University of California at San Diego – Department of Physics – Prof. John McGreevy

# Physics 215A QFT Fall 2021
# Assignment 5

**Due 11:59pm Thursday, October 28, 2021**

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1. **Brain-warmer: the identity does nothing twice.** Check our relativistic state normalization by squaring the expression for the identity in the 1-particle sector:

$$\mathbb{1}_1^2 \stackrel{!}{=} \mathbb{1}_1 = \int \frac{đ^d p}{2\omega_{\vec p}} |\vec p\rangle \langle \vec p| .$$

2. **Even more about QFT in $0+0$ dimensions.**

   In this problem we return to the simplest scalar QFT, namely a one-dimensional integral with quartic action.

   (a) By a change of integration variable show that

$$Z = \int_{-\infty}^{\infty} dq\; e^{-S(q)}$$

   with $S(q) = \frac{1}{2}m^2q^2 + gq^4$ is of the form

$$Z = \frac{1}{g^{1/4}} \mathcal{Z}\left(m^2/\sqrt{g}\right) .$$

   This means you can make your life easier by setting $g = 1$, without loss of generality.

   (b) Convince yourself (*e.g.* with Mathematica) that the integral really is expressible as a Bessel function.

   (c) It would be nice to find a better understanding for why the partition function of $(0+0)$-dimensional $\phi^4$ theory is a Bessel function. Then find a Schwinger-Dyson equation for this system which has the form of Bessel's equation for

$$K(x^2/a) \equiv e^{-x^2/a}(x^2)^{-1/4}\mathcal{Z}(x)$$

   for some constant $a$. (If you get stuck I can tell you what function to choose for the 'anything' in the S-D equation.)

   (d) Make a plot of the perturbative approximations to the 'Green function' $G \equiv \langle q^2 \rangle$ as a function of $g$, truncated at orders 1 through 6 or so. Plot them against the exact answer.

(e) (Bonus problem) Show that $c_{n+1} \sim -\frac{2}{3} n c_n$ at large $n$ (by brute force or by cleverness).

3. **The vacuum is a fluid with $p = -\rho$.** [Bonus problem]

We said in lecture that the vacuum energy density $\rho$ gravitates and that, when positive, its effect is to cause space to inflate – to expand exponentially in time. An important aspect of this phenomenon is that the vacuum fluctuations produce not only an energy density, but a *pressure*, $p = T^i_i$ (no sum on $i$), of the form $p = -\rho$, which is negative for $\rho > 0$. The vacuum therefore acts as a perfect fluid with $P = -\rho$. (The stress tensor for a perfect fluid in terms of its velocity field $u^\mu$ takes the form $T^{\mu\nu} = (p+\rho)u^\mu u^\nu + p g^{\mu\nu}$, so in a frame with $u^\mu = (1, \vec{0}^\mu)$, $T^0_0 = \rho, T^i_i = P$.) Solving Einstein's equations with such a source produces an inflating universe. In this problem we show that this is is the case.

(a) Show that the energy-momentum tensor for a free relativistic scalar field ($S[\phi] = \int d^D x \sqrt{g} \mathcal{L}, \mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{m^2}{2}\phi^2$) takes the form

$$T_{\mu\nu} = a \partial_\mu \phi \partial_\nu \phi - b g_{\mu\nu} \mathcal{L}$$

with some constants $a, b$.

(b) Reproduce the formal expression for the vacuum energy

$$\langle 0 | \mathbf{H} | 0 \rangle = V \int đ^d k \frac{1}{2} \hbar \omega_{\vec{k}}$$

using the two point function

$$\langle 0 | \phi(x)^2 | 0 \rangle = \langle 0 | \phi(0)\phi(0) | 0 \rangle = \lim_{\vec{x}, t \to 0} \langle 0 | \phi(x)\phi(0) | 0 \rangle$$

and its derivatives. ($V$ is the volume of space.)

(We will learn to draw this amplitude as a Feynman diagram which is a circle (a line connecting a point to itself).)

(c) Show that the vacuum expectation value of the pressure

$$\langle 0 | T_{ii} | 0 \rangle$$

(no sum on $i$) gives the same answer, up to a sign.

[Hints: You'll find a quite different looking integral from the vacuum energy. Use rotation invariance of the vacuum to simplify the answer. The claim is that however you regulate the integral vacuum pressure and $\frac{1}{2} \int đ^d k \omega_k$, you'll get the same answer. A convenient regulator is *dimensional regularization*: treat the dimension $d$ as an arbitrary complex number.]

(d) Show that the resulting vacuum energy momentum tensor ($T_{00} = \rho, T_{ii} = -\rho$ (no sum on $i$))is the same as the contribution to the energy-momentum tensor from an action of the form

$$S_{\text{cc}} = \int d^D x \sqrt{g} \Lambda$$

where $\Lambda$ is a constant (the cosmological constant).

(e) Argue that $p = -\rho$ is required in order that the vacuum energy does not specify a preferred rest frame.

4. **Non-Abelian currents.** On a previous homework, we studied a complex scalar field. Now, we make a big leap to *two* complex scalar fields, $\Phi_{\alpha=1,2}$, with

$$S[\Phi_\alpha] = \int d^d x dt \left( \frac{1}{2} \partial_\mu \Phi^\star_\alpha \partial^\mu \Phi_\alpha - V\left(\Phi^\star_\alpha \Phi_\alpha\right) \right)$$

Consider the objects

$$Q^i \equiv \frac{1}{2} \int d^d x \mathbf{i} \left( \Pi^\dagger_\alpha \sigma^i_{\alpha\beta} \Phi^\dagger_\beta \right) + h.c.$$

where $\sigma^{i=1,2,3}$ are the three Pauli matrices.

(a) What symmetries do these charges generate (*i.e.* how do the fields transform)? Show that they are symmetries of $S$.

(b) If you want to, show that $[Q^i, H] = 0$, where $H$ is the Hamiltonian.

(c) Evaluate $[Q^i, Q^j]$. Hence, non-Abelian.

(d) To complete the circle, find the Noether currents $J^i_\mu$ associated to the symmetry transformations you found in part 4a.

(e) Generalize to the case of $N$ scalar fields.

5. **Combinatorics from 0-dimensional QFT.** [This is a bonus problem. I will not post the solutions of this problem until later. If you have a hard time with it now, please try again in a week.]

Catalan numbers $C_n = \frac{(2n)!}{n!(n+1)!}$ arise as the answer to many combinatorics problems (beware: there is some disagreement about whether this is $C_n$ or $C_{n+1}$).

One such problem is: count random walks on a 1d chain with $2n$ steps which start at 0 and end at 0 without crossing 0 in between.

Another such problem is: in how many ways can $2n$ (distinguishable) points on a circle be connected by chords which do not intersect within the circle.

Consider a zero-dimensional QFT with the following Feynman rules:

- There are two fields $h$ and $l$.
- There is an $\sqrt{t}h^2 l$ vertex in terms of a coupling $t$.
- The bare $l$ propagator is 1.
- The bare $h$ propagator is 1.
- All diagrams can be drawn on a piece of paper without crossing.[1]
- There are no loops of $h$.

The last two rules can be realized from a lagrangian by introducing a large $N$ (below).

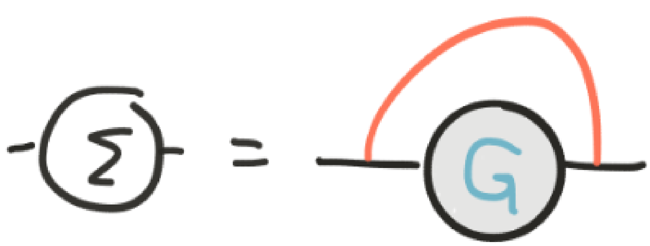
(a) Show that the full two-point green's function for $h$ is

$$G(t) = \sum_n t^n C_n$$

the generating function of Catalan numbers.

(b) Let $\Sigma(t)$ be the sum of diagrams with two $h$ lines sticking out which may not be divided into two parts by cutting a single intermediate line. (This property is called 1PI (one-particle irreducible), and $\Sigma$ is called the "1PI self-energy of $h$". We'll use this manipulation all the time later on.) Show that $G(t) = \frac{1}{1-\Sigma(t)}$.

(c) Argue by diagrams for the equation (sometimes this is also called a Schwinger-Dyson equation)




where $\Sigma$ is the 1PI self-energy of $h$.

[1]An annoying extra rule: All the $l$ propagators must be on one side of the $h$ propagators. You'll see in part 5f how to justify this.

(d) Solve this equation for the generating function $G(t)$.

(e) If you are feeling ambitious, add another coupling $N^{-1}$ which counts the crossings of the $l$ propagators. The resulting numbers can be called Touchard-Riordan numbers.

(f) How to realize the no-crossings rule? Consider

$$L = \frac{\sqrt{t}}{\sqrt{N}} l_{\alpha\beta} h_\alpha h_\beta + \sum_{\alpha,\beta} l_{\alpha\beta}^2 + \sum_\alpha h_\alpha^2$$

where $\alpha, \beta = 1 \cdots N$. By counting index loops, show that the dominant diagrams at large $N$ are the ones we kept above. Hint: to keep track of the factors of $N$, introduce ('t Hooft's) double-line notation: since $l$ is a matrix, its propagator looks like: $\begin{matrix} \alpha - - - - - - - - \alpha \\ \beta - - - - - - - - \beta \end{matrix}$, while the $h$ propagator is just one index line $\alpha\_\_\_\_\_\_\alpha$, and the vertex is ___!!___. If you don't like my ascii diagrams, here are the respective pictures: $\langle l_{\alpha\beta} l_{\alpha\beta} \rangle =$ 

,

$\langle h_\alpha h_\alpha \rangle =$ 

and the $hhl$ vertex is:  .

(g) Use properties of Catalan numbers to estimate the size of non-perturbative effects in this field theory.

(h) There are many other examples like this. Another similar one is the relationship between symmetric functions and homogeneous products. A more different one is the enumeration of planar graphs. For that, see BIPZ.