Path Integral Reminder for 0+1 dim'l QFT.

$$\langle q_f | e^{-iHt} | q_i \rangle = \int_{q(0)=q_i}^{q(t)=q_f} [dq] \, e^{iS[q]}$$

$$[dq] \equiv \prod_{i=1}^{N_t} dq_i$$

Appl 1: $\frac{\delta S}{\delta q(t)} = 0$ is stationary phase

Appl 2: groundstate of $H \equiv |0\rangle$

$$|0\rangle \propto e^{-HT} |\text{any}\rangle \quad \text{as } T \to \infty$$

euclidean time evolution

$$\mathbb{1} = \sum_{E_n} |E_n\rangle\langle E_n|$$

$$= \sum_{E_n} e^{-E_n T} |E_n\rangle C_n \qquad C_n = \langle E_n | \text{any} \rangle$$

if $C_0 \neq 0$: $\quad = e^{-E_0 T} \left(\underbrace{C_0 |0\rangle} + \mathcal{O}\left(e^{-\underbrace{(E_1 - E_0)}_{>0} T}\right) \right)$

$$e^{-HT} = \int [dq]\, e^{-\int_{-T}^{0} d\tau\, L_E(q(\tau), \dot{q}(\tau))} \, |q(0)\rangle\langle q(-T)|$$

euclidean action

$$\equiv S_E[q] = \int d\tau\, L_E$$

$$= \int d\tau \left[(\partial_\tau q)^2 + V(q) \right]$$

$$\langle f(\hat{q}) \rangle \equiv \frac{\langle 0| f(\hat{q}) |0\rangle}{\langle 0|0\rangle}$$

$$= \frac{1}{Z} \int [dq]\, e^{-S_E[q]} f(q(0))$$

$$Z = \int [dq)\, e^{-S_E[q]} = \frac{1}{Z} \int \prod_{i=1}^{M_t} dq_i\, e^{-\sum_i L_E(q_i)} f(q_0)$$

$$q(\tau_i) = q_i$$

If $V(q)$ is quadratic $\quad V(q) = \# q^2$

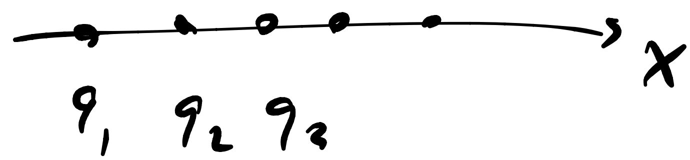
$$(\partial_\tau q)^2 \sim (q_i - q_{i-1})^2$$

$$\rightarrow \quad Z = \int \prod dq_i \; e^{-q_i D_{ij} q_j} \qquad D = D^T \text{ real.}$$

$$= \frac{2\pi^{M_t}}{\det D}.$$

GAUSSIAN.

1.2 Scalar Field Theory



$$Z = \int [dq_1 \dots dq_N] \, e^{iS[q]}$$

$$S[q] = \int dt \left(\sum_n \tfrac{1}{2} m_n \dot{q}_n^2 - V(q) \right)$$

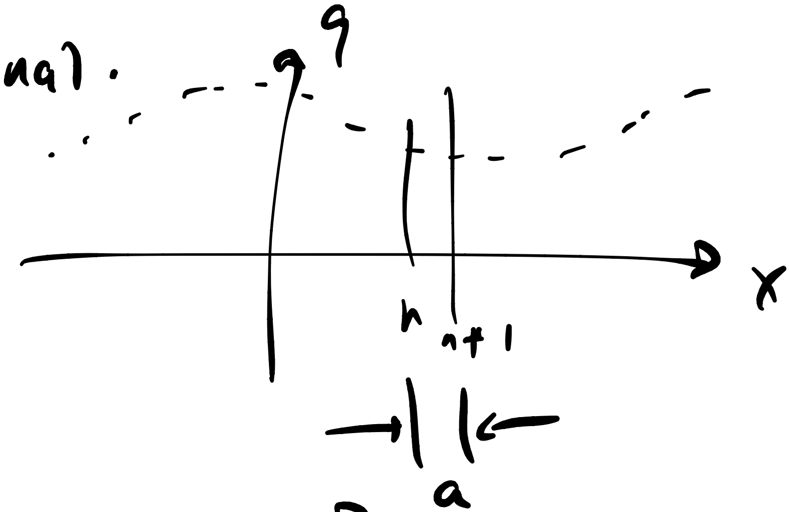
$$\equiv \int dt \; L(q, \dot{q}).$$

e.g.

$$\left(V(q) = \sum_n \frac{k}{2} (q_n - q_{n-1})^2 \right)$$

"Continuum Limit" "$\underline{a \to 0}$, $N \to \infty$"

$q_n \equiv q(x=na)$.



• $(q_n - q_{n-1})^2 = \left(q(x_n) - q(x_{n-1})\right)^2 = a^2 (\partial_x q)^2\Big|_{x=na} + \dots$

Taylor expand
$q(x_{n-1})$ about x_n.

$= q(x_n) + a \frac{\partial}{\partial x} q(x_n) + \dots$

• $a \sum_n f(q_n) \simeq \int dx\, f(q(x))$.

$$Z = \int [Dq]\, e^{iS[q]}$$

$$[Dq] = \prod_{i=1}^{M_t} \prod_{j=1}^{N=M_x} dq_{ij} \quad \left(q(t_i, x_j) = q_{ij}\right).$$

$$S[q] = \int dt \int dx\, \mathcal{L} \qquad L = \int dx\, \mathcal{L}$$

$\mathcal{L}$ ← Lagr. density.

$$\mathcal{L} = \frac{1}{2}\left(\mu (\partial_t q)^2 - \mu v_s^2 (\partial_x q)^2 - r q^2 - u q^4 - \dots \right)$$

$a^4 (\partial_x q)^4$

$\mu, v_s, r, u \dots$ are fns of $m, K, a \dots$

- $$0 = \frac{\delta S}{\delta q(x,t)} = -\mu \ddot{q} + \mu v_s^2 \partial_x^2 q - r q - 2u q^3 + \dots$$

[for phonon problem: $r = u = 0 \dots$

$L(q, \partial_\mu q) = L(\partial_\mu q)$.

q is a Goldstone mode.

- $$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_t q)} = \mu \dot{q}$$ field momentum density.

$$\left(\pi_n = \frac{\partial L}{\partial \dot{q}_n} = \mu \dot{q}_n \right)$$

$$H = \sum_n (p_n \dot{q}_n - L_n) = \int dx \left(\pi(x) \dot{q}(x) - \mathcal{L} \right)$$

$$= \int dx \left[\frac{\pi(x)^2}{2\mu} + \mu v_s^2 (\partial_x q)^2 + r q^2 + u q^4 + \dots \right]$$

$$\underline{\phi \equiv \sqrt{\mu}\, q} \longrightarrow \mathcal{L} = \frac{1}{2}\dot{\phi}^2 + \ldots$$

$$\boxed{L \to \infty} \quad \begin{cases} \frac{1}{L^d}\sum_k \ldots \overset{L\to\infty}{\rightsquigarrow} \int đ^d k \ldots \\ L^d \delta_{k,k'} \overset{L\to\infty}{\rightsquigarrow} (2\pi)^d \delta^{(d)}(k-k') \end{cases}$$

<u>Check</u>: $\forall k'$ $\quad 1 = \sum_k \delta_{k,k'} = \int đ^d k\, (2\pi)^d \delta^{(d)}(k-k')$ ✓

Continuum (free) scalar field thy in d+1 dims

$$S[\phi] = \int d^d x\, dt \left(\underbrace{\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}v_s^2 \vec{\nabla}\phi\cdot\vec{\nabla}\phi}_{} - V(\phi) \right)$$

$$= \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

$$= \frac{1}{2} \partial^\mu \phi \partial_\mu \phi$$

$$\partial_\mu \phi = (\partial_t \phi, \vec{\nabla}\phi)_\mu \qquad (v_s = c = 1)$$

Lorentz inv't.

$$0 = \frac{\delta S[\phi]}{\delta \phi(x)} = -\partial_t^2 \phi + v_s^2 \nabla^2 \phi - V'(\phi)$$
$$= -\partial_\mu \partial^\mu \phi - V'(\phi).$$

$\boxed{\text{FREE}}$ CASE: $V(\phi) = \frac{1}{2} m^2 \phi^2$.

eom: $0 = (\partial_\mu \partial^\mu \phi + m^2)\phi$ $\left(\begin{array}{c}\text{KG} \\ \text{eqn.}\end{array}\right)$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right)$$

Solve using canonical formalism: $\pi = \frac{\partial \mathcal{L}}{\partial \dot\phi} = \dot\phi$
w/ $L \to \infty$.

$$H = \int d^d x \left(\underbrace{\frac{\pi(x)^2}{2}}_{\geq 0} + \frac{1}{2} v_s^2 \underbrace{(\vec\nabla\phi \cdot \vec\nabla\phi)}_{\geq 0} + \underbrace{\frac{1}{2} m^2 \phi^2}_{\geq 0} \right)$$

$\mathcal{L}$ only depends on x
through $\phi(x)$.
$\Rightarrow$ translation inv't.

$p = \frac{\partial L}{\partial \dot q}$

$L = \frac{1}{2} m \dot q^2$

$p = m \dot q$

transl. inv't & linear $\Longrightarrow$ Fourier analyze.

step 1
$$\begin{cases} \phi(x) = \int đ^d k \, e^{-i\vec{k}\cdot\tilde{x}} \phi_k \\ \pi(x) = \int đ^d k \, e^{-i\tilde{k}\cdot\tilde{x}} \pi_k \end{cases}$$

$$\rightsquigarrow H = \int đ^d k \left(\tfrac{1}{2} \pi_k \pi_{-k} + \tfrac{1}{2} (v_s^2 k^2 + m^2) \phi_k \phi_{-k} \right)$$

$$k^2 \equiv (-i\tilde{k}) \cdot i\tilde{k} = \tilde{k} \cdot \tilde{k}$$

$$= \sum_k h_{k,-k}.$$

$\omega_k^2 = v_s^2 k^2 + m^2$.
(condition for $e^{i(\tilde{k}\cdot x - \omega_k t)}$ to solve KG eqn.)
"on-shell".

step 2
$$\begin{cases} \phi_k = \sqrt{\frac{\hbar}{2\omega_k}} (a_k + a_{-k}^\dagger) \\ \pi_k = \frac{1}{i} \sqrt{\frac{\hbar \omega_k}{2}} (a_k - a_{-k}^\dagger) \end{cases}$$

$$[\phi(x), \pi(y)] = i\delta^{(d)}(x-y) \Longleftrightarrow [a_{\vec{k}}, a_{\tilde{k}'}^\dagger] = \delta(k-k')$$
$$= (2\pi)^d \delta^{(d)}(k-k')$$

<u>WARNING</u>: normalization of $a, a^\dagger$ differs by $\left(\frac{2\pi}{2}\right)^{d/2}$.

$$H = \int d^d k \; \hbar\omega_k \left(a_k^\dagger a_k + \tfrac{1}{2} \right) \quad \checkmark$$

$$\begin{cases} \phi(\vec{x}) = \int d^d k \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{i\vec{k}\cdot\vec{x}} a_k + e^{-ik\cdot x} a_k^\dagger \right) \\ \pi(\vec{x}) = \frac{1}{i}\int d^d k \sqrt{\frac{\hbar\omega_k}{2}} \left(e^{ik\cdot x} a_k - e^{-ik\cdot x} a_k^\dagger \right) \end{cases}$$

- $\phi = \phi^\dagger$. $\quad \pi = \pi^\dagger$
- $\phi(\vec{x})$ is not enough to extract $a, a^\dagger$

  only $a_k + a_{-k}^\dagger$ appears

$$\phi(\vec{x},t) = e^{iHt}\phi(x)e^{-iHt}$$

$$= \int d^d k \sqrt{\frac{\hbar}{2\omega_k}} \left(\underbrace{e^{i(k\cdot x - \omega_k t)}}_{\omega>0} a_k + \underbrace{e^{-i(k x - \omega_k t)}}_{\omega<0} a_k^\dagger \right)$$

annihilates particle

creates antiparticle

w/

$$\pi(x,t) = \dot{\phi}(x,t).$$

- <u>Negative frequencies</u>. $(\partial_t^2 + \vec{k} + m^2)\phi_k = 0$

  has 2 solns $\quad \phi_k = e^{\pm i\omega_k t}$

"Momentum"

$$\left.\begin{array}{l} p_n \\ \pi(x) \end{array}\right\}$$ canonical field momenta

$\hbar\vec{k}$ ← 1-particle momentum

$\vec{P}$ ← generator of translations

[total momentum in the fields

$$\phi(x,t) = e^{-iHt}\phi(x,0)e^{iHt}$$

$$= e^{iP_\mu x^\mu}\phi(0,0)\,e^{-iP_\mu x^\mu}$$

$$P_\mu \equiv (H, \vec{P})_\mu$$

f'n of field operators.

$$H = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + \tfrac{1}{2}\right)$$

$|0\rangle$, $a_k^\dagger|0\rangle$, $a_k^\dagger a_{k'}^\dagger|0\rangle \ldots$

vac — 1 scalar particle

$E = E_0$ $\quad E - E_0 = \hbar\omega_k \quad E - E_0 = \hbar\omega_k + \hbar\omega_{k'}$

1.3 Quantum light : Photons

Maxwell's eqns

$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} = -\partial_t\vec{B} \quad \Longleftrightarrow \quad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma}$ *

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{B} = \partial_t E + 4\pi\vec{j} \quad \Longleftrightarrow \quad \partial^\mu F_{\mu\nu} = 4\pi j_\nu .$

$$\begin{cases} E^i = F^{0i} \\ \epsilon^{ijk} B^k = -F^{ij} \end{cases}$$

solve *

$$A_\mu = (\Phi, \vec{A})_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

$$\left(\begin{aligned} \vec{E} &= -\vec{\nabla}\Phi - \partial_t\vec{A} \\ \vec{B} &= \vec{\nabla}\times\vec{A} \end{aligned} \right)$$

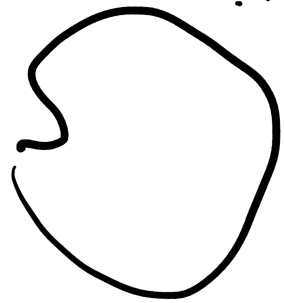
Gauge Redundancy

Potentials related by $\begin{cases} \vec{A} \to \vec{A} - \vec{\nabla}\lambda \\ \Phi \to \Phi + \partial_t \lambda \end{cases}$ represent the same physical configuration.

$$\left(A_\mu \to A_\mu - \partial_\mu \lambda \right)$$

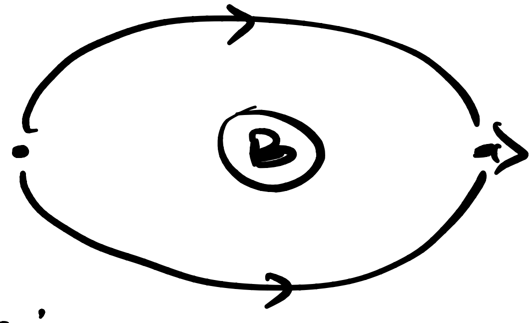
$$\oint_C A_\mu dx^\mu - \oint A_\mu^\lambda dx^\mu = \oint \frac{\partial \lambda}{\partial x^\mu} dx^\mu \underset{\uparrow \atop \text{F.T.C.}}{=} 0.$$

$$\left(A_\mu \to A_\mu^\lambda \equiv A_\mu - \partial_\mu \lambda \right)$$



phase of a charged particle moving along path C in spacetime is $e^{ie \oint A}$

Bohm-Aharonov



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partially Fix the gauge redundancy:

$$\vec{\nabla} \cdot \vec{A} = 0. \quad \text{(Coulomb gauge)}$$

if $\rho = \vec{j} = 0$, also $\Phi = 0$.

Ampere $\Rightarrow$

$$c^2 \nabla \times (\nabla \times A) = c^2 \nabla \cdot (\nabla \cdot A) - c^2 \nabla^2 A = -\partial_t^2 A$$

$$\Longrightarrow \quad \partial_t^2 \vec{A} - c^2 \nabla^2 \vec{A} = 0.$$

- in 3d
- $\phi \to \vec{A}$
- $v_s \to c$
- $\vec{\nabla} \cdot \tilde{A} = 0 \Rightarrow \vec{A}(r) = \int d^3k \, e^{i\tilde{k}\cdot\tilde{x}} \vec{A}(\tilde{k})$

$$\tilde{k} \cdot \vec{A}(k) = 0$$

" $\tilde{A}$ is transverse. "

$$S[A] = \int d^4\underline{\underline{x}} \; \mathcal{L}\left(A_\mu(x), \partial_\nu A_\mu(x)\right)$$

LOCALITY

- made from A_μ, ∂_μ
- Lorentz inv't (contract all indices)
- gauge inv't

$$F_{\mu\nu} \to F^\lambda_{\mu\nu} = F_{\mu\nu} + \cancel{[\partial_\mu, \partial_\nu]}\lambda$$ 0

$$\partial^\mu \partial^\nu \underbrace{F_{\mu\nu}}_{AS} = 0$$

$$\mathcal{L}_{\text{maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(E^2 - B^2)$$

[$\exists$ other possible terms

eg $\partial_\rho F_{\mu\nu} \partial^\rho F^{\mu\nu}$

ignore for now.]