Generalization of Weyl Integral Formula and Abelianization of 2d Yang-Mills Theory

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We briefly review how to generalize Weyl integral formula to functional integral and show that the generalized formula can be used to compute partition function of 2d Yang-Mills theory on closed surface.

INTRODUCTION

In group theory, we can simplify the integral of a conjugation invariant function over a compact Lie group G by restricting the domain of integral to T which is a maximal torus of G:

$$\int_G dg f(g) = \int_T dt \; det\Delta_W(t) f(t) \tag{1}$$

where Δ_w(t) is the Weyl determinant. This relies on the fact that every element of G is conjugate to some element of T(in other words every $g \in$ G can be 'diagonalized' in some basis). Since T is Abelian, this formula seems to imply that an integral of a function with a non-Abelian (conjugation) symmetry can be reduced to an integral of a function with an Abelian symmetry.

GENERALIZATION OF WEYL INTEGRAL FORMULA

One natural question is how can we generalize (1) to functional integral so that we can simplify the path integral of gauge theory. So we first need to answer if a smooth map g $\in$ Map(M,G) can be written as

$$g(x) = h(x)t(x)h(x)^{-1} \tag{2}$$

for t$\in$ Map (M,T) and h $\in$ Map(M,G). We can always do this pointwise but there are topological obstructions to achieving this globally.
As an example consider a map from $S^2 \rightarrow$ SU(2). We can parametrize elements of SU(2) as

$$x_4\mathbf{1} + \sum_{k=1}^{3} x_k\sigma_k = \begin{pmatrix} x_4 + ix_3 & x_1 + ix_2 \\ -x_1 + ix_2 & x_4 - ix_3 \end{pmatrix} \tag{3}$$

Here we are using the convention that Tr$\sigma_k\sigma_l$=-2δ_{kl}. The winding number of this map n(g)=1.
Now suppose we can smoothly deform map g into U(1) via some h, h^{-1}gh=t. As the space of maps from S^2 to SU(2) is connected, g is homotopic to t and one has n(g)= n(f). But, since g^2=-$\mathbf{1}$, t is a constant map so that n(t)=0. This contradiction shows that there can be no smooth or continuous h satisfying h^{-1}gh=t.

In fact one can further consider generalization n(f,A) of n(f),

$$n(f,A) = -\frac{1}{32\pi}\int_{S^2} trf[df,df] - \frac{1}{2\pi}\int_{S^2} tr[d(fA)] \tag{4}$$

depending on both f and an SU(2) connection A. It turns out that the first Chern class of the U(1) component of the gauge field A^h is equal to the winding number of the original map. In this case it is just the pull-back of the U(1) bundle SU(2)$\rightarrow$ $SU(2)/U(1) \sim S^2$ via g and this turns out to be more or less what happens in general.
The correct generalization of (1) is

$$Z[P_G] = \sum_{l\in[P_T;P_G]} \int_{A[l]} d[A^t] \int_{B[l]} d[A^k] \int d[\phi^t] \Delta_w[t] e^{iS[\phi^t,A^t,A^k]} \tag{5}$$

where we denote the space of connections on P_G and on a principle T bundle P_T^l representing an element l$\in$ $[P_T;P_G]$ by $\mathcal{A}$ and A[l] respectively and the space of one-form with values in the section of $P_T^l \times T$ k by $\mathcal{B}$[l].

ABELIANIZATION OF 2D YANG-MILLS THEORY

Now we can use the generalized Weyl integral formula to study Yang-Mills theory on a 2-d closed surface Σ_g with genus g.

$$Z_{\Sigma_g}(\epsilon) = \int_{\mathcal{A}} [dA] e^{\frac{1}{2\epsilon}\int_{\Sigma_g} trF_A\star F_A} \tag{6}$$

where ϵ represents the coupling constant of the theory.
We can rewrite this integral by introducing a $\mathbf{g}$-valued scalar field $\phi \in$ Map$(\Sigma_g, \mathbf{g})$, so that the partition function becomes

$$Z_{\Sigma_g}(\epsilon) = \int_{\mathcal{A}} [dA] \int_{Map(\Sigma_g,\mathbf{g})} d[\phi] e^{\int_{\Sigma_g} tr[i\phi F_A + \frac{\epsilon}{2}\phi\star\phi]} \tag{7}$$

Since the action is invariant under gauge transformation

$$S[g^{-1}\phi g, A^g] = S[\phi, A] \tag{8}$$

we can proceed by using the gauge freedom to conjugate ϕ into Map(Σ_g,$\mathbf{t}$). The reduced action is

$$S[\phi^t, A^t, A^k] = tr\int_{\Sigma_g} \phi^t dA^t + \frac{1}{2}[\phi^t, A^k]A^k + \frac{\epsilon}{2}\phi^t \star \phi^t \tag{9}$$

We can integrate out A^k since the action is quadratic in A^k

$$\int d[A^k] \Rightarrow det^{-1/2} ad(\phi^t)|_{\Omega^1(\Sigma_g,k)} \qquad (10)$$

This almost cancels against the Weyl determinant(contribution from ghost)as one-form in 2d has as many degrees of freedom as two scalars. The zero modes surplus is one constant scalar mode minus g harmonic one-form modes so that the combined determinant would simply reduce to a finite dimensional determinant,

$$\frac{det\ ad(\phi^t)|_{\Omega^0(\Sigma_g,k)}}{det^{1/2} ad(\phi^t)|_{\Omega^1(\Sigma_g,k)}} = (det\ ad(\phi^t)|_k)^{\chi(\Sigma_g)/2} \qquad (11)$$

We have assumed that only constant modes of τ contribute to the path integral.
Putting the above together, we are left with an Abelian functional integral over ϕ^t and A^t and a sum over the topological sectors,

$$Z_{\Sigma_g}(\epsilon) = \sum_{l\in[P_T;P_G]} \int_{\mathcal{A}[l]} d[A^t](det\ ad(\phi^t)|_k)^{\chi(\Sigma_g)/2} e^{\int_{\Sigma_g} tr(i\phi^t dA^t + \frac{\epsilon}{2}\phi^t \star \phi^t)} \qquad (12)$$

We can expand the gauge A^t and ϕ^tas

$$A^t = i\alpha_l A^l, \phi^t = i\phi_l \lambda^l \qquad (13)$$

where λ^t is a basis of fundamental weights.
A^t are connections on torus bundles over Riemann surface Σ_g and such bundles are completely classified by their first chern class. That is for a given torus bundle, we have a set of integers $n^l, l = 1, ..., r$ with

$$\int_{\Sigma_g} F^l_A = 2\pi n^l \qquad (14)$$

In order to do the integral, we can split the gauge field A^t into a classical A_c and a quantum parA^q. The quantum part is a torus valued one-form while the classical piece may be taken to satisfy

$$dA^l_c = 2\pi n^l \omega \qquad (15)$$

which clear obeys (14).
The path integral over the torus gauge field can be performed and yield a constraint

$$d\phi^l = 0 \qquad (16)$$

This implies ϕ^t must be space-time independent and the final integral reduces to a finite integral.
Putting everything together, the complete integral is

$$\prod_{l=1}^{r}\sum_{n_l}\int d\phi^l\ det_k(ad(\phi^t))^{\chi(\Sigma_g)/2} exp(-i\frac{\phi^l n_l}{2\pi} - \frac{\epsilon\phi^2}{16\pi^2}) \qquad (17)$$

The sum over n_l yields a periodic delta function on ϕ

$$\prod_{l=1}^{r}\sum_{n_l} exp(i\frac{\phi^l n_l}{2\pi}) = \prod_{l=1}^{r}\sum_{n_l}\delta(\phi^l - 4\pi^2 n^l) \qquad (18)$$

Substituting this expression into (17) gives us

$$\prod_{l=1}^{r}\sum_{n_l}\int d\phi^l\ det_k(ad(\phi^t))^{\chi(\Sigma_g)/2} exp(-\epsilon\pi^2 n_l^2) \qquad (19)$$

The sum over Chern classes may be thought of as a sum over the weight lattice. That is, one sets $\lambda = \sum_l n_l \lambda^l$ with the weight lattice given by

$$\Lambda = Z[\lambda^1, ..., \lambda^r] \qquad (20)$$

. In this way we obtain,

$$\sum_\lambda \prod_\alpha < \alpha, \lambda >^{\chi(\Sigma_g)/2} exp(-2\pi^2\epsilon < \lambda, \lambda >) \qquad (21)$$

If we shift the weight λ by the Weyl vector ρ we obtain

$$\sum_\lambda \prod_\alpha < \alpha, \lambda + \rho >^{\chi(\Sigma_g)/2} exp(-2\pi^2\epsilon < \lambda + \rho, \lambda + \rho >) \qquad (22)$$

We can further factor out the action of Weyl group by a sum over highest weights and the final formula for the partition function is given by

$$Z_{\Sigma_g} = \sum_\lambda d(\lambda)^{\chi(\Sigma_g)} exp(-2\pi^2\epsilon < \lambda + \rho, \lambda + \rho >) \qquad (23)$$

where $d(\lambda)$ is the dimension of the irreducible representation labelled by the weights λ,

$$d(\lambda) = \prod_{\alpha>0} < \alpha, \lambda + \rho > / \prod_{\alpha>0} < \alpha, \rho > \qquad (24)$$

This result can be compared with the one obtained by using cutting and pasting techniques and the difference between them can be absorbed by a redefinition of counter terms.

CONCLUSION

We briefly reviewed how to generalize Weyl formula and how to apply it to study 2d Yang-Mills theory. In particular we checked that this method succeeds in reproducing previous result obtained by cutting and pasting techniques. And this should be suffice to prove the correctness of this method.

REFERENCE

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