- NO LECTURE THURS (veterans day)
- If you want, hand in HW07 on Friday.

Recap:

$\sigma, \dfrac{d\sigma}{d\Omega}, \Gamma$ ← measureable

$\uparrow$ final state phase space (to do)

S_{fi}

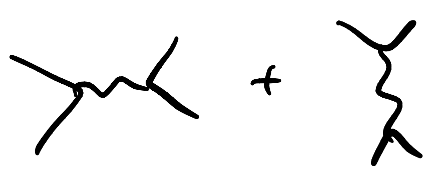
$\uparrow$ LSZ (today) ←

$G^{(n)} = \langle \Omega | T \, \phi_H \cdots \phi_H | \Omega \rangle$

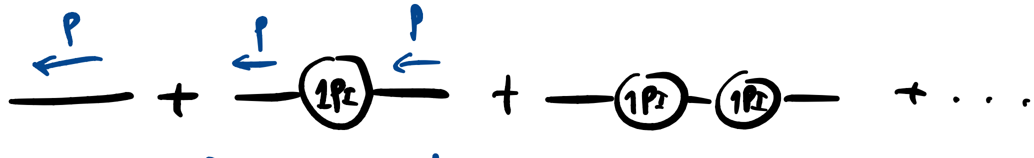
$\uparrow$ $T \to \infty(1 - i\epsilon)$ Feynman contour

$\langle 0 | T \, \phi_I \cdots \phi_I \, e^{-i\int V} | 0 \rangle$

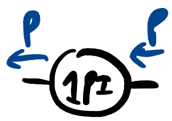
$\uparrow$ wick, exp of vac. bubbles.



Feynman diagrams for S-matrix elts

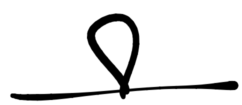
$\tilde{G}^{(2)}(p) =$ 

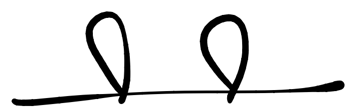
"self-energy"

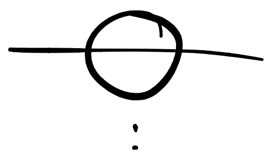


$\equiv -i\Sigma(p) \equiv \sum_i \Big($ all 1PI diagrams w/ 2 nubbins & momentum p $\Big)$

1PI $\equiv$ can't be disconnected by cutting a single propagator.

e.g.: 
yes 1PI


not 1PI.

In terms of $\Sigma(p)$, 

$$\Delta_0(p) \equiv \frac{i}{p^2 - m_0^2 + i\epsilon}$$

$$\tilde{G}(p) = \Delta_0(p) + \Delta_0(p)\left(-i\Sigma(p)\right)\Delta_0(p) + \Delta_0(-i\Sigma)\Delta_0(-i\Sigma)\Delta_0 + \dots$$

$$= \Delta_0(p)\left(1 + \frac{\Sigma}{p^2 - m_0^2} + \left(\frac{\Sigma}{p^2 - m_0^2}\right)^2 + \dots\right)$$

$$\tilde{G}^{(2)}(p) = \frac{i}{p^2 - m_0^2} \frac{1}{1 - \frac{\Sigma}{p^2 - m_0^2}}$$

$$= \frac{i}{p^2 - m_0^2 - \Sigma(p^2)}$$

Lorentz-inv $\Rightarrow$ $\Sigma(p^\mu) = \Sigma(p^2)$

location of pole m_0^2 (bare mass) $\rightsquigarrow$ $\boxed{m_0^2 + \Sigma(m^2) \equiv m^2}$

near the pole : $\tilde{G}^{(2)}(p) = \frac{iZ}{p^2 - m^2} + \underbrace{\text{regular bits}}$

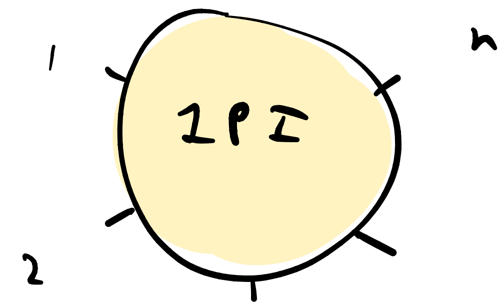
$Z \equiv$ wavef'n renormalization factor.

$=$ amplitude for ϕ to create $\underline{1}$ particle.

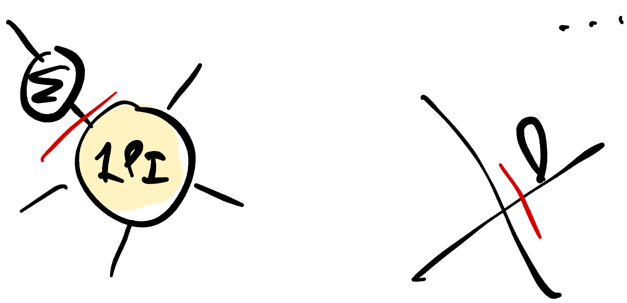
$(|Z| < 1)$

amplitude for ϕ to do something else.

$$\tilde{G}^{(n)}_{1PI}(p_1 \dots p_n) =$$

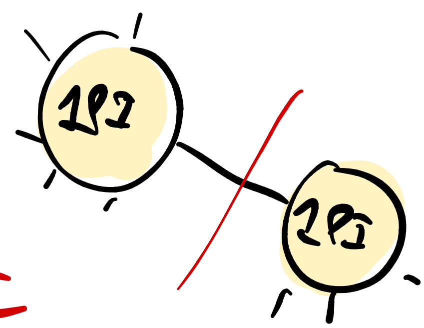


not 1PI:



$$1PI \implies \underline{\underline{\text{amputated}}} = \text{no external legs.}$$

$$\text{amputated} \not\Rightarrow 1PI$$



amputated but not 1PI

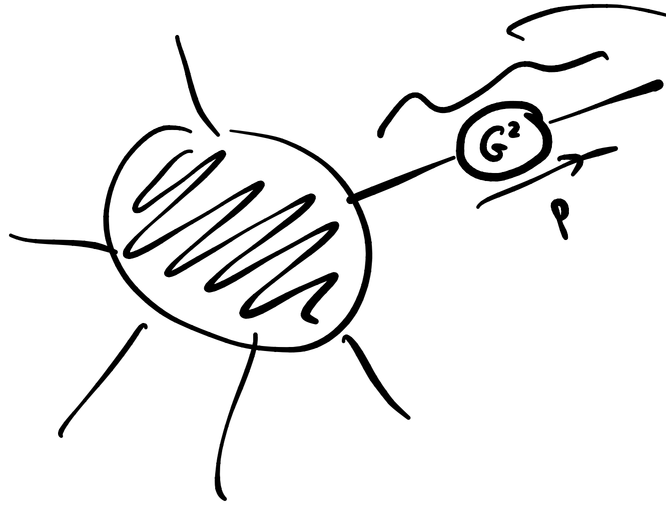
claim:

$$\prod_{a=1}^{n+m} \underbrace{\lim_{p_a^0 \to E_{\vec{p}_a}} \frac{p_a^2 - m^2}{i\sqrt{Z}}}_{\text{amputate external legs.}} \underline{\tilde{G}^{(n+m)}(k_1 \dots k_m, -p_1 \dots -p_n)} \qquad p_a \in \{k, p\}$$

[LSZ reduction formula]

$$= \langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{k}_1 \dots \vec{k}_m \rangle = S_{fi}$$

$\tilde{G}^{(n)} =$



$$\frac{i}{p^2 - m_0^2} \to \frac{i}{p^2 - m_0^2 - \Sigma(p^2)} \overset{p^2 \to m^2}{\sim} \frac{iZ}{p^2 - m^2}$$

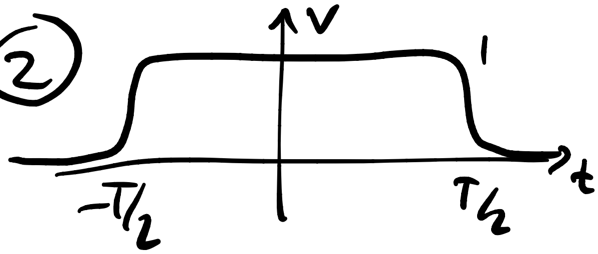
Why is LSZ true?

① for a free field:

$$\sqrt{2\omega_k}\, a_k = i \int d^dx\, e^{ikx} (-i\omega_k + \partial_0) \phi_{free}(t, \vec{x})$$

$$\sqrt{2\omega_k}\, a_k^\dagger = -i \int d^dx\, e^{-ikx} (+i\omega_k + \partial_0) \phi_{free}(t, \vec{x})$$

2



$$\phi(x) \begin{cases} \xrightarrow{t\to-\infty} Z^{1/2}\phi_{in}(x) \\ \xrightarrow{t\to+\infty} Z^{1/2}\phi_{out}(x) \end{cases}$$

★

free fields "at $t=\pm\infty$"

$\Rightarrow$

$$\sqrt{2\omega_k}\, a^{\dagger}_{(in)} = -i \int d^dx\, e^{-ikx}(i\omega_k + \partial_0)\phi_{in}(x)$$

$$\overset{★}{=} \frac{-i}{\sqrt{Z}} \int d^dx\, e^{-ikx}(i\omega_k + \partial_0)\phi(x)\Big|_{t\to-\infty}$$

3 $\sqrt{2\omega_k}\left(a^{\dagger}_{(in)} - a^{\dagger}_{(out)}\right)$

$$\overset{FTC}{=} i\frac{1}{\sqrt{Z}}\int_{-\infty}^{\infty} dt\, \frac{\partial}{\partial t}\left(\int d^dx\, e^{-ikx}(i\omega_k + \partial_0)\phi(x)\right)$$

$$= iZ^{-1/2}\int d^Dx\left(e^{-ikx}\partial_0^2\phi - \phi\,\underbrace{(-\omega_k^2)}_{= -(\vec{\nabla}^2 - m^2)} e^{-ikx}\right)$$

$$\overset{\text{IBP in space}}{=} iZ^{-1/2}\int d^Dx\, e^{-ikx}(\Box + m^2)\phi(x).$$

④ $\langle p_1 \dots p_n | \hat{S} | k_1 \dots k_m \rangle$ assume: $p_i \neq k_j$.

$$= \prod \sqrt{2\omega} \langle \Omega | \prod a_p^{out} \, \hat{S} \, \prod (a_k^{in})^\dagger | \Omega \rangle$$

$$= \prod \sqrt{2\omega} \langle \Omega | T \prod a_p^{out} \, \hat{S} \, \prod (a_k^{in})^\dagger | \Omega \rangle$$

$$= \prod \sqrt{2\omega} \langle \Omega | T \left[\prod a_p^{out} \, \hat{S} \left(a_{k_1}^{in\,\dagger} - \underset{\substack{\uparrow \\ \text{at } t = +\infty}}{a_{k_1}^{out\,\dagger}} \right) \prod_2^m a_k^{in\,\dagger} \right] | \Omega \rangle$$

$$= \prod_2^{n+m} \sqrt{2\omega} \; i Z^{-1/2} \int d^D x_1 \, e^{-ik_1 x_1} (\square_1 + m^2)$$
$$\langle \Omega | T \left(\prod a_p^{out} S \, \phi(x_1) \prod_2 a_k^{in\,\dagger} \right) | \Omega \rangle$$

⑤ Same thing for particles $2 \dots n+m$: $+ \; X$

from $(\square + m^2)$ hits T.

$$\langle p_1 \dots p_n | \hat{S} | k_1 \dots k_m \rangle =$$

$$\prod_{j=1}^m \int d^{d+1} y_j \, e^{+i p_j y_j} \, i \frac{1}{\sqrt{Z}} (\square_j + m^2)$$

$$\prod_{i=1}^n \int d^{d+1} x \, e^{-i k_i x_i} \, i \frac{1}{\sqrt{Z}} (\square_i + m^2) \langle \Omega | T \phi(x_1) \dots \phi(y_n) S | \Omega \rangle$$

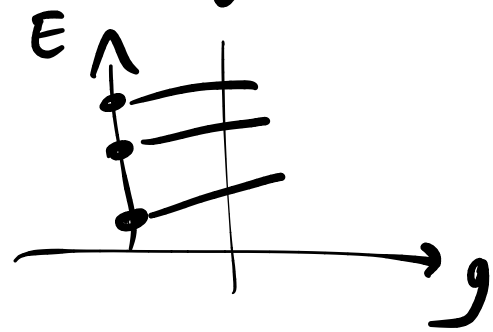
$$+ X'$$

$$\tilde{G}^{(n+m)}(k_1 \dots k_m, -p_1 \dots -p_n) \overset{p_a^2 \to m^2}{\simeq} \prod_{a=1}^{n+m} \frac{i\sqrt{Z}}{p_a^2 - m^2} \langle p_1 \dots p_n | S | k_1 \dots k_m \rangle$$

$+$ less singular terms ∎

Comments about LSZ:

① essential ingredient: label eigenstate of $H_0 + V$ by eigenstates of H_0 by adiabatic continuation.



Doesn't always work:

- eigenstates of H may not be particle states
  e.g.: Conformal field theory (CFT) "unparticles". (not particle excitations)
- asymptotic states may be particles but not quanta of ϕ.

- e.g.: • boundstates

• QCD : asymptotic states = hadrons
$\neq$
fields in $\mathcal{L}$ = quarks & gluons

② Suppose: we have any local operator $\mathcal{O}$ s.t.

$$\langle p | \mathcal{O}_a(x) | \Omega \rangle = Z_{\mathcal{O}_a} e^{ipx}$$

1-particle state

"interpolating operator"

$$G_{\mathcal{O}}^{(n)}(1 \ldots n) \equiv \langle \Omega | T\, \mathcal{O}_1(x) \ldots \mathcal{O}_n(x_n) | \Omega \rangle$$

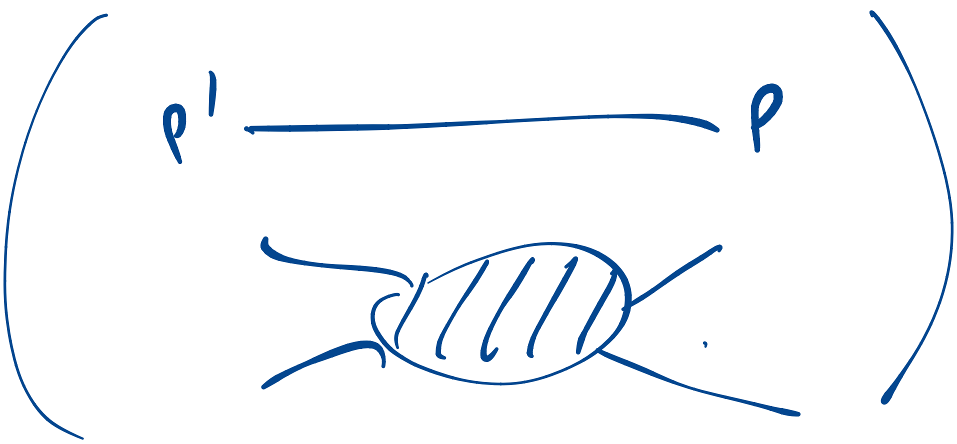
Then:
$$\prod_{a \in i} \left(Z_a^{-1/2} i \int d^D x_a \, e^{-i p_a x_a} (\Box_a + m_a^2) \right)$$
$$\prod_{b \in f} \left(Z_b^{-1/2} i \int d^D x_b \, e^{+i k_b x_b} (\Box_b + m_b^2) \right)$$
$$\times \quad G_{\mathcal{O}}^{(n)}(1 \ldots n)$$
$$= \langle \{p_f\} | \hat{S} | \{k_i\} \rangle .$$

eg: $\mathcal{O}_p(x) = \bar{\Psi}_e(x)\Psi_e(x)$ can create a positronium atom.

on-shell $\equiv$ $p^2 \to m^2$.

$$\langle p_1' p_2' | S | p_1 p_2 \rangle = \delta(p_1 - p_1')\,\delta(p_2 - p_2') \, \#$$

$$+ \quad \underline{\qquad\qquad\qquad\qquad}$$



S-matrix from Feynman diagrams:

$$\langle f | (S - \mathbb{1}) | i \rangle \equiv (2\pi)^D \delta^D\Big(\sum_f p_f - \sum_i p_i\Big)\, i\mathcal{M}_{fi}$$

Rules for $i\mathcal{M}$:

① draw all amputated diagrams w/ the given initial & final states
on R (initial), on L (final)

[leave off ext. lines
not: ]

② for each vertex — impose momentum conservation

— $\cdot\, (-i\lambda)$

③ for each internal line, $\cdot\, \Delta(p) = \dfrac{i}{p^2 - m_0^2 + i\epsilon}$

④ for each loop $\cdot \int d^D k$.

⑤ $\cdot\, S(A)$

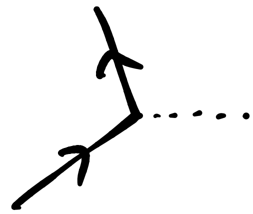
eg: "Nucleon scattering"

$$\mathcal{L}_I = g\,\Phi^* \Phi\, \phi$$

 $= \dfrac{i}{p^2 - m^2}$

 $= \dfrac{i}{p^2 - M^2}$.

$$\begin{cases} \Phi \sim b + c^\dagger \\ \Phi^\dagger \sim c + b^\dagger \\ \phi \sim a + a^\dagger \end{cases}$$

 $= -ig$.

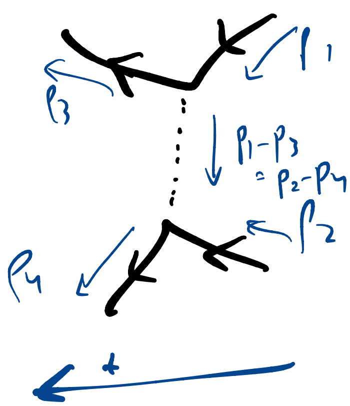
eg:

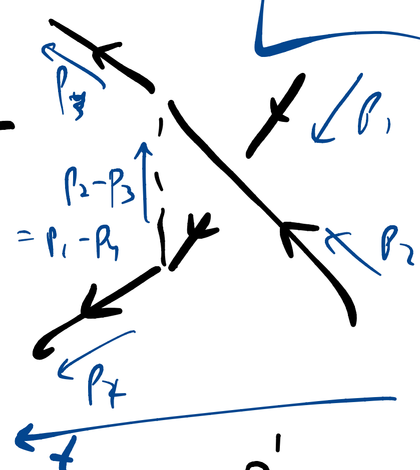
$|i\rangle = |\vec{p}_1, \vec{p}_2\rangle$

$|f\rangle = |\vec{p}_3, \vec{p}_4\rangle$

$= \sqrt{2E_{p_3}}\sqrt{2E_{p_4}}\; \underline{b^\dagger_{p_3} b^\dagger_{p_4}} |0\rangle$.

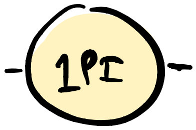
note: arrows on Φ props keep track of charge.

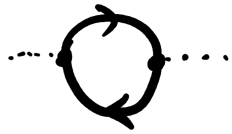
$i\mathcal{M}_{f\leftarrow i} =$ 

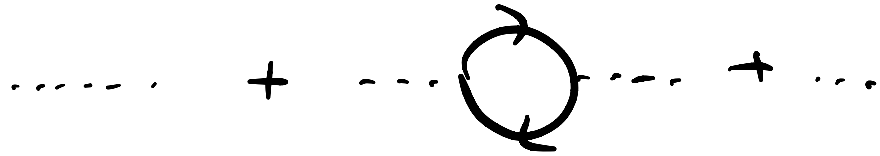
$+$ 

$$= (-ig)^2 \left(\frac{i}{(p_1 - p_3)^2 - M^2} + \frac{i}{(p_2 - p_3)^2 - M^2} \right)$$

Related by Bose sym of initial state.

$$-i\Sigma_\phi(p^2) = $$ 

$$=$$  $$+ \dots$$

$$\tilde{G}^{(2)}_\phi(p) =$$ 

$$= \frac{i}{p^2 - M^2 - \Sigma(p^2)}$$

$$\langle\Omega|\,\phi(x)\,\phi(y)\,|\Omega\rangle$$

$$\mathbb{1} = \sum_n |n\rangle\langle n|$$

$$= |\Omega\rangle\langle\Omega| + \int d^3k\, |k\rangle\langle k| + \text{multiparticle states}$$

$$= \int d^3k \underbrace{\langle\Omega|\,\phi(x)\,|\vec{k}\rangle}_{\equiv \sqrt{Z}\, e^{-ik^\mu x_\mu}\big|_{k^0=\omega_{\vec{k}}}} \underbrace{\langle\vec{k}|\,\phi(y)\,|\Omega\rangle}_{= \sqrt{Z}\, e^{iky}} + \dots$$

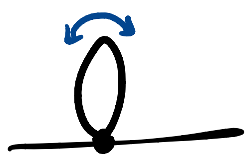
$\int d^4x \, e^{ipx}$ ()

$$= \frac{iZ}{p^2 - m^2}$$

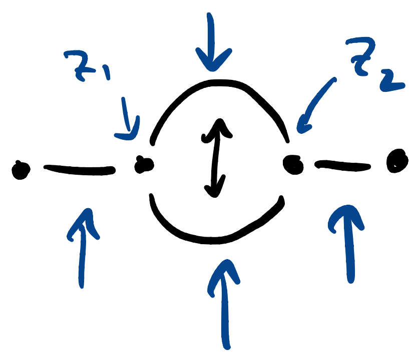
To determine symmetry factors:

method 1: find all perms. of the edges & vertices that preserve the diagram.

fix ext. legs.

eg: 

$S(A) = \frac{1}{\# \ldots}$



method 2:

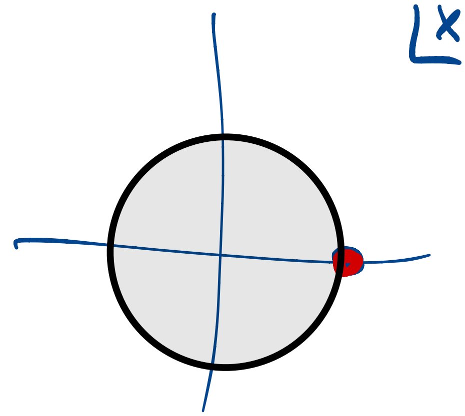
$V = \frac{g\phi^3}{3!}$

$$\langle T\phi_1 \phi_2 e^{-i\int V}\rangle = \langle T(1 - i\int V - \ldots)\phi_1\phi_2\rangle$$

$$= \ldots + \frac{1}{2!}\frac{(-ig)^2}{(3!)^2}\iint_{z_1 z_2} \langle T\phi_1\phi_2 \; \phi^3(z_1)\; \phi^3(z_2)\rangle$$

$$\sum_{n=0}^{\infty} x^n = \underbrace{\frac{1}{1-x}}_{} \quad \text{for } |x| < 1.$$

has a pole at $\underline{x=1}$.



but is analytic everywhere else.

The BHS agree in an open set $\Rightarrow$ agree everywhere.

$$\langle 0| T\left(e^{-i\int g\phi^3}\right)|0\rangle = e^{\sum\left(\overset{\text{connected}}{\text{vac. bubbles}}\right)}$$