# Recap / generalization

$$H = \sum_{n=1}^{N} \frac{p_n^2}{2m_n} + \frac{1}{2} K_{nm} q_n q_m$$

$$\begin{cases} \text{only } \dfrac{K_{nm}+K_{mn}}{2} \text{ appears} \quad (\Leftarrow [q_n, q_m]=0) \\ K_{nm} = K^*_{nm} \Rightarrow K = K^\dagger . \end{cases}$$

simplify: $Q_n \equiv \sqrt{m_n}\, q_n \quad P_n \equiv \dfrac{p_n}{\sqrt{m_n}} \quad V_{mn} \equiv \dfrac{K_{mn}}{\sqrt{m_n m_m}}$

$$\Rightarrow H = \sum_n \frac{P_n^2}{2} + Q_n V_{nm} Q_m$$

$Q^\dagger = Q$, $P^\dagger = P$

$$= \sum_n \frac{P_n^\dagger P_n}{2} + Q_n^\dagger V_{nm} Q_m \quad \underline{\geq 0}.$$

$$[q_n, p_m] = i\delta_{nm} \Rightarrow [Q_n, P_m] = i\delta_{nm}$$

$V = V^\dagger \Rightarrow$ can be diagonalized

$\exists$ unitary $N \times N$ $U$ $\quad (U^\dagger U = UU^\dagger = \mathbb{1})$

s.t. $UVU^\dagger$ is diagonal.

$U^\dagger U = \mathbb{1}$ and $U U^\dagger = \mathbb{1}$

$$\sum_n U_{\alpha n} (U^\dagger)_{n\beta} = \delta_{\alpha\beta}$$

and: $$\sum_\alpha (U^\dagger)_{n\alpha} U_{\alpha m} = \delta_{nm}$$

"$U V U^\dagger$ is diagonal"

$$\sum_{nm} U_{\alpha n} V_{nm} (U^\dagger)_{m\beta} = \omega_\alpha^2 \delta_{\alpha\beta}$$

evals of $V$.

Let: $$\tilde{Q}_\alpha \equiv \sum_n U_{\alpha n} Q_n \qquad \tilde{P}_\alpha \equiv \sum_n U_{\alpha n} P_n$$

($\geq 0$ or else unstable.)

$U^\dagger_x$ BHSE

$$\sum_\alpha (U^\dagger)_{n\alpha} \tilde{Q}_\alpha = Q_n .$$

$$\sum_n P_n^2 = \sum_n P_n^\dagger P_n = \sum_{\alpha\beta} \underbrace{\sum_n U_{\alpha n} (U^\dagger)_{n\beta}}_{=\delta_{\alpha\beta}} P_\alpha^\dagger P_\beta$$

$$= \sum_\alpha P_\alpha^\dagger P_\beta .$$

$$\sum Q_n^+ V_{nm} Q_m = \sum_{\alpha\beta} \underbrace{\sum_{nm} U_{\alpha n} V_{nm} (U^+)_{n\beta}}_{= \omega_\alpha^2 \delta_{\alpha\beta}} \tilde{Q}_\alpha^+ \tilde{Q}_\beta$$

$$= \sum_\alpha \omega_\alpha^2 Q_\alpha^+ Q_\alpha .$$

<u>Special case</u>: $m_n = m$, $\underline{K_{nm} = (T - \mathbb{1})_{nm}}$

$$T_{nm} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 0 & 1 & \\ & & 0 & 0 & 1 \\ & & & & \ddots \\ 1 & & & & \end{pmatrix}_{nm}$$

① <u>LOCALITY</u>

$K_{nm} \neq 0$ only for $|n-m| <$ something.

② <u>translation invariance in x</u>: $K_{nm}$ depends only on $n-m$.

$$\Rightarrow \quad U_{kn} = \frac{e^{ikna}}{\sqrt{N}} .$$

and $(U^+)_{kn} = U_{-kn}$

$Q_k^+ = Q_{-k}$

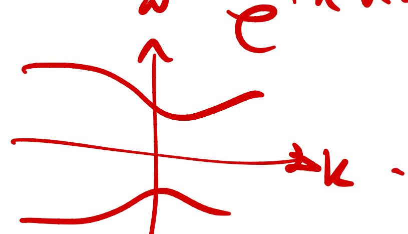
$P_k^+ = P_{-k}$ .

(Fourier analysis)

<u>e.g.</u>:

$$K = \begin{pmatrix} 3 & 1 & & & 1 \\ 1 & 3 & 1 & & \\ & 1 & 3 & & \\ & & & \ddots & \\ 1 & & & & \end{pmatrix}$$

$$\begin{cases} K_{nn} = 3 \\ K_{n,n\pm1} = 1 \\ K_{n,m} = 0 \text{ else} \end{cases} \Longrightarrow \text{eigenvectors are } e^{ikna}$$



③ nonlinearly realized continuous sym

$$q_n \to q_n + \epsilon \,.$$

sym because $V(q) = V(q_n - q_{n-1})$

$\Longrightarrow$ Goldstone Mode

$(\omega = 0)$

---

Eqns of Motion (eom) :

method 1 : Heisenberg eqns (Hamilton's eqn) :

$$i\partial_t \mathcal{O} = [\mathcal{O}, H]$$

$$\begin{cases} i\partial_t q_n = [q_n, H] = i\dfrac{p_n}{m} \\ i\partial_t p_n = [p_n, H] = -i\dfrac{\partial}{\partial q_n} V(q) \end{cases}$$

method 2 :

$$L = \sum p_n \dot{q}_n - V \Big|_{\dot{q} = p/m}$$

$$= \underline{\underline{\sum_n m_n \frac{\dot{q}_n^2}{2}}} - \sum_{nm} \frac{1}{2} K_{nm} q_n q_m .$$

$$S = \int dt \, L$$

Dirac delta

$$\boxed{\frac{\delta q_m(s)}{\delta q_n(t)} = \delta_{mn} \delta(t-s)}$$

eom : $$0 = \frac{\delta S}{\delta q_n(t)}$$

$$0 = \int ds \left[ m_n \dot{q}_n(s) \frac{\partial}{\partial t} \delta(t-s) - K_{nm} q_m \right]$$

$$\overset{IBP}{=} - m_n \ddot{q}_n - K_{nm} q_m .$$

H, L, S

quadratic in q $\Rightarrow$ eom are linear .

special case $m_n = m$, $k_n = T - I$

$$m\ddot{q}_n = -k\,(2q_n - q_{n-1} - q_{n+1})$$

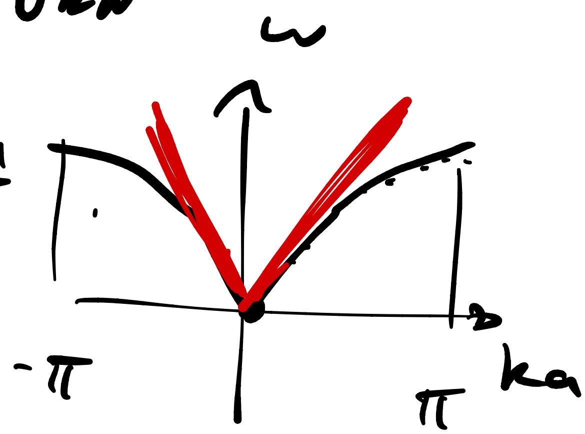
$$\longrightarrow m\ddot{q}_k = -k\,(2 - 2\cos ka)\,q_k$$

$$q_n(t) \equiv \sum_\omega e^{-i\omega t} q_{k\omega}$$

$$\Rightarrow \quad 0 = (\omega^2 - \omega_k^2)\, q_{k\omega}$$

w/ $\omega_k^2 = 4\frac{k}{m}\sin^2\frac{ka}{2}$

$$\simeq v_s^2 k^2 + \mathcal{O}(ka)^4$$




$$0 = (\partial_t^2 - v_s^2\partial_x^2)\, q(x,t)$$

$$+ \quad (a\partial_x)^4 q + \dots$$

QM: $[q_n, p_m] = i\delta_{nm} \mathbb{1}$

$$\Rightarrow [q_k, p_{k'}] = \ldots = i\delta_{k,-k'} \mathbb{1}.$$

For $k \neq 0$:
$$\begin{cases} q_k \equiv \sqrt{\frac{\hbar}{2m\omega_k}} \left(a_k + a_{-k}^\dagger\right) \\ p_k \equiv \frac{1}{i}\sqrt{\frac{\hbar m \omega_k}{2}} \left(a_k - a_{-k}^\dagger\right) \end{cases}$$

$$[a_k, a_{k'}^\dagger] = \delta_{kk'} \mathbb{1}.$$

$$\rightsquigarrow H = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + \tfrac{1}{2}\right) + \frac{p_0^2}{2m}$$

$[p_0, q_0] = -i$

$\frac{\partial V}{\partial q_0} = 0$

The discovery of Fock space:

groundstate

$$|g_0\rangle = |p_0 = 0\rangle \otimes |0\rangle$$

$$a_k |0\rangle = 0 \quad \forall k \neq 0$$

excitation of oscillators:

$$a_k^\dagger |0\rangle \propto \left| \begin{matrix} \text{one phonon of} \\ \text{momentum } \hbar k \end{matrix} \right\rangle$$

has energy $\hbar\omega_k$ $\left(\begin{matrix} \text{because} \\ [N_k, a_k^\dagger] = a_k^\dagger \end{matrix}\right)$

$N_k \equiv a_k^\dagger a_k$

"$|k\rangle$" in QM.

---

$$\left| \begin{matrix} \text{one phonon at} \\ \text{position } x \end{matrix} \right\rangle = \sum_k e^{ikx} \left| \begin{matrix} \text{one phonon} \\ \text{of momentum} \\ \hbar k \end{matrix} \right\rangle$$

$$\sim \sum_k e^{ikx} a_k^\dagger |0\rangle .$$

"$|x\rangle$".

(particle $\equiv$ thing that can be localized.)

$$|k,k'\rangle = a_k^\dagger a_{k'}^\dagger |0\rangle \leftarrow \left|\begin{matrix}\text{two}\\ \text{phonons}\end{matrix}\right\rangle$$

is an eigenstate of $H$ w/ energy $\hbar\omega_k + \hbar\omega_{k'}$

$\vdots$

$$\mathcal{H}_{N\text{ particles}} = \mathcal{H}_{\text{com}} \otimes \mathcal{H}_{\text{Fock}}$$

rep of $[q_0, p_0] = i$

$$\mathcal{H}_{\text{Fock}} = \text{span}\left\{ (a_{k_1}^\dagger)^{n_{k_1}} (a_{k_2}^\dagger)^{n_{k_2}} \cdots |0\rangle \right.$$

$$\left. \equiv |\{n_{k_1}, n_{k_2} \cdots\}\rangle \right\}$$

$\Rightarrow$ phonons are indistinguishable bosons. $\quad \left( [a_k^\dagger, a_{k'}^\dagger] = 0. \right)$

Lessons: ① particles $\rightarrow$ field $\rightarrow$ particles (collective)

② Lorentz inv. can emerge

$$0 = (\partial_t^2 - v^2 \partial_x^2)\phi(x) = \partial_\mu \partial^\mu \phi$$

(achyun modes w/ $ka \ll 1$) $= \eta^{\mu\nu}\partial_\mu\partial_\nu\phi$

③ At finite N everything is finite.

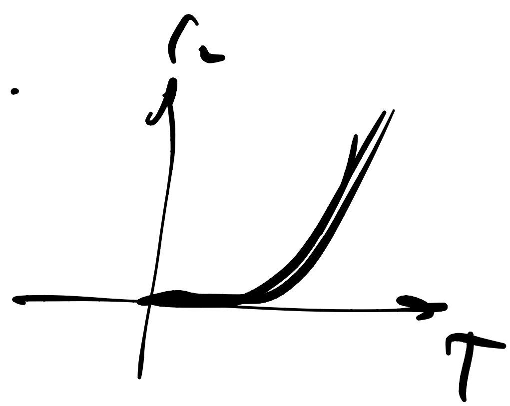
expt'l verification: • phonons are real.

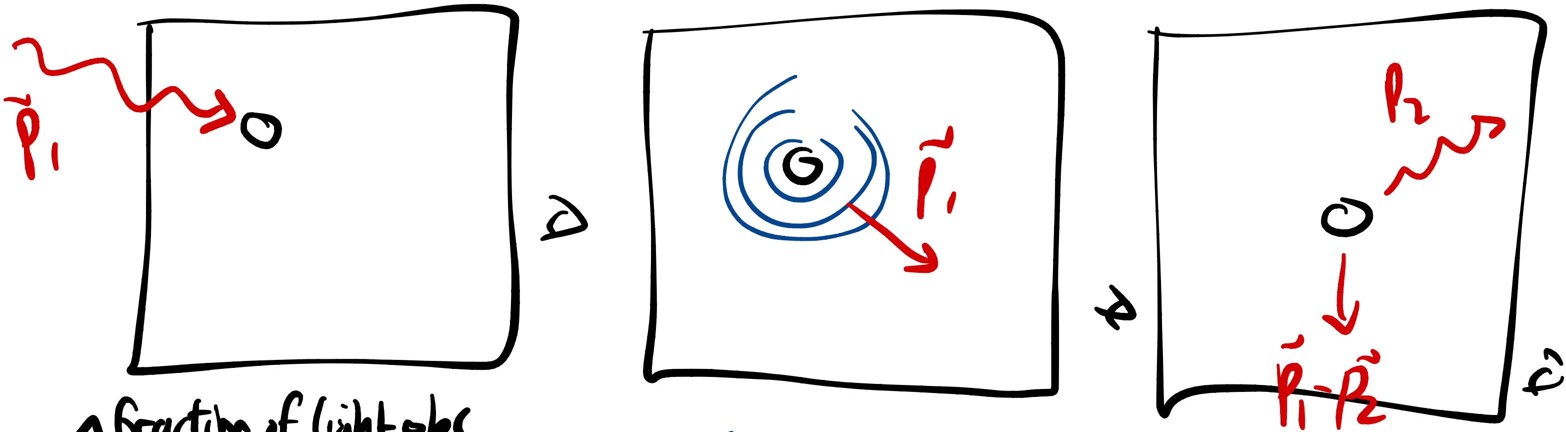
$$E(T) = \sum_k \frac{\hbar\omega_k}{e^{\hbar\omega_k/k_BT} - 1}$$

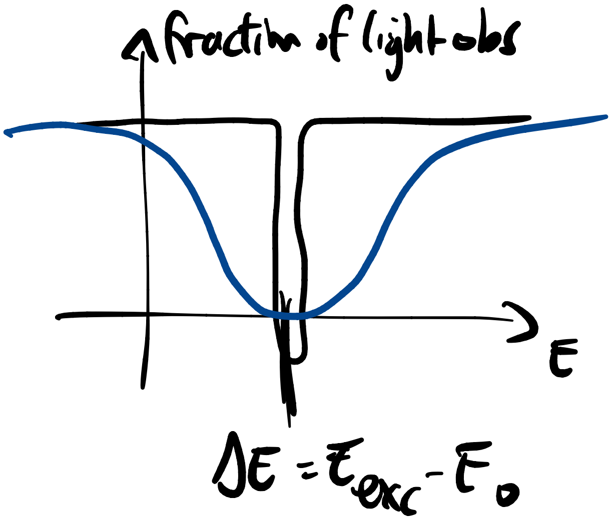
$$\overset{L\to\infty}{=} \int d^dk \frac{v_s|k|}{e^{v_s|k|/T} - 1}$$

$q = \frac{v_s k}{T}$ $= T^{d+1} \#.$

$c_v \sim T^d.$






vs:

Recoil $\Rightarrow$

$$E^{recoil}_{max} = \frac{(2P_1)^2}{2m} = \frac{(2E_0/c)^2}{2M}$$

Resolution: scattering off a solid

1) $\exists$ phonons. $\left| A_{\substack{zero \\ phonons}} \right|^2 > 0$.

2) only the com moves.

$q_n \simeq q_n + a$. $\qquad q_0 \simeq \frac{1}{\sqrt{N}} \left( \sum_n q_n + Na \right)$

$\Rightarrow q_0 \simeq q_0 + \sqrt{N} a$. $\qquad e^{i p_0 q_0} \overset{!}{=} e^{i p_0 (q_0 + a\sqrt{N})}$

$\Rightarrow p_0 \in \frac{2\pi \mathbb{Z}}{\sqrt{N}\, a}$

$$\left.\frac{p_0^2}{2m}\right|_{\text{smallest allowed}} = \frac{1}{2}\frac{1}{Nm}\left(\frac{2\pi}{a}\right)^2$$

$Nm$ = mass of whole solid!

---

Interactions: $\lambda = 0$ is special!

$$\left(H = \sum p^2 + q^2 + \lambda q^3 \cdots\right)$$

① additivity of energy

② $[H, N_k] = 0 \quad \forall k.$ $\quad \underline{\underline{N_k = a_k^\dagger a_k}}$

BIG SYMMETRY!

broken by $\Delta H = \sum_{nml} \lambda_{nml} q_n q_m q_l$

even break $N = \sum_k N_k$. $\rightsquigarrow$ $= \underline{\underline{a^\dagger a^\dagger a^\dagger}} + \underline{\underline{a^\dagger a^\dagger a}} + a^\dagger a a + aaa$

$$\sum_n q_n^4 = \sum a^{+4} + \sum \cdots a^+_{k_1} a_{k_2} a^+_{k_3} a_{k_4}$$

not every QFT = free springs.

## Towards scalar field theory :

$$\begin{cases} q_n = \sqrt{\frac{\hbar}{2m}} \sum_k \frac{1}{\sqrt{\omega_k}} \left( e^{ikx_n} a_k + e^{-ikx_n} a_k^+ \right) + \frac{1}{\sqrt{N}} q_0 \\ p_n = \frac{m}{i} \sqrt{\frac{\hbar}{2m}} \sum_k \sqrt{\omega_k} \left( e^{ikx_n} a_k - e^{-ikx_n} a_k^+ \right) + \frac{p_0}{\sqrt{N}} \end{cases}$$

$$= m \dot{q}_n$$

$$\phi \equiv \sqrt{m}\, q$$

# Path Integral Reminder

eg $\quad H = \frac{p^2}{2m} + V(q)$

for real-time propagator:

$$\langle q | e^{-iHt} | q_0 \rangle \equiv \int_{q(0)=q_0}^{q(t)=q} [Dq] \, e^{i \underbrace{\int_0^t dt \left( \frac{1}{2}\dot{q}^2 - V(q) \right)}_{\equiv S[q]}}$$

$$[Dq] \equiv \mathcal{N} \prod_{\ell=1}^{M_t} dq(t_\ell)$$

$$\left( M_t = \frac{t}{\Delta t} \to \infty \quad \Delta t \to 0, \; t \text{ fixed} \right)$$

$$\langle q_2 | e^{-iH\Delta t} | q_1 \rangle = \langle q_2 | e^{-i\Delta t \frac{\hat{p}^2}{2m}} \, \mathbb{1} \, e^{-i\Delta t V(\hat{q})} | q_1 \rangle + \mathcal{O}(\Delta t^2)$$

$$\left( e^{A+B} \neq e^A e^B \text{ if } [A,B] \neq 0 \right)$$

$$\mathbb{1} = \mathbb{1}^2 = \left( \left( \int dp \, |p\rangle\langle p| \right) \left( \int dq \, |q\rangle\langle q| \right) \right)$$

$$\langle p | q \rangle \propto e^{ipq}$$

## Quick Applications:

① PI **explains** role of solns of eom.

stationary phase $\Longleftrightarrow$ $0 = \dfrac{\delta S}{\delta q(t)}$ $\forall t$.

meaning of $\dfrac{\delta}{\delta q(t)}$:

at finite $M_t$, $q(t_\ell) = q_\ell$

stationary phase $\Longleftrightarrow$ $0 = \dfrac{\partial S}{\partial q_\ell}$ $\forall \ell$.

$$\frac{\partial q_i}{\partial q_\ell} = \delta_{i\ell} \xrightarrow{M_t \to \infty} \frac{\delta q(t)}{\delta q(s)} = \delta(t-s).$$

② euclidean path integrals compute groundstate expectation values.

Schrod eqn: $i\partial_t |\psi\rangle = \underline{\hat{H}} |\psi\rangle$.