Casimir force using hard cutoff:

Schwartz § 15.2

2.1 Fields Mediate Forces, cont'd.

$m^2 \leadsto m^2 - i\epsilon$

Recap: $e^{iW[J]} \equiv Z[J] = \int [D\phi]\, e^{i\left(S_0[\phi] + \int \phi J\right)}$

$$\boxed{S[\phi] \equiv -\frac{1}{2}\int \phi\,(\partial^2 + m^2)\,\phi}$$

$$= \int \prod_n^N dq_n\, e^{\frac{i}{2} q_n A_{nm} q_m + i q_n J_n}$$

$A_{nm} \leadsto A_{nm} + i\epsilon$

$$= \sqrt{\frac{(2\pi i)^N}{\det A}}\, e^{-\frac{i}{2} J_n (A^{-1})_{nm} J_m}$$

$$A_{xy} = -\delta^D(x-y)\,(\partial^2 + m^2)$$

$$(A^{-1})_{y+x,y} \equiv D(x) = \int d^{d+1}k\, \frac{e^{ik_\mu x^\mu}}{k^2 - m^2 + i\epsilon}$$

$$= -i\int d^dk \left(\theta(t)\, \frac{e^{-i(\omega_k t - \vec{k}\cdot\vec{x})}}{2\omega_k} + \theta(-t)\, \frac{e^{+i(\omega_k t - \vec{k}\cdot\vec{x})}}{2\omega_k}\right)$$

$$\left(\omega_k = \sqrt{\vec{k}^2 + m^2}.\right)$$

Propagator:

$$(A^{-1})_{nm} = \frac{\int \prod dq \; e^{-\frac{1}{2} q A q} \, q_n q_m}{Z}$$

$$= \frac{\partial}{\partial J_n} \frac{\partial}{\partial J_m} \Bigg|_{J=0} \ln \Bigg(\underbrace{\int \prod dq \; e^{-\frac{1}{2} qAq + q \cdot J}}_{\text{generating function(al)}} \Bigg)$$

→ CLAIM:

$$D(x-y) \stackrel{?}{=} \cancel{?} \; (-i)? \; \checkmark \; \langle 0 | T \hat{\phi}(x) \, \hat{\phi}(y) | 0 \rangle$$

vac — destroy that exc. — create some exc. — vac

t ←

$$= \theta(x^0 - y^0) \langle 0 | \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \hat{\phi}(y) \hat{\phi}(x) | 0 \rangle$$

"<u>time-ordered</u> propagator":

$$t \equiv x^0 - y^0 > 0 .$$

$$\hat{\phi}(x) = \int \frac{d^dk}{\sqrt{2\omega_k}} \left(e^{-ik_\mu x^\mu}\big|_{k^0=\omega_k} a_k + h.c. \right)$$

$$\langle 0| \hat{\phi}(x) \hat{\phi}(y) |0\rangle$$

$$\boxed{\langle 0| \left(\cancel{a} + \cancel{a^\dagger} \right) \left(\cancel{a} + \cancel{a^\dagger} \right) |0\rangle}$$

$$= \int \frac{d^dk\, d^dq}{\sqrt{2\omega_k 2\omega_q}} e^{-ikx+iqy} \underbrace{\langle 0| a_k a_q^\dagger |0\rangle}_{=(2\pi)^d \delta^d(k-q)}$$

$$= \int \frac{d^dk}{2\omega_k} e^{-ik(x-y)} \Big|_{k^0=\omega_k} .$$

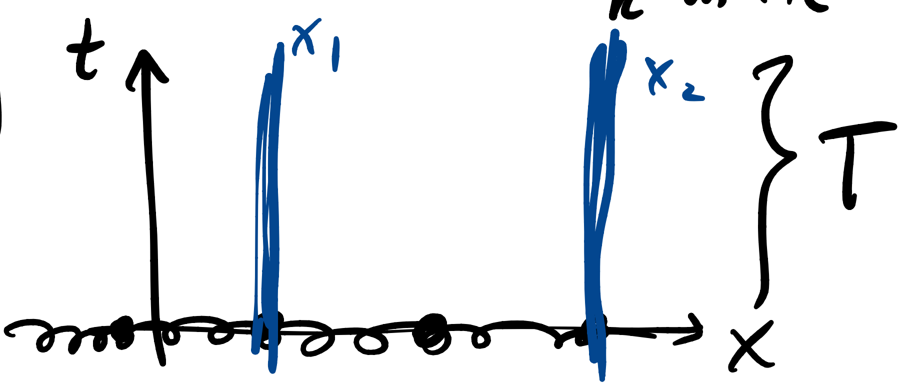
$$W[J] - \underline{W[0]} = -\frac{1}{2} \iint d^{d+1}x\, d^{d+1}y\, J(x) D(x-y) J(y)$$

gaussian integral

$$\boxed{\begin{gathered} J(x) \equiv \int d^{d+1}k\, e^{ikx} J_k \\ J_k^* = J_{-k} \end{gathered}}$$

$$= -\frac{1}{2} \int d^{d+1}k\, J_k^* \frac{1}{k^2 - m^2 + i\epsilon} J_k .$$

we pick J_k!



$$J(x) = J_1(x) + J_2(x)$$

$$J_{a=1,2}(x) = \delta^3(x - x_a)$$

$$\Longrightarrow J_k = \int dx^0\, e^{-ik^0x^0}\left(e^{i\vec{k}\cdot\vec{x}_1} + e^{i\vec{k}\cdot\vec{x}_2}\right)$$

$$W[J=J_1+J_2) = -\frac{1}{2}\Big[\cancel{J_1 D J_1} + \cancel{J_2 D J_2} + 2\, J_1 D J_2\Big]$$

$$= -\underbrace{\int dx^0}_{\equiv T}\int dy^0 \int đk^0\, e^{ik^0(x^0-y^0)} \cdot \int đ^3k\, \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{k^2 - m^2 + i\epsilon} + \dots$$

$$\int dy^0\, e^{ik^0(x^0-y^0)} = 2\pi\,\delta(k^0)\,\cancel{e^{ik^0x^0}}$$

$$= -\underbrace{\int dx^0}_{\equiv T} \int đ^3k\, \frac{e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{-\vec{k}^2 - m^2 + i\epsilon}$$

$H_J|\Omega_J\rangle = E_{gs}[J)|\Omega_J\rangle.$

Conseq. q iϵ

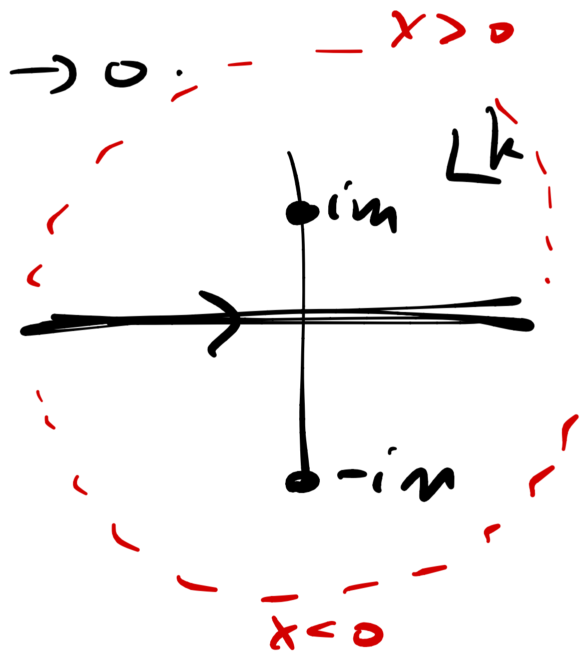
$$e^{iW[J]} = \langle\Omega_J|\, e^{-iHT}\, |\Omega_J\rangle = e^{-iE_{gs}(J)\,T}.$$

$$W(J) = -E_{gs}(J)\, T .$$

$$E_{gs}(J) = -\int d^d k \, \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + m^2}$$

$$\vec{r} \equiv \vec{x}_1 - \vec{x}_2 .$$

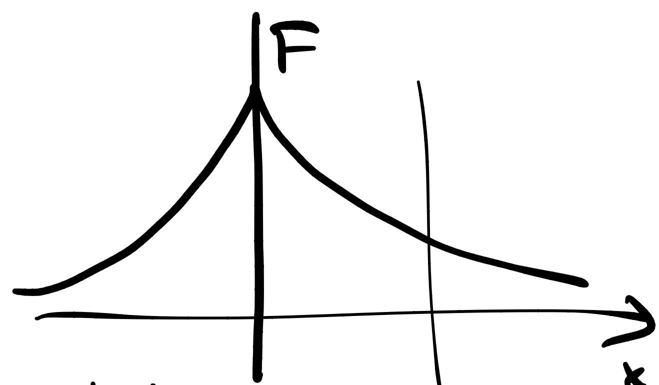
$\epsilon \to 0$.



$$\boxed{d=1} \qquad -\int \frac{dk}{2\pi} \, \frac{e^{ikx}}{k^2+m^2}$$

$$= -\frac{2\pi i}{2\pi} \, \frac{e^{-m|x|}}{2im}$$

$$= -\frac{e^{-m|x|}}{2m} .$$



$$F = -\partial_x E_{gs}(x) = \frac{1}{2} e^{-m|x|}$$

$\underline{\underline{1/m}}$.

Range of interaction $\sim m^{-1}$

$m =$ mass of the <u>force carrier</u>.

$$E_{3d}(\vec{r}) = \int d^3k \, \frac{e^{i\vec{k}\cdot\vec{r}}}{k^2+m^2} \qquad y \equiv \cos\theta$$

$$= \frac{1}{(2\pi)^2}\int \frac{k^2 dk}{k^2+m^2} \underbrace{\int_{-1}^{1} dy \, e^{iky r}}_{= \frac{\sin kr}{kr}}$$

$= \ldots$ residues

$$= \frac{e^{-mr}}{4\pi r}.$$

Yukawa

attractive!

2.2 Euclidean path Integral & Wick rotation

one mode: $S[q] = \frac{1}{2}\int dt\,\left((\partial_t q)^2 - \Omega^2 q^2\right) + \int Jq$

(eg $\Omega^2 = k^2 + m^2$ for some k)

$\boxed{\tau \equiv it}$.

$S[q] = \frac{1}{2}\, i\int d\tau\left(-(\partial_\tau q)^2 - \Omega^2 q^2\right) + i\int d\tau\, Jq$.

$= -i \int d\tau\left[\left(\frac{(\partial_\tau q)^2 + \Omega^2 q^2}{2}\right) - Jq\right]$.

$Z[J] = \int [Dq]\, e^{iS[q]} = \int [Dq]\, e^{-S_E[q]}$

$S_E[q] = \int d\tau\left(\frac{1}{2}\left((\partial_\tau q)^2 + \Omega^2 q^2\right) - Jq\right)$

$\overset{IBP}{=} \int d\tau\left[\frac{1}{2}\, q\,\underbrace{(-\partial_\tau^2 + \Omega^2)}_{>0}\, q - Jq\right]$

a Positive operator.

$$(-\partial_\tau^2+\Omega^2)\,G_E(\tau,\sigma) = \delta(\tau-\sigma) \quad .$$

$$\left(\quad (A)_{nm}\,(A^{-1})_{m\ell} = \delta_{n\ell} \quad\right)$$

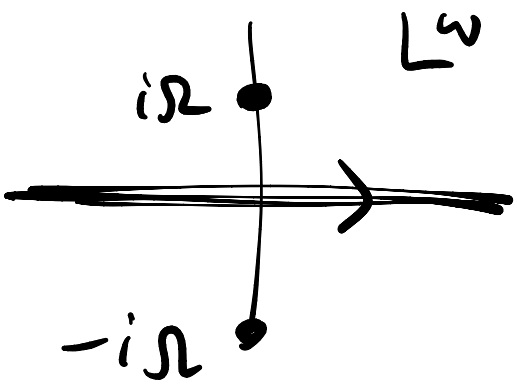
time transl inv. $\Rightarrow$ $G_E(\tau,\sigma) = G_E(\tau-\sigma)$

$$G_E(\sigma) = \int d\omega\; e^{i\omega\sigma}\, G_\omega$$

$$\Rightarrow \quad G_\omega = \frac{1}{\omega^2+\Omega^2}$$

$$G_E(\sigma) = \int d\omega\, \frac{e^{i\omega\sigma}}{\omega^2+\Omega^2} = \frac{e^{-\Omega|\sigma|}}{2\Omega} \;.$$

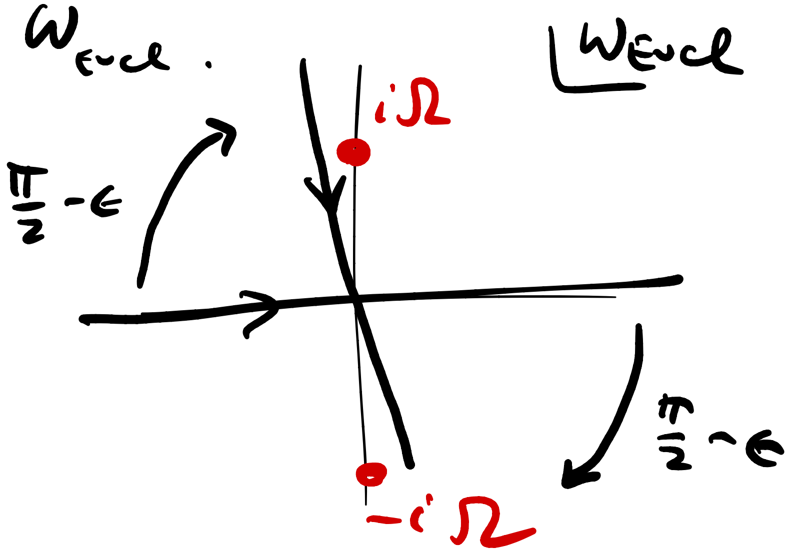
same as Yukawa force

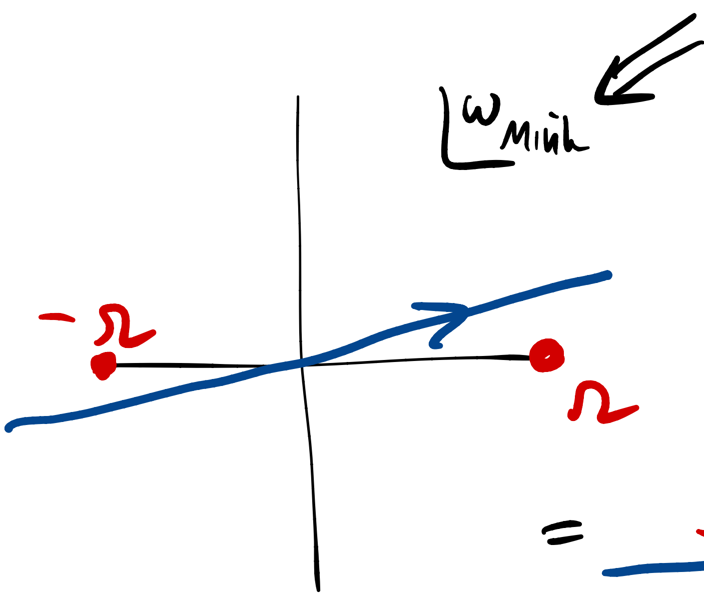


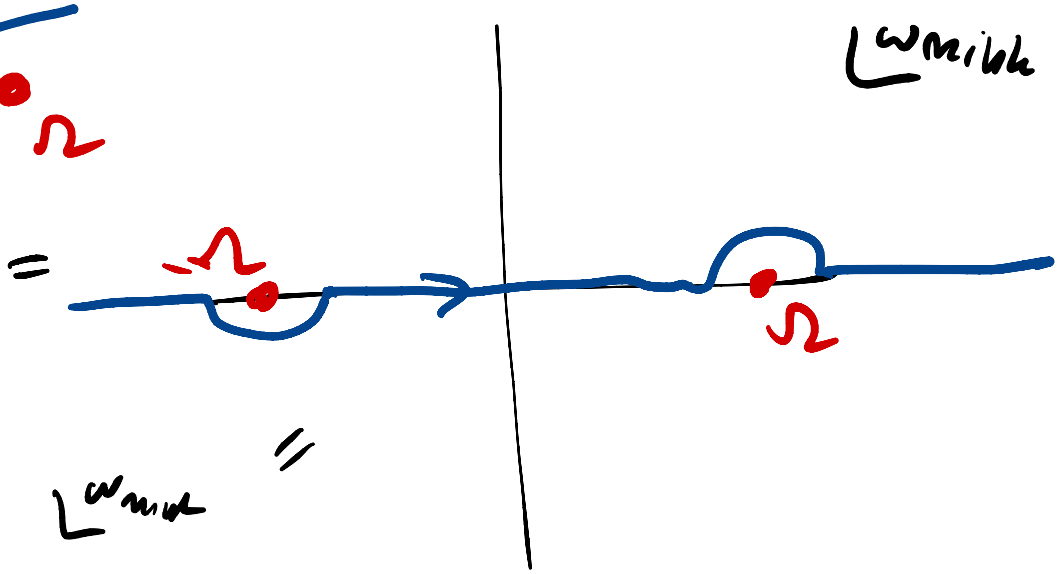
$$\left[\; \int q\,(-\partial_\tau^2+\Omega^2)\,q = \int q\,(\underbrace{-\partial_\tau^2-\vec{\partial}_x^2}_{\text{hence Euclidean.}}+m^2)\,q \;\right]$$

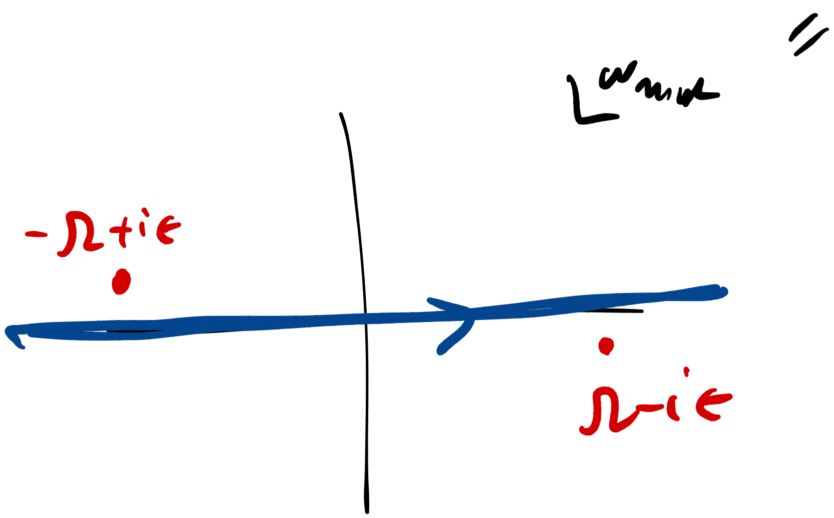
CLAIM: the real-time vacuum expectation value is the analytic continuation of the euclidean amplitude.

$$\omega_{Mink} = e^{-i\left(\frac{\pi}{2}-\epsilon\right)}\,\omega_{Eucl}\,.$$









$\longleftarrow \quad m^2 \rightarrow m^2 - i\epsilon$

<u>But</u>: $\langle q_0 | e^{-\beta H} | \text{any} \rangle \propto \psi_{gs}(q_0)$

$$= \int_{q(\beta)=q_0} [dq]\, e^{-S_E[q]}.$$

$\Rightarrow$ $i\epsilon$ prescription gives

c-#s.

$$\frac{1}{Z}\int [D\phi]\, e^{iS[\phi]}\, \phi(x_2)\,\phi(x_3)\,\phi(x_1)\cdots$$

$m^2 \to m^2 - i\epsilon$

operators

$$= \langle gs | T\left(\hat\phi(x_1)\hat\phi(x_2)\hat\phi(x_3)\ldots\right) | gs \rangle$$

time ordering.

$$Z[J] = \langle 0 | T e^{i\int Jq} | 0 \rangle$$

$$\Rightarrow \langle 0 | T\phi(x_1) \ldots | 0 \rangle = -i \frac{\delta}{\delta J(x_1)}\Big|_{J=0} \cdots \ln Z[J].$$

[other real-time G's are also useful

(e.g. $G_R = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$ $\wedge$ $\theta(x^0 - y^0)$)

ORIGIN OF DISTINCTION:

$A_E = -\partial_\tau^2 + \ell^2$ is a positive operator

$\Longrightarrow$ no zero evals, no kernel.

$$A_E = \sum_n \lambda_n |n\rangle\langle n| \qquad \lambda_n > 0$$

$$\Longrightarrow A_E^{-1} = \sum_n \frac{1}{\lambda_n} |n\rangle\langle n| \quad \text{is well-def'd.}$$

In real time A has a kernel.

$$A \sim \delta(x-y)\left(\partial_\mu \partial^\mu - m^2\right)$$

= $\underline{\underline{\text{states}}}$! satisfy $\underline{\omega^2 - \tilde{k}^2 - m^2 = 0}$.

<u>we've shown</u>: $i\epsilon$ prescription

$$m^2 \rightsquigarrow m^2 - i\epsilon$$

<u>OR</u> $\omega^2 \rightsquigarrow \omega^2 + i\epsilon$

is the same as Wick rotation.

$$\begin{cases} \omega_{eucl} = e^{-i(\frac{\pi}{2}-\epsilon)}\,\omega_{Mink} \\ \& \quad \omega_{eucl}\, t_{eucl} \stackrel{!}{=} \omega_{Mink}\, t_{Mink} \\ \Rightarrow t_{eucl.} = e^{+i(\frac{\pi}{2}-\epsilon)}\, t_{Mink}. \end{cases}$$

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special case of eucl. path integral:

$$\sum_f \langle f | e^{-\beta H} | f \rangle \equiv \mathrm{tr}\, e^{-\beta H}$$

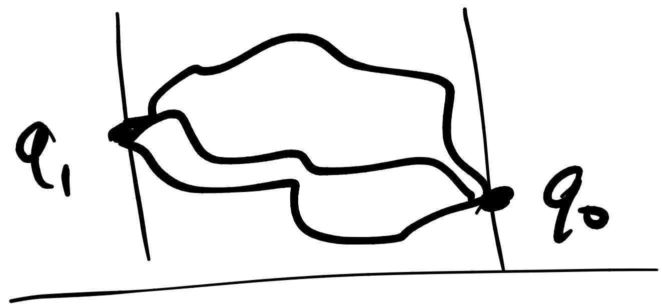
$$= Z(\beta) = \int_{q(0)=q(\beta)} [dq]\; e^{-\int_0^\beta d\tau\, L_E[q]}$$

thermal partition f'n

periodic b.c. in eucl. time.

as $\beta \to \infty$ only g.s. contributes.

$$\langle q_1 | e^{-H\beta} | q_0 \rangle \overset{F\text{-}K}{=} \int_{\substack{q(0)=q_0 \\ q(\beta)=q_1}} [Dq)\, e^{-\int_0^\beta d\tau\, L_E[q]}$$



$$\mathrm{tr}\,(\dots) = \int dq_0 \langle q_0 | \cdots | q_0 \rangle$$

2.3 Feynman diagrams from path Integral.

Brave: allow q^4 interaction terms

Cowardly: in QFT in 0+0 dimensions.

$$Z(J) = \int_{-\infty}^{\infty} dq\; e^{-\frac{1}{2}m^2q^2 - \frac{g}{4!}q^4 + Jq} = \int dq\, e^{-S(q)}.$$

$$u_i = x_i - A^{-1}_{ji} J_i$$

$$v_i A_{ij} u_j$$

<u>step 1:</u> $\int \prod dx_j \; e^{-\frac{1}{2} x_i A_{ij} x_j}$

$$= \int \prod du_i \; e^{-\frac{1}{2} u_i a_i u_i}$$

$$= \left(\prod \int du \; e^{-u_i^2 a_i} \right)$$

$$= \prod \sqrt{\frac{2\pi}{a_i}} \Big) \quad \checkmark$$

<u>step 2:</u> $\int \prod dx_j \; e^{-\frac{1}{2} x_i A_{ij} x_j - x \cdot J}$

$$= \int \underbrace{\prod dx}_{\prod du} \; e^{-\, u A u + \frac{J A^{-1} J}{2}}$$

$$\underline{\underline{= x - A^{-1} J}}$$

$$= \left(\text{step } 1 \right) \underline{\underline{e^{J A^{-1} J}}}$$

$$z(J) = \langle e^{q \cdot J} \rangle$$

$$\langle f(x_i) \rangle = \frac{1}{Z} \int \prod dx_i \, e^{-\frac{1}{2} x A x} f(x_i)$$

$$A^{-1}_{ij} = \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \ln Z$$

$$= \langle x_i \, x_j \rangle$$

① $$\frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \ln \left(\int dx \, e^{xAx + Jx} \right)$$

$$\|$$

② $$\frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \ln \left(\# \, e^{J A^{-1} J} \right)$$

$$= (A^{-1})_{ij} \quad \checkmark$$

$$\ln Z[J] = \frac{1}{2} J A^{-1} J + \text{cst.}$$

$$U(t) = e^{-iHt}$$

$$i\partial_t |\psi\rangle = H|\psi\rangle.$$

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle.$$

$$O(t) = U^{\dagger} O(0) U(t).$$

$$\partial_t O = +i[H, O].$$

$$i\partial_t O = -[H, O].$$

Schrod   Hei.

$$\langle\psi(t)| O |\psi(t)\rangle = \langle\psi| O(t) |\psi\rangle$$

$$= \langle\psi(0)| U^{\dagger} O U |\psi\rangle$$

$$\langle 0| e^{ikq} |0\rangle = \frac{1}{Z}\int \mathcal{D}q \, e^{iS + ikq}$$

$$S = \int \sum_i \left(\dot{q}_i^2 - q_i A_{ij} q_j \right)$$

$$q = \sum (a + a^\dagger)$$

$$= \sum q_n A_{nm} q_m$$

$$e^{k(a + a^\dagger)}$$

$$= e^{k a^\dagger} e^{k a} e^{-\# k^2}$$

$$\underline{\forall k} : \sum_s \vec{e}_s(\hat{k})_i \, \vec{e}_s^{\,*}(\hat{k})_j = \delta_{ij} - \hat{k}_i \hat{k}_j .$$

$$\underline{[a_{ks}, a^\dagger_{k's'}]} = \delta_{ss'} \underline{(2\pi)^d \delta^d(k - k')}$$