prod_{b\in p_1} dU(b) d_f \mathrm{Tr} U_{p_1} d_f \mathrm{Tr} U_{p_1}^\dagger = d_f^2 \tag{0.11}$$

So the result for the cube is

$$\Phi(\mathcal{G}) = d_f^2 a_f(\beta)^6 \tag{0.12}$$

Consider more complicated situations. Since we have

$$\int dU \chi_r(U) = \delta_{r,0} \tag{0.13}$$

If we want the integral below not vanishing, the Kronecker product $r_1 \otimes r_2 \otimes \cdots \otimes r_n$ should contain the trivial representation. We can get the selection rule:
1. The support $\|\mathcal{G}\|$ (the support of graph means the plaquettes which are associated with nontrivial representations) is a closed surface
2. The Kronecker product of the occurring representations must contain the trivial representation.
Otherwise, the contribution of the graph $\mathcal{G}$ vanishes.
For the graphs which consist of several disjoint components:

$$|\mathcal{G}| = |X_1| \cup \cdots \cup |X_n| \tag{0.14}$$

with $|X_i| \cap |X_j| = 0$, for $i \neq j$. In this case the contribution is

$$\Phi(\mathcal{G}) = \prod_i \Phi(X_i) \tag{0.15}$$

Given an arbitrary graph, we can decompose it into connected parts (polymers) X_i.

CLUSTER EXPANSION FOR THE FREE ENERGY

First, we need the concept of moments and cumulants. Let I be a set. Then the moment is a sequence of symmetric, real functions over I

$$< \alpha, \ldots, \beta > \in \mathbf{R} \tag{0.16}$$

The cumulant of the moment $<>$ satifies

$$< \alpha, \ldots, \zeta > = \sum_P [\alpha, \ldots, \beta][\gamma, \ldots, \delta] \ldots [\mu, \ldots, \nu] \tag{0.17}$$

where P goes over all partitions. Now we proceed to the graphical expansion of the free energy. The partition function in the terms of polymers is

$$Z = c_0^{6\Omega}\{1 + \sum_{n=1}^{\infty} \sum_{X_1,\ldots,X_n} \frac{1}{n!} \Phi(X_1) \cdot \cdots \cdot \Phi(X_n)\} \tag{0.18}$$

We can define the following moment:

$$< X_1, \ldots, X_n > = 1, \text{if every pair} X_i, X_j \text{is disconnected}$$

The expression of Z is

$$Z = c_0^{6\Omega}\{1 + \sum_{n=1}^{\infty} \sum_{X_1,\ldots,X_n} \frac{1}{n!} < X_1, \ldots, X_n > \Phi(X_1) \cdot \cdots \cdot \Phi(X_n)\} \tag{0.19}$$

Using the main theorem of the moment-cumulant formalism, we can write the free energy in the form

$$\begin{aligned} F &= -\frac{1}{\Omega} \ln Z \\ &= -6\ln c_0 - \frac{1}{\Omega} (\sum_{n=1}^{\infty} \sum_{X_1,\ldots,X_n} \frac{1}{n!} [X_1, \ldots, X_n] \\ &\Phi(X_1) \cdot \cdots \cdot \Phi(X_n)) \end{aligned} \tag{0.20}$$

We define a cluster of C to be a connected collection of polymers. If a cluster C contains polymers X_i with possible multiplications n_i, we write

$$C = (X_1^{n_1}, X_2^{n_2}, \ldots) \tag{0.21}$$

Then the free energy expansion can be written in terms of clusters

$$F = -6\ln c_0 - \frac{1}{\Omega} \sum_{C=(X_1^{n_1},\ldots,X_k^{n_k})} a(C) \Phi(X_1)^{n_1} \cdot \cdots \cdot \Phi(X_k)^{n_k} \tag{0.22}$$

This is the cluster expansion for the free energy. In the presence of translation invariance, such as a finite lattice with periodic boundary conditions, we can simplified it as

$$F = -6\ln c_0 - \sum_{C=(X_i^{n_i})} a(C) \prod_i \Phi(X_i)^{n_i} \tag{0.23}$$

Now we can see the example of SU(2) lattice gauge theory expanded up to $\mathcal{O}(\beta^1 2)$:
1. The single polymers like cube and double-cube has the combinatorial factor $a(C) = 1$.
2. Clusters consisting of two distinct polymers:
a) X_1 and X_2 are both cubes but in different places, we have $C = (X_1, X_2)$ and $a(C) = 1$. The contribution is

$$-(4a_{1/2}(\beta)^6)^2 \tag{0.24}$$

b) X_1 and X_2 are the same cube, then $C = (X_1^2)$, $a(C) = -\frac{1}{2!}$, and the contribution is

$$-\frac{1}{2}(4a_{1/2}(\beta)^6)^2 \tag{0.25}$$

Finally, we will get the expansion of F in terms of the coefficients a_j:

$$F = -6\ln c_0 - \sum a_{1/2}^k a_1^l a_{3/2}^m \cdots \tag{0.26}$$

This can be reexpanded in powers of β for numerical purposes.

CONCLUSION

In contrast to perturbation theory, the strong-coupling expansion has a finite range of convergence. It is possible to prove various properties of the theory in a rigorous way by the strong-coupling expansion, such as Existence of a mass gap and Static quark confinement.

[1] Istvan Montvay, and Gernot Munster. *Quantum Field on a Lattice*. Cambridge Press, 1993.