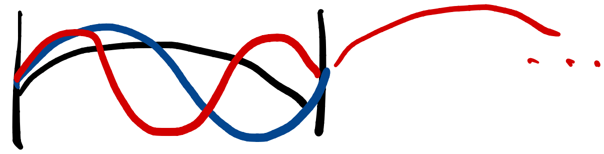
1.5 Casimir Effect



$\longleftarrow d \longrightarrow$

$$E_0(d) = \sum_{\{k\}} \frac{1}{2}\hbar\omega_k$$

$\{k\}$ allowed by box

$$\underline{F = -\partial_d E_0(d)}$$

<u>simple case</u>:
- in 1+1 dims
- (massless) scalar field

* $|\leftarrow d \rightarrow| \longleftarrow L-d \longrightarrow |$

only move this.

<u>$d \ll L$.</u>

<u>Assume</u>: <u>perfectly conducting walls</u> $\phi(0) = \phi(d) = \phi(L)$.

modes in left cavity: $\phi_j(x) = \sin\left(\frac{j\pi x}{d}\right)$ $j = 1, 2, 3 \ldots$

$$k_j = \frac{j\pi}{d} \Rightarrow \omega_j = \frac{\pi j}{d} \cdot c.$$

$$E_0(d) = f(d) + f(L-d)$$

$$f(d) = \frac{1}{2}\hbar \sum_{j=1}^{\infty} \omega_j = \frac{\hbar c \pi}{2d} \underbrace{\sum_{j=1}^{\infty} j}_{} \overset{???}{=} \frac{\infty}{d} \Big]$$

$$\left(\begin{aligned} \zeta(s) &= \sum_{j=1}^{\infty} \frac{1}{j^s} \Big] \\ \zeta(-1) &= -\frac{1}{12}. \end{aligned} \right)$$

WRONG!

error: Real plates don't affect highest-k modes!

i.e. $\omega_j \gg \pi/a$ (set $c=1$)

Regulator: $f(d) \rightsquigarrow f(a,d) = \frac{\hbar\pi}{2d} \sum_{j=1}^{\infty} j e^{-\omega_j a/\pi}$

$$= -\frac{\pi\hbar}{2} \frac{\partial}{\partial a} \Big(\underbrace{\sum_{j=1}^{\infty} e^{-aj/d}}_{\frac{1}{1-e^{-a/d}} - 1} \Big)$$

$\frac{a}{d} \ll 1$

$$= + \frac{\pi\hbar}{2d} \frac{e^{a/d}}{(e^{a/d}-1)^2}$$

Series[$g(x)$, {x, 0, 3}] $x = a/d \ll 1$.

$$f(a,d) \simeq \hbar \left(\underbrace{\frac{\pi d}{2a^2}}_{\to \infty \text{ when } a\to 0} - \underbrace{\frac{\pi}{24 d}}_{\to \text{cst}} + \underbrace{\frac{\pi a^2}{480 d^3} + \dots}_{\to 0 \text{ when } a \to 0} \right)$$

$$E_0(d,a) = f(a,d) + f(a, L-d) = \underbrace{\frac{\hbar \pi}{2a^2} L}_{\text{ind of } d\,!!} + \text{FINITE}$$

thing we can measure :

$$F = -\partial_d E_0 = -\left(f'(d) - f'(L-d) \right)$$

$$= -\hbar \left[\left(\cancel{\frac{\pi}{2a^2}} + \frac{\pi}{24 d^2} + O(a^2) \right) - \left(\cancel{\frac{\pi}{2a^2}} + \frac{\pi}{24 (L-d)^2} + O(a^2) \right) \right]$$

$a \ll d \ll L$

$$\overset{a\to 0}{=} -\frac{\pi \hbar}{24} \left(\frac{1}{d^2} - \frac{1}{(L-d)^2} \right)$$

ind of a!

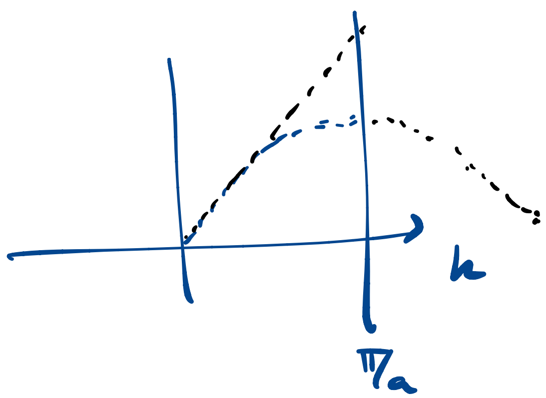
$$\overset{L \gg d}{=} -\frac{\pi \hbar c}{24 d^2} \left(1 + O(d/L) \right).$$

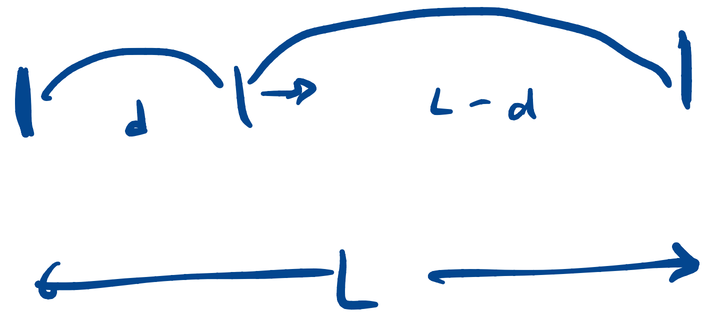
Attractive force!

• Ans. to physics Q's is independent of Λ regulators.

(symmetric. smooth?)

(Physics claim)

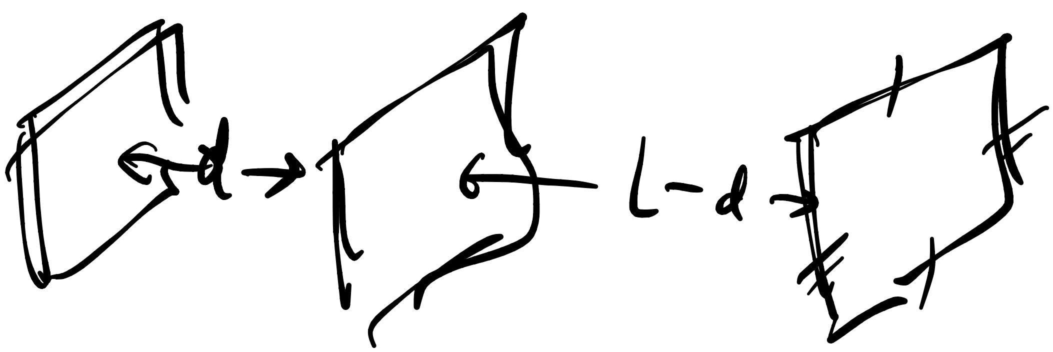




$$\frac{1}{d}\sum_{j=1}^{N} j = \frac{N(N+1)}{2d} .$$

?

3+1d: $\frac{F}{A} = P = \gamma \frac{\hbar c}{d^4}$. $\gamma \neq 0.$



Comment about E_0 .

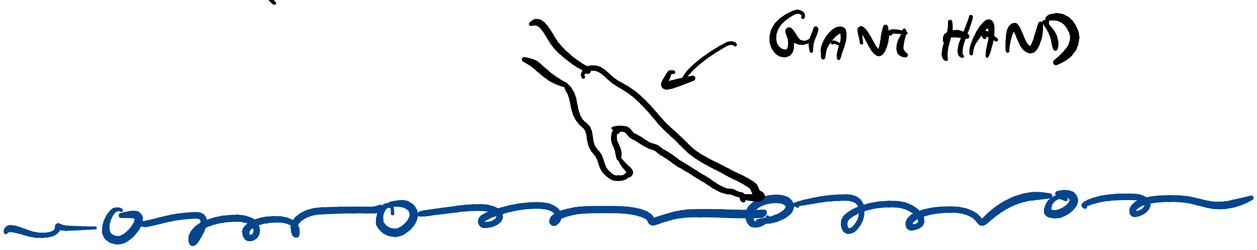
① it gravitates .
$E_0 > 0 \underset{GR}{\Longrightarrow}$ inflation

② it is very UV sensitive!!

$$E_0 = \sum_k \hbar\omega_k = \underbrace{L^d}_{\uparrow} \underbrace{\int^{\Lambda} d^dk\, k}_{\sim \Lambda^{d+1}}$$

(2) The path integral makes some things easier.
(propagators)

2.1 Fields mediate forces.



$$H(t) = \sum_n \frac{p_n^2}{2m_n} + K_{nm} q_n q_m + \delta V(t)$$

$$\delta V(q) = - \underbrace{J_n(t)}_{\text{chosen by HAND}} q_n(t)$$

"background field".

$$\langle F | e^{-i\int_0^T dt H(t)} | I \rangle = \int_{I \to F} [D\phi]\, e^{i\int dt\, d^dx \left(\mathcal{L} + J(x,t)\phi(x,t)\right)}.$$

e.g. suppose ADIABATIC $\left(\frac{|\dot{J}|}{|J|} \ll \Delta E \right)$

$$\langle 0| e^{-i\int_0^T dt\, H(t)} |0\rangle \simeq e^{-i\int_0^T dt\, E_{gs}(t)}$$

" vac. persistence amplitude".

$$\mathcal{L}(\phi) = \frac{1}{2}\left(\partial_\mu \phi \partial^\mu \phi - m^2\phi^2\right)$$

$$\overset{\text{IBP}}{=} -\frac{1}{2}\underline{\underline{\phi\,(\partial^2 + m^2)}}\,\phi \quad + \text{ total deriv.}$$

$$\underline{\underline{e^{iW[J]}}} \equiv \int [D\phi]\, e^{i\int(\mathcal{L} + J\phi)}$$

$$= \int_{-\infty}^{\infty} \prod_{n,t}^{N, M_t} dq_{n,t}\; e^{\frac{i}{2} q_x A_{xy} q_y + i J_x q_x}$$

$$= \sqrt{\frac{(2\pi i)^{NM_t}}{\det A}}\; e^{-\frac{i}{2} J_x A^{-1}_{xy} J_y}$$

$$q_x A_{xy} q_y \equiv \int d^{d+1}x\, d^{d+1}y\; \phi(x) A_{xy} \phi(y)$$

$$A_{xy} = -\delta^{d+1}(x-y)(\partial_x^2 + m^2)$$

$$A_{xz}(A^{-1})_{zy} = \delta_{xy}$$

$$\Leftrightarrow -(\partial^2 + m^2)\underbrace{D(x-y)} = \delta^{d+1}(x-y) \quad \circledast$$

"D is a Green's fn for $-(\partial^2+m^2)$".

$$A^{-1}_{xy} = D(x,y) \underset{\substack{\uparrow \\ \text{transl. inv.}}}{=} D(x-y) .$$

one way to define the integral $\int [Dq]$:

eg: $$\int_{\mathbb{R}} dq\, e^{-\frac{1}{2} q A q} = \sqrt{\frac{\pi^{\#}}{\det A}}$$

<u>overkill</u>: $\longrightarrow$

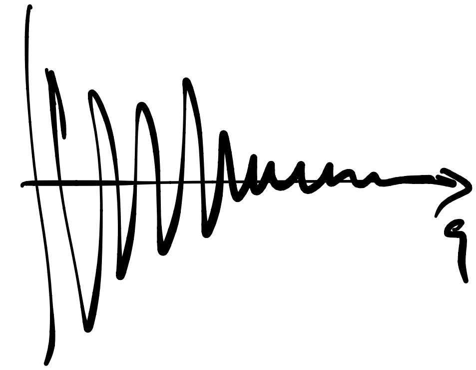
$$\int_{\mathbb{R}} dq\, e^{i\frac{1}{2} q (A + i\epsilon) q}$$ is enough.

ϵ is infinitesimal: $\underline{\epsilon^2 = 0}$. $a\epsilon = \epsilon$, $a \in \mathbb{R}_+$.

$\underline{\underline{\epsilon > 0}}$

$$\int_{-\infty}^{\infty} dq_{nt}\, e^{-\epsilon q_{nt}^2 + i q \dots}$$

damps large $|q|$



Replace $m^2 \rightsquigarrow m^2 - i\epsilon$.

$*$ is transl inv & linear:

$$D(x) = \int \bar{d}^{d+1}k \; e^{ik_\mu x^\mu} D_k$$

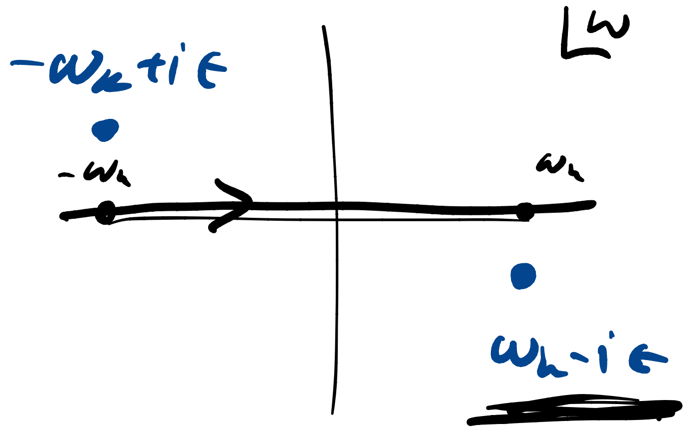
$$\delta^{d+1}(x) = \int \bar{d}^{d+1}k \; e^{ik_\mu x^\mu}$$

$$* \Leftrightarrow \quad 1 = (k^2 - m^2 + i\epsilon) D_k .$$

$$\Rightarrow D(x) = \int \bar{d}^{d+1}k \; \frac{e^{ik_\mu x^\mu}}{k^2 - m^2 + i\epsilon}$$

$$k^2 - m^2 + i\epsilon = \omega^2 - (\underbrace{\vec{k}^2 + m^2}) + i\epsilon$$
$$= \omega^2 - (\omega_k^2 - i\epsilon)$$

$$\int d\omega \, \frac{e^{ikx}}{\omega^2 - (\omega_k^2 - i\epsilon)}$$

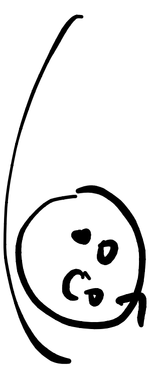


Poles at

$$\omega = \pm\sqrt{\vec{k}^2 + m^2 - i\epsilon}$$
$$= \pm\sqrt{\omega_k^2 - i\epsilon} = \pm\left(\sqrt{\omega_k^2} - \frac{i\epsilon}{\omega_k} + O(\epsilon^2)\right)$$
$$= \pm(\omega_k - i\epsilon).$$

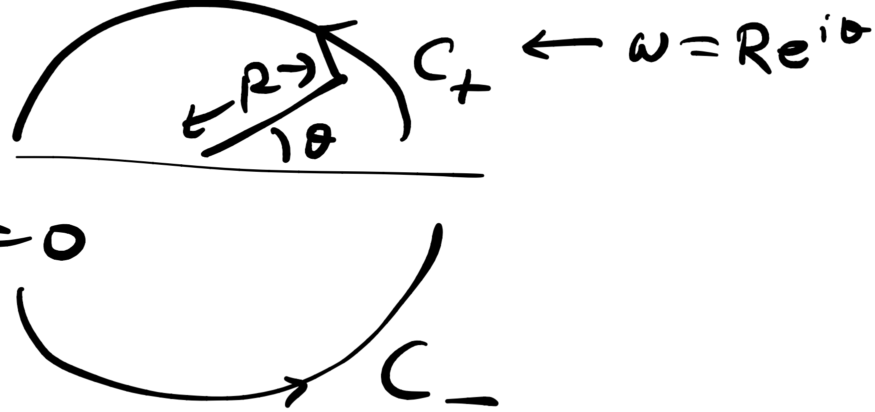
$$\oint_C dz\, f(z) = 2\pi i \sum_{z_j} \mathrm{Res}_{z=z_j} f$$

$$\oint_{C_0} \frac{dz}{z} = i\int_0^{2\pi} d\theta = 2\pi i$$



$$\int d\omega \, \frac{e^{i(\omega t - \vec{k}\cdot\vec{x})}}{\omega^2 - (\omega_k^2 - i\epsilon)}$$

If $t > 0$



$$\int_{C_+} \mathcal{I}(\omega)\,d\omega = 0$$

If $t < 0$

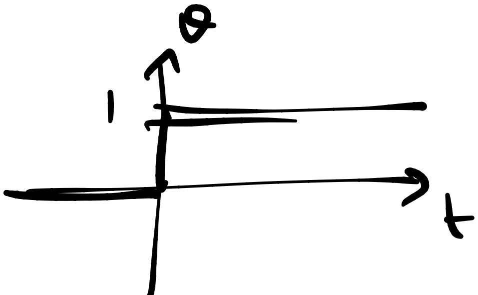
$$\int_{C_-} \mathcal{I}(\omega)\,d\omega = 0.$$

$$e^{i\omega t} = e^{iRe^{i\theta}t}$$

$$= e^{-R\sin\theta\, t}$$

on C_+ $\sin\theta > 0$.

$$D(x) = -i\int d^dk\left(\theta(t)\,\frac{e^{-i(\omega_k t - \vec{k}\cdot\vec{x})}}{2\omega_k} + \theta(-t)\,\frac{e^{i(\omega_k t - i\vec{k}\cdot\vec{x})}}{2\omega_k}\right)$$



"time-ordered".