$$x_i = (t_i, \vec{x}_i)$$

$$\frac{1}{Z}\int \underbrace{[D\phi]}_{=\prod_{i=1} d\phi_i} e^{iS[\phi]}\, \phi(x_1)\cdots\phi(x_n) \Big|_{m^2\to m^2 - i\epsilon}$$

$$= \langle gs | T\left(\hat\phi(x_1)\cdots\hat\phi(x_n)\right)|gs\rangle$$

$$t_{i_1} < t_{i_2} < \dots$$

$i\epsilon$ is the same as analytic continuation from eucl. path integral ("Wick rotation")

2.3 Feynman diagrams in $D=0+0$.

$$Z(J) = \int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2} q m^2 q - \frac{g}{4!}q^4 + Jq} \equiv \int dq\, e^{-S(q)}$$

suppose $g \ll 1$.

$$= \int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2 q^2 + Jq}\left(1 - \frac{g}{4!}q^4 + \frac{1}{2}\left(\frac{g}{4!}q^4\right)^2 - \frac{1}{3!}\left(\ \right)^3 + \dots\right.$$

$$\int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2 q^2 + Jq}\, q^{4n} = \underline{\left(\frac{\partial}{\partial J}\right)^{4n}} \underbrace{\int_{-\infty}^{\infty} dq\, e^{-\frac{1}{2}m^2 q^2 + Jq}}_{Z_0[J]}$$

$$Z(J) \overset{?}{=} \underline{\sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{g}{4!}\right)^n \left(\frac{\partial}{\partial J}\right)^{4n}} Z_0(J)$$

$$= e^{-\frac{g}{4!}(\partial_J)^4} Z_0(J)$$

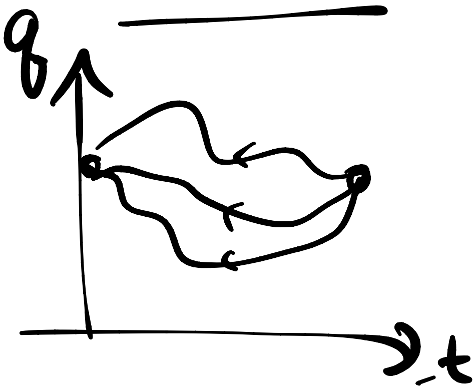
$$= \sqrt{\frac{2\pi}{m^2}}\, e^{\frac{1}{2} J \frac{1}{m^2} J}$$

$$= \underline{Z_0(0)\, e^{\frac{J^2}{2m}}}$$

$W(J)$

<u>QFT D=0+1</u> : $\underline{\langle f | e^{-iHt} | i \rangle}$

$$= \int [Dq]\, e^{iS[q]}$$



$\underline{q(t)}$

$$P(q) = \frac{1}{Z}\, e^{-S(q)}.$$

$$\int_{-\infty}^{\infty} dq\, P(q) = 1.$$

calculate coeff of $\underline{g J^4}$ in $\tilde{Z}(J) = \frac{Z(J)}{Z(0)}$.

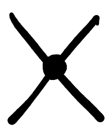
$$\tilde{Z}(J) \cong \underline{e^{-\frac{g}{4!}\partial_J^4}}\ \underline{e^{J^2/2m}}$$

$$= \left(1 - \frac{g}{4!}\underline{\partial_J^4} + \dots\right)\left(1 + \dots + \underline{\frac{1}{4!}\left(\frac{J^2}{2m}\right)^4} + \dots\right)$$

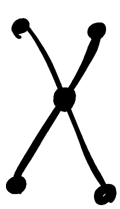
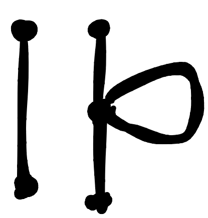
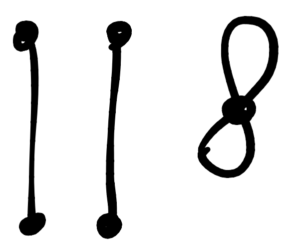
$$= \dots + \# \frac{J^4 g}{m^8} + \dots$$

Diagrams:

 $\rightsquigarrow 1/m^2$

 $\rightsquigarrow -g$

 $\rightsquigarrow J$

 +  + 

$$\# \frac{J^4 g}{m^8} + \#' \frac{J^4 g}{m^8} + \#'' \frac{J^4 g}{m^8}$$

Generalize:

$$Z(J) = \int \prod_{n=1}^{N} dq_n \, e^{-\frac{1}{2} q_n A_{nm} q_m + J_n q_n - \frac{g}{4!} \sum_n q_n^4}$$

$$= \int \prod dq \, e^{-\frac{1}{2} q A q + J q} \left(1 - \frac{g}{4!} (\sum q^4) + \dots\right)$$

Wick's Thm:

$$\frac{1}{Z(0)} \int_{-\infty}^{\infty} \prod_n dq_n \, e^{-\frac{1}{2} q_n A_{nm} q_m} \underline{q_{n_1} \cdots q_{n_k}} = \begin{cases} 0 & \text{if } k \text{ odd} \\ \sum (\text{Contractions}) & \end{cases}$$

$$\sum_{\text{Contractions}} (A^{-1})_{12} (A^{-1})_{34} \cdots$$

Contractions $\equiv$ pairing up of $\{q_{n_1} \cdots q_{n_k}\}$

eg: $$\frac{1}{Z(0)} \int \prod dq \, e^{-\frac{1}{2} q A q} \, \underline{q_{n_1} q_{n_2}} = (A^{-1})_{n_1 n_2} \checkmark$$

$$\equiv \langle q_{n_1} q_{n_2} \rangle$$

$$\frac{1}{Z(0)}\int \prod dq \; e^{-\frac{1}{2} qAq} \, q_{n_1} q_{n_2} q_{n_3} q_{n_4} \equiv \langle q_{n_1} \cdots q_{n_4} \rangle$$

$$= (A^{-1})_{n_1 n_2} (A^{-1})_{n_3 n_4} + (A^{-1})_{n_1 n_3} (A^{-1})_{n_2 n_4}$$

$$+ (A^{-1})_{n_1 n_4} (A^{-1})_{n_2 n_3} .$$

for one q: $\frac{1}{Z(0)} \int dq \, e^{-q^2 m^2/2} \, q^4 = \frac{1}{m^2} + \frac{1}{m^2} + \frac{1}{m^2}$

$$= \frac{3}{m^2} .$$

Pf: $\frac{1}{Z(0)} \int \prod dq \; e^{-\frac{1}{2} qAq} \, q_{n_1} \cdots q_{n_k}$

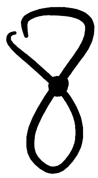
$$= \frac{1}{Z(0)} \frac{\partial}{\partial J_{n_1}} \cdots \frac{\partial}{\partial J_{n_k}} Z(J) \Big|_{J=0}$$

$$= \frac{\partial}{\partial J_{n_1}} \cdots \frac{\partial}{\partial J_{n_k}} e^{J A^{-1} J/2} \Big|_{J=0} = \cdots$$

coeff of gJ^0.

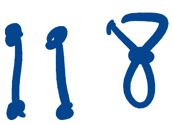
$$\frac{1}{Z(0)}\int dq\, e^{-\frac{1}{2}qAq}\left(1+qJ+\frac{(qJ)^2}{2!}+\dots\right)\left(1-\frac{g}{4!}q^4+\dots\right)$$

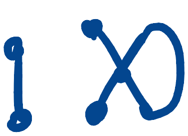
$$= \langle q^4\rangle\left(-\frac{g}{4!}\right)=\frac{3}{m^4}\left(-\frac{g}{4!}\right).$$

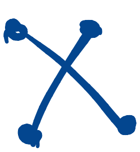
$=$  .

coeff of gJ^4:

$$\frac{1}{Z(0)}\int dq\, e^{-S_0}\left(\dots\ \frac{1}{4!}J^4 qqqq+\dots\right)\left(\dots\ -\frac{g}{4!}qqqq+\dots\right).$$

\# 

\+ \# 

\+ \# 

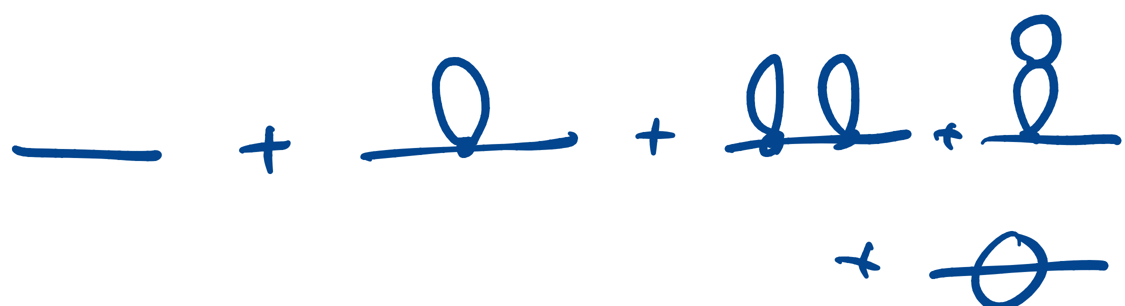
$g^2 J^6$:

$$\int dq\, e^{-S_0}\left(\dots \frac{1}{6!}J^6 q^6 + \dots\right)\left(\dots + \frac{1}{2!}\left(\frac{g}{4!}q^4\right)^2 + \dots\right)$$

back to one q:

$$G \equiv \langle q^2 \rangle \Big|_{J=0} = \frac{\int dq\, q^2 e^{-S(q)}}{\int dq\, e^{-S(q)}}$$

$$= -2\frac{\partial}{\partial m^2}\log Z(J=0)$$

$G \simeq$  $+\mathcal{O}(g^3)$

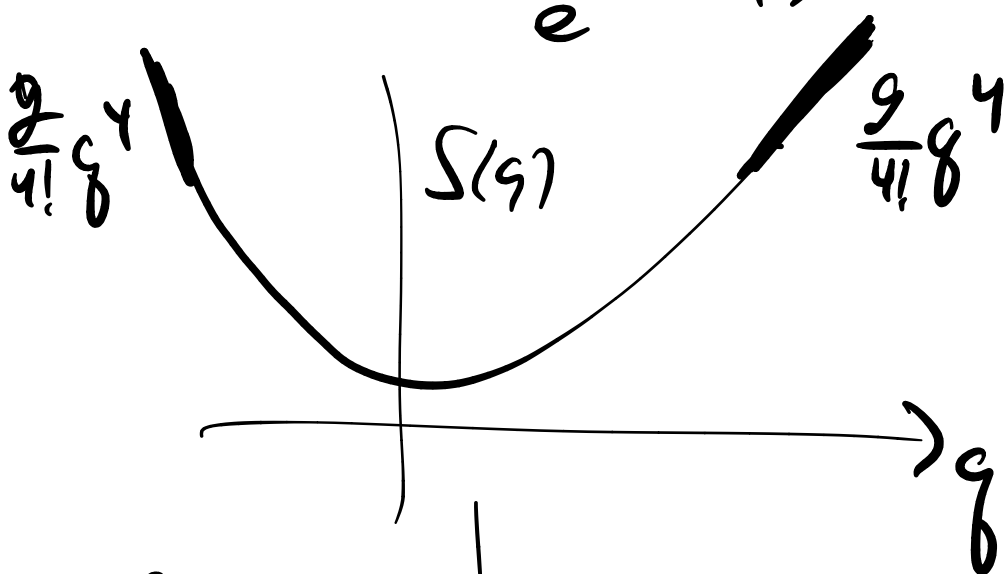
$$= m^{-2}\left(1 - \underline{\underline{\tfrac{1}{2}}}\, g\, m^{-4} + \overbrace{\underline{\underline{\tfrac{2}{3}}}\, g^2 m^{-8}} + \mathcal{O}(g^3)\right)$$

2.4 Large-order pert. thy.

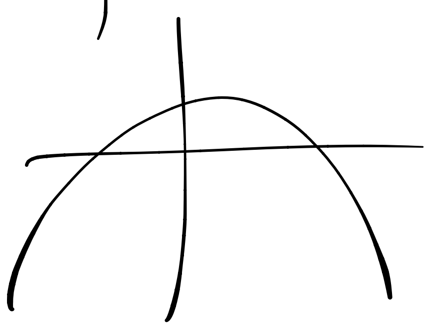
- This expansion in g **DOES NOT CONVERGE!**

Proof: $$Z(g) = \int_{-\infty}^{\infty} dq \; \underbrace{e^{-\frac{1}{2}q^2 m^2 + Jq - \frac{g}{4!}q^4}}_{e^{-S(q)}}$$

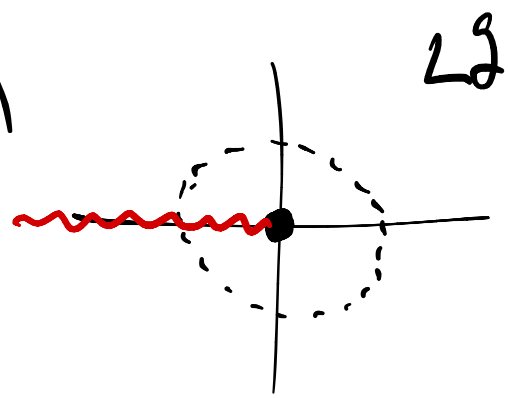
For $g>0$:



if $g = -\epsilon$



$Z = \infty$.



$\Rightarrow$ radius of convergence $= 0$

[Dyson].

- $Z(J=0) = \frac{2}{\sqrt{m^2}} \sqrt{\rho}\, e^{\rho} \underline{\underline{K_{1/4}}}(\rho)$, $\rho \equiv \frac{3m^4}{4g}$.

$(\mathrm{Re}\sqrt{\rho}>0)$

$$K_\nu(\rho) \overset{\rho\to 0}{\sim} \rho^{\nu}.$$

- $G \simeq m^{-2} \sum_{n=0}^{\infty} c_n \left(\frac{g}{m^4}\right)^n$ $\quad c_n$ known.

$$\underline{\underline{c_{n+1} \overset{n\gg}{\simeq} -\frac{2}{3} n c_n}}.$$

$$\Rightarrow |c_n| \sim n!$$

$$\sim \sum_{n=0}^{\infty} \underline{\underline{n!}} \left(\frac{g}{m^4}\right)^n \quad \text{not convergent!}$$

$c_n \sim$ # of diagrams at order n.

~~QFT~~ $\sim n!$

- There's a <u>best</u> order of pert thy for <u>given g/m^4</u>!

<u>Best</u>: $c_{n+1}\left(\frac{g}{m^4}\right)^{n+1} \overset{!}{\sim} c_n\left(\frac{g}{m^4}\right)^n$

$$\Rightarrow \quad \boxed{n_* \sim \frac{3m^4}{2g}} \, .$$

- What does pert. thy miss?

$$S(q) = m^2 q^2 + g q^4$$

$$\tilde{q} \equiv g^{1/4} q \qquad = \frac{m^2}{g^{1/2}} \tilde{q}^2 + \tilde{q}^4 \, .$$

Saddle point: $0 = S'(q_*) = m^2 q_* + \frac{g}{3!} q_*^3$.

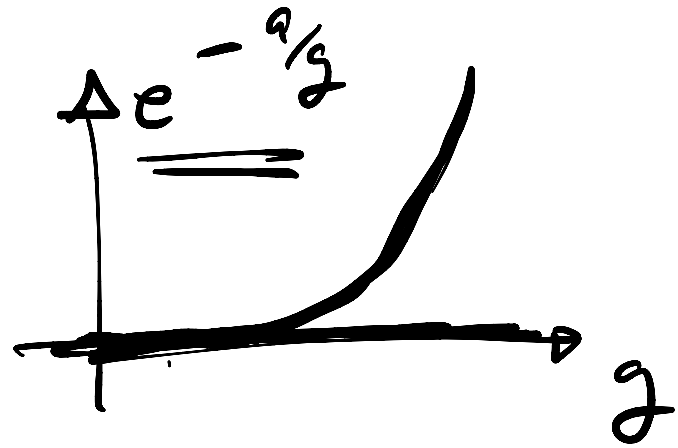
3 sol'n: $q_* = 0$, $S(\checkmark) = 0$; $q_* = \pm i \sqrt{\frac{3! m^2}{g}}$.

$$S\left(q_* = \pm i \sqrt{\frac{3! m^2}{g}}\right) = -\frac{3m^4}{2g} \, .$$

$$\Rightarrow \int dq \, e^{-S(q)} \sim \sum_{\text{saddles } q_*} e^{-S(q_*)} (1 \ldots)$$

other saddles (instantons) contribute

$$\underbrace{e^{-\#/g}}_{e^{-S(q_+)}} \left(1 + \dots\right)$$



Q: what is the Taylor expansion of

$$e^{-1/g}$$

about $g = 0$? 0.

$\Rightarrow$ invisible in pert theory!

- $\exists$ a technique <u>Borel resummation</u>

$$\underline{\underline{B(z)}} \equiv \sum_{m=0} \frac{c_m}{m!} z^m$$

<u>CLAIM</u>:

$$Z(g) = \frac{1}{g} \int dz \, B(z) e^{-z/g} \quad \text{if converges...}$$

• <u>Why</u> does Z satisfy Bessel's eqn?

$$0 \overset{\text{Stokes}}{=} \int_{-\infty}^{\infty} dq \, \frac{\partial}{\partial q} \left(\text{anything} \; e^{-S(q)} \right)$$

<u>eg:</u> $0 = \int dq \, \frac{\partial}{\partial q} \left(q \, e^{-S(q)} \right)$

$\Longrightarrow$ Bessel's eqn.

Schwinger-Dyson eqn.

$$0 = \int [D\phi] \, \underline{\frac{\delta}{\delta \phi(x)}} \left(\underline{\text{anything}} \; \underline{e^{iS[\phi]}} \right)$$

eg: anything $= \phi(y)$

$\Longrightarrow$ equations of motion.

$$|0\rangle = \mathcal{N} e^{-HT/2}|\text{any}\rangle$$

T/2

R

$$= \mathcal{N}\int Dq\, e^{-S}\, \underline{|q_i\rangle}$$

$$\langle 0| = \langle \text{any}|\, e^{-HT/2} = \int Dq\, e^{-S} \underline{\langle q_f|}$$

T/2

L

$$\langle 0|\, f(\hat{q})\, |0\rangle$$

$$= \int Dq\, e^{-S} \int Dq\, e^{-S}\, f(q_f)\,\delta(q_i - q_f)$$

L q_f q_i R

$$= \int_{L+R} Dq\, e^{-S}\, f(\underline{q_f})$$

$$\underline{\mathbb{1} = \int dq\, |q\rangle\langle q|}$$