University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2021
Assignment 11 ("Final Exam")

Due 11:59pm Thursday, December 9, 2021

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1. **Brain-warmers.**

   (a) Check that
   $$(p \cdot \sigma)\,(p \cdot \bar{\sigma}) = p^2.$$

   (b) Use the previous part to show that if
   $$u_r(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi_r \\ \sqrt{p\cdot\bar{\sigma}}\,\xi_r \end{pmatrix} \quad \text{and} \quad v_r(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta_r \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta_r \end{pmatrix}$$
   with $p^2 = m^2$ (solutions of the Dirac equation with mass m), then
   $$\bar{u}_r(\vec{p})u_s(\vec{p}) = 2m\xi_r^\dagger \xi_s \quad \text{and} \quad \bar{v}_r(\vec{p})v_s(\vec{p}) = -2m\eta_r^\dagger \eta_s$$
   (where $\bar{u} \equiv u^\dagger \gamma^0$ as usual).

   (c) Show that $\bar{u}_r(\vec{p})v_s(\vec{p}) = 0$ and $u_r(\vec{p})^\dagger v_s(-\vec{p}) = 0$ but $u_r(\vec{p})^\dagger v_s(\vec{p}) \neq 0$.

2. **Other bases for gamma matrices.**

   Many different bases of gamma matrices are frequently used by humans. You may read on the internet someone telling you that the gamma matrices are
   $$\tilde{\gamma}^0 = \begin{pmatrix} 1\!\!1_{2\times 2} & 0 \\ 0 & -1\!\!1_{2\times 2} \end{pmatrix}, \quad \tilde{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$
   and think that I have lied to you. This basis is useful for studying the non-relativistic limit. The Weyl basis which we introduced in lecture instead makes manifest the reducibility of the Dirac spinor into L plus R Weyl spinors. Find the unitary matrix U which relates them $\tilde{\gamma}^\mu = U\gamma^\mu U^\dagger$.

3. **Symmetries of the Dirac lagrangian.**

   (a) Find the Noether currents j^μ and j_5^μ associated with the transformations $\Psi \to e^{-\mathbf{i}\alpha}\Psi$ and $\Psi \to e^{-\mathbf{i}\alpha\gamma^5}\Psi$ of a free Dirac field. Show by explicit calculation that the former is conserved and the latter is conserved if $m = 0$.

(b) Find the conserved currents associated with the Lorentz symmetry $\Psi \mapsto \Lambda_{\frac{1}{2}}(\theta, \beta)\Psi$ of the Dirac Lagrangian. Show that the conserved charge takes the form mentioned in lecture

$$J^{\mu\nu} = \int_{\text{space}} \left(\mathcal{J}^{\mu\nu}_{\text{orbital}} + \Psi^\dagger J^{\mu\nu}_{\text{Dirac}}\Psi\right)$$

where $\mathcal{J}^{\mu\nu}_{\text{orbital}}$ has the form it would have for a scalar field, and $J^{\mu\nu}_{\text{Dirac}} \equiv \frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu]$ are the matrices satisfying the Lorentz algebra.

Convince yourself that the latter matrix specifies how the current acts in the one-particle sector.

4. **Meson scattering.**

Consider the Yukawa theory with fermions, with $\mathcal{L}_{\text{int}} = -g\bar{\Psi}\Psi\phi$, where Ψ is a Dirac fermion field and ϕ is a real scalar field.

(a) Draw the Feynman diagram that gives the leading contribution to the scattering amplitude for the process $\phi\phi \to \phi\phi$.

(b) Derive the correct sign of the amplitude by considering the relevant matrix elements of powers of the interaction hamiltonian.

(c) Evaluate the diagram in terms of a spinor trace and a momentum integral. Do not do the momentum integral. Suppose that the integral is cutoff at large k by some cutoff Λ. Estimate the dependence on Λ.

5. **The magnetic moment of a Dirac fermion.**

In this problem we consider the hamiltonian density

$$\mathfrak{h}_I = q\bar{\Psi}\gamma^\mu\Psi A_\mu \ .$$

This describes a local, Lorentz invariant, and gauge invariant interaction between a Dirac fermion field Ψ and a vector potential A_μ. In this problem, we will treat the vector potential, representing the electromagnetic field, as a fixed, classical background field.

Define single-particle states of the Dirac field by $\langle 0|\Psi(x)|\vec{p}, s\rangle = e^{-\mathbf{i}px}u^s(p)$. We wish to show that these particles have a magnetic dipole moment, in the sense that in their rest frame, their (single-particle) hamiltonian has a term $h_{NR} \ni \mu_B\vec{S}\cdot\vec{B}$ where $\vec{S} = \frac{1}{2}\vec{\sigma}$ is the particle's spin operator.

(a) q is a real number. What is required of A_μ for $H_I = \int d^3x \mathfrak{h}_I$ to be hermitian?

(b) How must A_μ transform under parity P and charge conjugation C in order for H_I to be invariant? (To answer this, you'll have to find out how the spinor bilinear transforms, *e.g.* from Peskin.) How do the electric and magnetic fields transform? Show that this allows for a magnetic dipole moment but not an electric dipole moment.

(c) Show that in the non-relativistic limit

$$\bar{u}(p')\gamma^{\mu\nu}u'(p)F_{\mu\nu} = a\xi^\dagger\sigma\cdot\vec{B}\xi'$$

for some constant a (find a). Recall that $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$. Here u, u' are positive-energy solutions of the Dirac equation with mass m and

$$u \overset{NR}{\to} \sqrt{m}(\xi,\xi), u' \overset{NR}{\to} \sqrt{m}(\xi',\xi')$$

in the non-relativistic limit.

(d) Suppose that A_μ describes a magnetic field $\vec{B}$ which is uniform in space and time.

Show that in the non-relativistic limit

$$\langle\vec{p}', s'|\, H_I\, |\vec{p}, s\rangle = \not{\delta}^3\,(\vec{p}-\vec{p}')\, h(\xi,\xi',\vec{B})$$

and find the function $h(\xi,\xi',\vec{B})$. You may wish to use the Gordon identity. Rewrite the result in terms of single-particle states with non-relativistic normalization (*i.e.* $\langle\vec{p}|\vec{p}'\rangle_{NR} = \not{\delta}^3(p-p')$). Interpret h as a non-relativistic hamiltonian term saying that the gyromagnetic ratio of the electron is $-g\frac{|q|}{2m}$ with $g = 2$.

(e) [optional] How does the result change if we add the term

$$\Delta H = \frac{c}{M}\bar{\Psi}F_{\mu\nu}[\gamma^\mu,\gamma^\nu]\Psi \quad ?$$

6. **Non-relativistic interactions from QFT.**

(a) **Coulomb potential.**

Derive from QED that the force between non-relativistic electrons is a repulsive $1/r^2$ force law!

(b) **Pseudoscalar Yukawa theory.**

Consider the theory of a massive Dirac fermion Ψ and a massive pseudoscalar φ interacting via the term

$$V_5 \equiv g_5\bar{\Psi}\gamma^5\Psi\varphi.$$

Convince yourself that this theory is parity invariant (for some assignment of the action of parity on the fields).

List the Feynman rules.

Draw and evaluate the diagrams contributing to $\Psi\Psi \to \Psi\Psi$ scattering at leading order in g_5.

Consider the non-relativistic limit, $m \gg |\vec{p}|$ and find the effective interaction hamiltonian. If you happen to find zero for the leading term, then it's not the leading term.

7. **Supersymmetry.** [Bonus problem] A continuous symmetry that mixes bosons and fermions is called *supersymmetry*.

   (a) The simplest example of a supersymmetric field theory is the theory of a free complex boson and a free Weyl fermion, with Lagrangian is

   $$\mathcal{L} = \partial_\mu \phi^\star \partial^\mu \phi + \chi^\dagger \mathbf{i} \bar{\sigma}^\mu \partial_\mu \chi + F^\star F.$$

   Here F is an auxiliary field whose purpose is to make the supersymmetry transformations look nice. Show that the action is invariant under

   $$\delta\phi = -\mathbf{i}\epsilon^T \sigma^2 \chi, \delta\chi = \epsilon F + \sigma \cdot \partial\phi \sigma^2 \epsilon^\star, \delta F = -\mathbf{i}\epsilon^\dagger \bar{\sigma} \cdot \partial\chi \tag{1}$$

   where the symmetry parameter ϵ is a 2-component spinor of Grassmann numbers.

   (b) Show that the term

   $$\Delta\mathcal{L} = \left(m\phi F + \frac{1}{2}\mathbf{i} m \chi^T \sigma^2 \chi \right) + h.c.$$

   is also invariant under the transformation (1). Eliminate F from the full Lagrangian $\mathcal{L} + \Delta\mathcal{L}$ by solving its equations of motion, and show that the fermion and boson fields are given the same mass.

   (c) We can include supersymmetric interactions as well. Show that the following field theory is supersymmetric:

   $$\mathcal{L} = \partial_\mu \phi_i^\star \partial^\mu \phi^i + \chi_i^\dagger \mathbf{i} \bar{\sigma} \cdot \partial \chi_i + F_i^\star F_i + \left(F_i \partial_{\phi_i} W + \frac{\mathbf{i}}{2} \partial_{\phi_i} \partial_{\phi_j} W \chi_i^T \sigma^2 \chi_j + h.c. \right)$$

   where $i = 1..n$ and $W = W(\phi)$ is an arbitrary function of the ϕ_i, called the *superpotential*. For the simple case $n = 1$ and $W = g\phi^3/3$ write out the field equations for ϕ and χ after eliminating F.