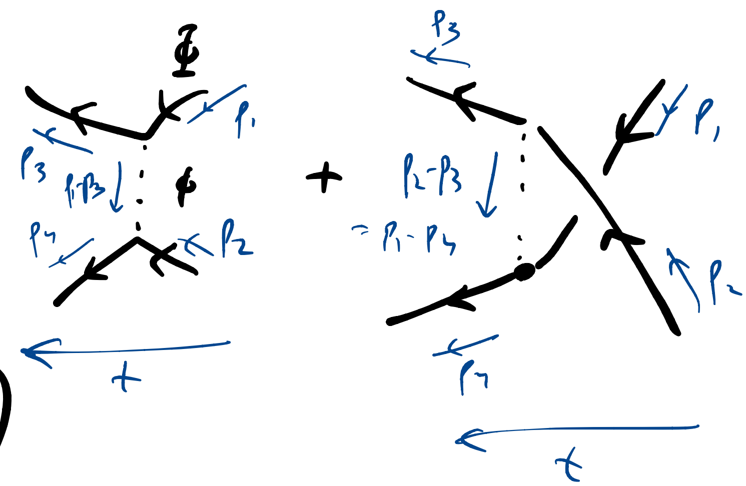
S-Matrix from Feynman diagrams cont'd:

$$\langle f | (\hat{S} - \mathbb{1}) | i \rangle \equiv \delta^D \left(\sum p_f - \sum p_i \right) \, i\mathcal{M}_{fi}$$

$\sum$ (amputated diagrams)

e.g.: $\mathcal{L}_{int} = \phi \, \Phi^* \Phi$

$$i\mathcal{M}_{\Phi\Phi \leftarrow \Phi\Phi} =$$



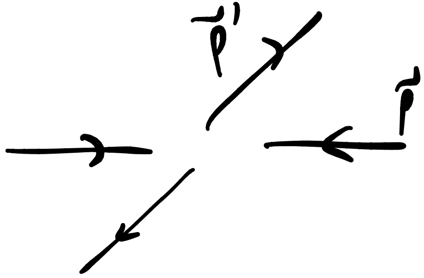
$p_1 - p_3 = -(p_2 - p_4)$

$$= (-ig)^2 \left(\frac{i}{(p_1 - p_3)^2 - M^2 + i\epsilon} + \frac{i}{(p_1 - p_4)^2 - M^2 + i\epsilon} \right)$$

Bose statistics

NR Limit: In com frame: $\begin{cases} \vec{p} \equiv \vec{p}_1 = -\vec{p}_2 \\ \text{and } \vec{p}\,' \equiv \vec{p}_3 = -\vec{p}_4 \end{cases}$

$$|\vec{p}| \ll m \implies p_1^0 = \sqrt{\vec{p}^{\,2} + m^2} \approx m \left(1 + \frac{\vec{p}^{\,2}}{2m^2} + \ldots \right)$$



$$\hat{p}_1 + \hat{p}_2 = \hat{p}_3 + \hat{p}_4$$

$$\Rightarrow \quad |\vec{p}\,'| = |\vec{p}| \ll m .$$

$$(p_1 - p_3)^2 = (\underbrace{p_1^0 - p_3^0}_{m-m})^2 - (\vec{p}-\vec{p}\,')^2 \simeq -(\vec{p}-\vec{p}\,')^2 < 0$$

$$(p_1 - p_4)^2 = (p_1^0 - p_4^0) - (\vec{p}+\vec{p}\,')^2 \simeq -(\vec{p}+\vec{p}\,')^2 < 0$$

$$i\mathcal{M}_{\Phi\Phi\leftarrow\Phi\Phi} \longrightarrow ig^2\left(\frac{1}{(\vec{p}-\vec{p}\,')^2 + M^2} + \frac{1}{(\vec{p}+\vec{p}\,')^2 + M^2} \right)$$

<u>Compare to NRQM:</u>

$$i\mathcal{A}_{Born}(\vec{p}\to\vec{p}\,') = -i \; {}_{NR}\langle \vec{p}\,' | \mathcal{U}(\vec{r}) | \vec{p} \rangle_{NR}$$

$$= -i \int d^d r \; \mathcal{U}(\vec{r}) e^{-i(\vec{p}-\vec{p}\,')\cdot r}$$

$$|\vec{p}\rangle_{NR} = \frac{1}{\sqrt{2E_1}\sqrt{2E_2}} |p_1 p_2\rangle$$

$$\Rightarrow (2m)^2 \, i\mathcal{A}_{Born}(p \to p') = \frac{+ig^2}{(\vec{p}-\vec{p}\,')^2 + M^2}$$

$$\Rightarrow \int d^d r \; \mathcal{U}(r) e^{-i(\vec{p}-\vec{p}\,')\cdot\vec{r}} = -\mathcal{M}_{\substack{NR\\ \text{limit}}} = -\frac{(g/2m)^2}{(\vec{p}-\vec{p}\,')^2 + M^2}.$$

$$\Rightarrow U(\vec{r}) = -\frac{(g/2m)^2}{4\pi r} e^{-Mr}$$

(attractive Yukawa force.)

In d=3 :

$$0 = \left[\int d^4x \; g\phi|\Phi|^2\right]$$

$[\phi]=[\Phi]=1$

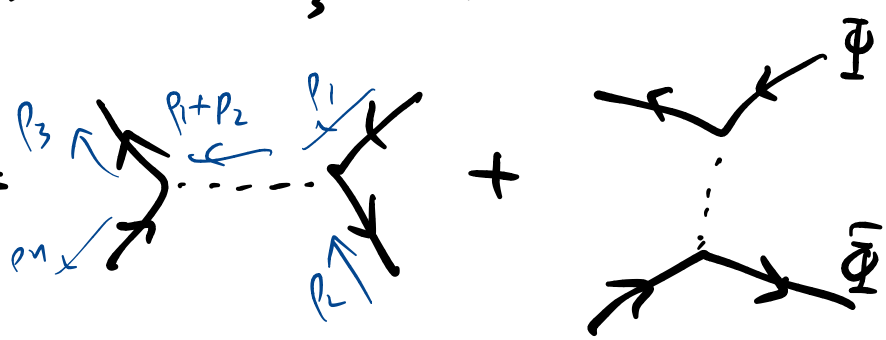
$$= -4 + [g] + 3$$

$\Rightarrow$ g/m is dimless.

$$|i\rangle = \sqrt{2E_{\vec{p}_1}}\sqrt{2E_{\vec{p}_2}}\; b^\dagger_{\vec{p}_1}\; c^\dagger_{\vec{p}_2}\,|0\rangle$$

$$|f\rangle \qquad {}_3 \qquad {}_4 \quad b^\dagger_3 \quad c^\dagger_4\,|0\rangle$$

$i\mathcal{M}_{\Phi\bar{\Phi}\leftarrow\Phi\bar{\Phi}} =$



's channel' 't channel'

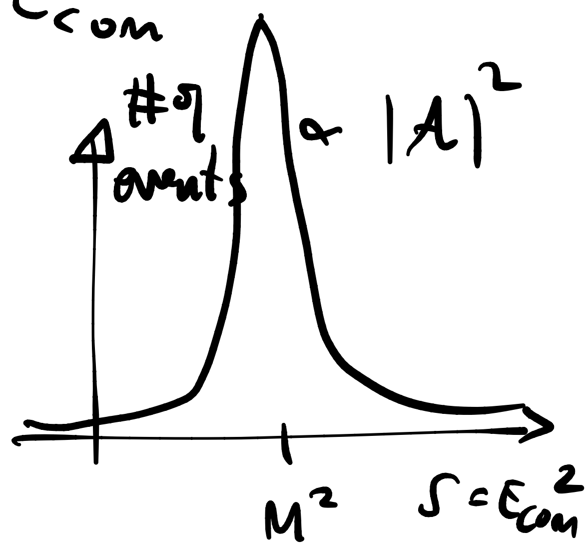
$$= (-ig)^2\left(\frac{i}{(p_1+p_2)^2 - M^2 + i\epsilon} + \frac{i}{(p_1-p_3)^2 - M^2 + i\epsilon}\right)$$

CLAIM: if $p_1^2 = m^2$, $p_2^2 = m^2$

it's possible for $(p_1+p_2)^2 = M^2$.

$$s \equiv (p_1+p_2)^2 \underset{\text{com}}{=} E_{com}^2$$

$$\mathcal{A} \propto \frac{1}{s - M^2 + i\epsilon}$$



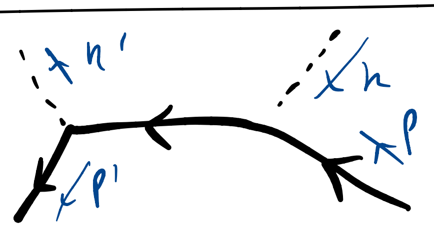
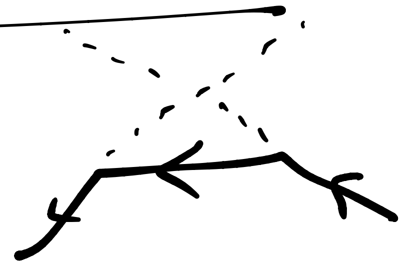
resonance

CLAIM: If $p_1^2 = m^2$, $p_2^2 = m^2$

then $(p_1 - p_3)^2 \equiv t$

and $(p_1 - p_4)^2 \equiv u$

cannot equal M^2

$i\mathcal{M}_{\Phi\phi \leftarrow \Phi\phi} =$  $+$ 

$$= (-ig)^2 \left(\frac{i}{(p+k)^2 - m^2 + i\epsilon} + \frac{i}{(p-k')^2 - m^2 + i\epsilon} \right)$$

Notice: Internal line represents a "virtual particle" $k^2 \neq \text{mass}^2$.

4.7 S-matrix → observable physics

Lifetimes: unstable particle
(stable in free theory, can decay because of V)

in its rest frame $p^\mu = (M, \vec{0})^\mu$.

let $dP \equiv$ prob. (particle decays into $\{f\}$ during time T.)

decay rate: $d\Gamma = \dfrac{dP}{T}$

$$\Gamma = \int_{\substack{\text{final}\\\text{states}}} d\Gamma = \frac{\#\text{ of decays per unit time}}{\#\text{ of particles}} \equiv \frac{1}{\tau}$$

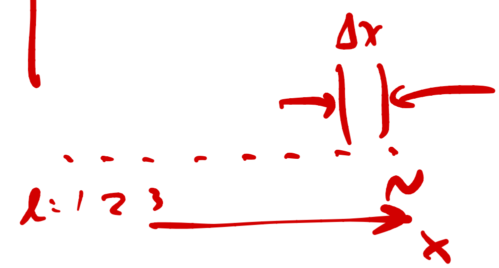
lifetime

$$dP = \frac{|\langle f|\hat{S}|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}\, d\Pi_f \quad \checkmark$$

$$S_{n\leftarrow 1}\left(\{p_j\}_{j=1}^{n} \leftarrow (M,\vec{0})\right)$$

① $\underline{d\Pi_f}$ $\equiv$ volume of the region of final-state phase space.

$$\boxed{\int_{\text{anyone}} d\Pi = 1.} \qquad d\Pi_f \propto \prod_{j=1}^{n} d^d p_j$$



In a box on a lattice: $\Delta x = L/N$

$$\begin{cases} x_l = \dfrac{l}{N} L \\[2mm] p^l = \dfrac{2\pi}{L}\dfrac{l}{N} \end{cases} \qquad l = 1..N.$$

$$\Delta x \sum_l = \frac{L}{N}\sum_l \underset{L\text{ fixed}}{\overset{N\to\infty}{\rightsquigarrow}} \int dx = L$$

$$\frac{1}{2\pi}\Delta p \sum_l = \frac{1}{2\pi}\frac{2\pi}{LN}\sum_l \underset{L\text{ fixed}}{\overset{N\to\infty}{\rightsquigarrow}} \int dp = \frac{1}{L}.$$

$$\Longrightarrow d\Pi = \prod_{j=1}^{n}\left(V\, d^d p_j\right) \qquad (V \equiv L^d)$$

② $\langle i|i\rangle\langle f|f\rangle$?

$$|\vec{p}\rangle = \sqrt{2\omega_p}\, a^\dagger_{\vec{p}}|0\rangle$$

$$\langle \vec{k}|\vec{p}\rangle = \sqrt{2\omega_p 2\omega_k}\ \underbrace{\langle 0|\, a_k a_p^\dagger|0\rangle}_{}$$

$$= \langle 0|[a_k, a_p^\dagger]|0\rangle$$

$$= \delta^d(\vec{k}-\vec{p})$$

$$= 2\omega_p\, \delta^d(\vec{k}-\vec{p})$$

$$\langle \vec{p}|\vec{p}\rangle = 2\omega_p\, \delta^d(0) \quad ?!$$

$$= 2\omega_p \left(\int dx\, e^{i(p-0)x}\right)^d$$

$$= 2\omega_p V .$$

$\Rightarrow \boxed{n \leftarrow 1}$ in the rest frame

$$|i\rangle = \sqrt{2M}\, a^\dagger_{\vec{0}}|0\rangle \Rightarrow \langle i|i\rangle = 2MV$$

$$|f\rangle = |\{p_j\}\rangle \rightarrow \langle f|f\rangle = \prod_j^n (2\omega_j V)$$

$$\langle f|(S-\mathbb{1})|i\rangle = i\,\delta^D(p_T)\,\langle f|\mathcal{M}|i\rangle$$

$$\uparrow\ (p_T = \sum p_f - \sum p_i)$$

$$|\langle f|(S-\mathbb{1})|i\rangle|^2 = \delta^D(0)\,\delta^D(p_T)\,|\mathcal{M}_{fi}|^2$$

$$= \underline{\underline{VT}}\,\delta^D(p_T)\,|\mathcal{M}_{fi}|^2$$

$$\Rightarrow dP = \frac{|\langle f|(S-\mathbb{1})|i\rangle|^2}{\langle i|i\rangle\langle f|f\rangle}\,d\Pi_f$$

$$= \frac{|\mathcal{M}|^2\,\cancel{V}T\,\delta^D(p_T)}{2M\cancel{V}\,\prod_j^n(2\omega_j\cancel{V})}\prod_j^n(\cancel{V}\,d^dp_j)$$

$$\begin{cases} d\Gamma = \dfrac{dP}{T} = |\mathcal{M}|^2\,\dfrac{d\Pi_{LI}}{2M} \\[2ex] d\Pi_{LI} \equiv \prod\limits_{\text{final state } j}^{n} \dfrac{d^dp_j}{2\omega_j}\,\delta^D(p_T) \end{cases}$$

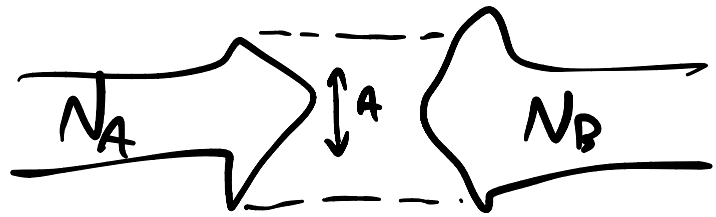
Lorentz-invariant measure on final-state phase space

$$\boxed{d\Gamma = \frac{|M|^2}{2M}\, d\Pi_{LI}}$$

decay rate in the rest frame.

$$\left(\frac{\Gamma^{\text{rest frame}}}{\Gamma} = \frac{E}{m} = \gamma \geq 1. \right)$$

CROSS SECTIONS.



$$\underline{\underline{\text{\# of events of interest}}} = \frac{N_A N_B}{A}\, \underline{\underline{d\sigma}}$$

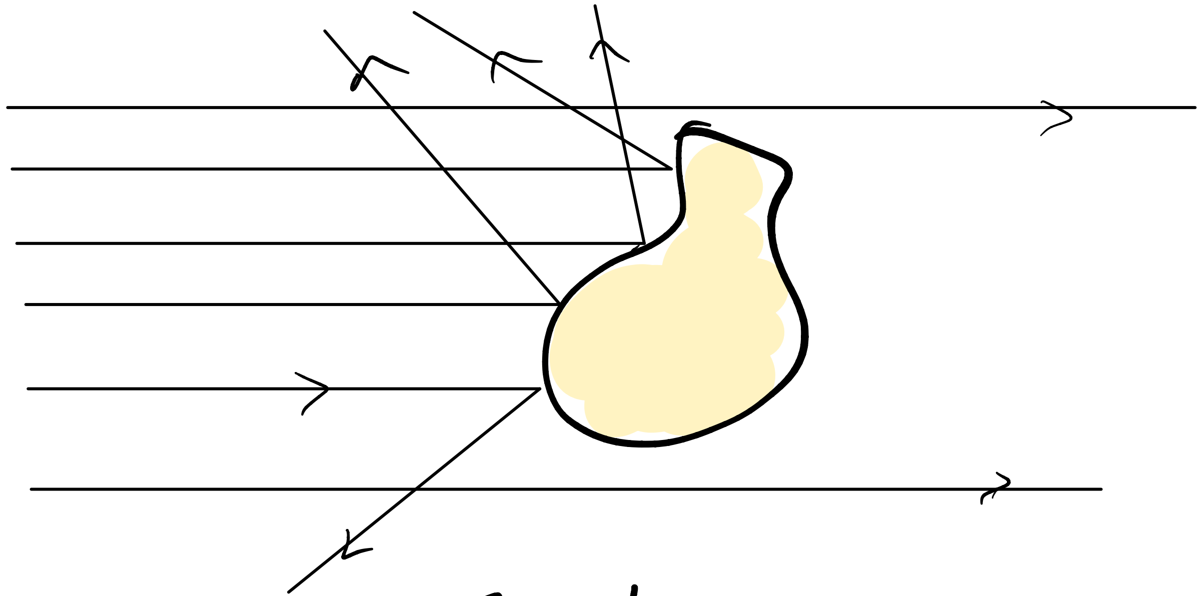
↑ diff'l scattering cross-section.

"of interest" could be

$\underline{\underline{\text{eg}}}$:

particles that go into solid angle $d\Omega$

$$\sigma = \int_{\text{all dirs}} \frac{d\sigma}{d\Omega}\, d\Omega .$$



Relate σ to S-matrix :

scattering rate : $dw_{fi} \equiv \frac{dP_{fi}}{T}$

intensive version of the def of σ : $d\sigma = \frac{dw}{j}$

$j \equiv$ particle current density aka particle flux $= \frac{\# \text{ of particles}}{\text{area}}$

$= \frac{\text{relative velocity of A \& B}}{\text{volume}} = \frac{|\vec{v}_A - \vec{v}_B|}{V}$

$$d\sigma = \frac{dw_{fi}}{j} = \frac{dP_{fi}}{T\,j} = \frac{V}{T}\frac{1}{|\vec{v}_A - \vec{v}_B|}\,dP_{fi}$$

$$dP = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}\,d\Pi_f$$

$$|i\rangle = |\vec{p}_A, \vec{p}_B\rangle \Rightarrow \langle i|i\rangle = (2\omega_A V)(2\omega_B V)$$

$$\Rightarrow dP = \frac{TV\ |\mathcal{M}|^2}{V^2\ 2\omega_A 2\omega_B}\,d\Pi_{LI}$$

$$\Rightarrow d\sigma = \frac{V}{T}\frac{1}{|v_A - v_B|}\,dP$$

$$\boxed{d\sigma = \underbrace{\frac{1}{2\omega_A 2\omega_B}\frac{1}{|\vec{v}_A - \vec{v}_B|}\,d\Pi_{LI}}_{\text{kinematics}}\ |\mathcal{M}|^2}$$

dynamics

Two-body phase space $(n=2)$:

$$d\Pi_{LI}^{n=2} = \underline{\delta}^{D}(\underbrace{P_T}) \frac{d^d p_1}{\underline{2}E_1} \frac{d^d p_2}{\underline{2}E_2}$$

$$= \frac{1}{4(2\pi)^{2d-(d+1)}} \frac{\underline{\delta}(E_1+E_2-\underline{E}_{cm})}{E_1 E_2} d^d p_1$$

$$E_i = \sqrt{\vec{p}_i^{\,2}+m^2}$$

$$\hat{p}_1 + \hat{p}_2 = \hat{p}_{cm}$$

$$\Rightarrow \vec{p}_2 = \vec{p}_{com} - \vec{p}_1$$

$$d\Omega\, dp_1\, p_1^{d-1}$$

$$= \frac{1}{4(2\pi)^{d-1}} \frac{d\Omega\, p_1^{d-1} dp_1}{E_1 E_2} \theta(p_1)\delta(x)$$

$$x(p_1) = E_1(p_1) + E_2(p_2 = p_{cm} - p_1) - E_{com}$$

$$\overset{\text{com}}{=} \frac{1}{4(2\pi)^{d-1}} \frac{d\Omega\, p_1^{d-2}}{E_1 E_2} \cdot \underbrace{dx\, \delta(x)}_{=1} \cdot \frac{E_1 E_2}{E_{cm}} \theta(E_{cm} - m_1 - m_2)$$

$$= \frac{1}{4(2\pi)^{d-1}} \frac{d\Omega\, p_1^{d-2}}{E_{cm}} \theta(E_{cm} - m_1 - m_2).$$

So in $d=3$: COM: $\vec{k}_A = -\vec{k}_B$ initial momenta.

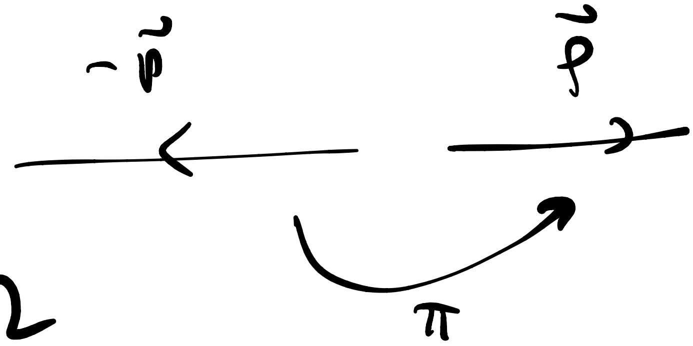
$$|v_A - v_B| = \left| \frac{|k_A|}{E_A} + \frac{|k_B|}{E_B} \right|$$

$$= |k_A| \cdot \frac{E_{cm}}{E_A E_B}$$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{d\Omega}\right)^{2\leftarrow 2}_{COM} = \frac{1}{64\pi^2 E_{cm}^2} \frac{|\vec{p}_1|}{|\vec{k}_A|} |\mathcal{M}|^2 \, \theta(E_{cm} - m_1 - m_2)}$$

$$\sigma = \int_{all} \frac{d\sigma}{d\Omega} d\Omega$$

WARNING: if the 2 particles in the final are identical



$$\Rightarrow \sigma = \frac{1}{2} \int_{4\pi} \frac{d\sigma}{d\Omega} d\Omega.$$

Interlude: old-fashioned pert thy.

$H = H_0 + V$. suppose continuous spectrum.

$$\Rightarrow \begin{cases} H_0|\varphi\rangle = E|\varphi\rangle \leftarrow \text{known} \\ H|\psi\rangle = E|\psi\rangle \end{cases}$$

$$\Rightarrow |\psi\rangle = |\varphi\rangle + \underbrace{\frac{1}{E - H_0 + i\epsilon}}_{\Pi} V|\psi\rangle$$

Lippman-Schwinger eqn.

Let $V|\psi\rangle \equiv T|\varphi\rangle$
'transfer matrix'

$$\Rightarrow |\psi\rangle = |\varphi\rangle + \Pi T|\varphi\rangle$$

$$= (1 + \Pi T)|\varphi\rangle$$

V(BHS) $\Rightarrow V|\psi\rangle = T|\varphi\rangle$

$$= V|\varphi\rangle + V\Pi T|\varphi\rangle$$

$\Leftrightarrow \quad T = V + V\Pi T$

$$= V + V\Pi(V + V\Pi T)$$

$$= V + V\Pi(V + V\Pi(V + V\Pi \cdots))$$

$$= \left(\frac{1}{1 - V\Pi}\right) V$$

let $\mathbb{1} = \sum_i |\varphi_i\rangle\langle\varphi_i|$ eigenstates of H

$\rightarrow \quad T_{fi} \equiv \langle \varphi_f | T | \varphi_i \rangle$

$$= V_{fi} + \underbrace{V_{fj}\,\Pi(j)\,V_{ji}}_{} + \cdots$$

$\uparrow$ Born approx

$V_{fj} = \langle \varphi_f | V | \varphi_j \rangle$

$\Pi(j) \equiv \dfrac{1}{E - E_j + i\epsilon} \qquad E = E_i = E_f.$

eg: $V = \frac{e}{2}\int d^dx\, \Phi(x)\,\Phi^\dagger(x)\,\phi(x)$.

$|i\rangle = |\vec{p}_1\,\vec{p}_2\rangle$, $|f\rangle = |\vec{p}_3, \vec{p}_4\rangle$

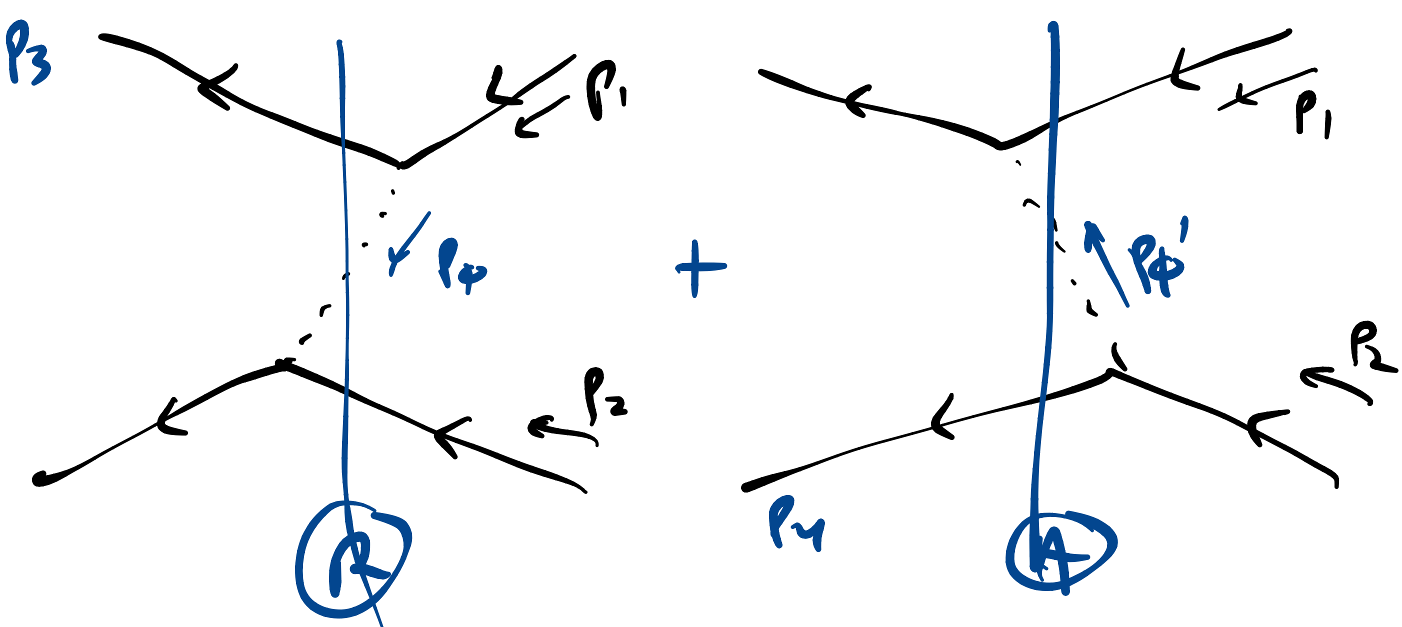
of Φ's

$$\Rightarrow T_{fi} = \underbrace{V_{fi}}_{=0} + \sum_n V_{fn}\frac{1}{E_i - E_n + i\epsilon}V_{ni} + \dots$$

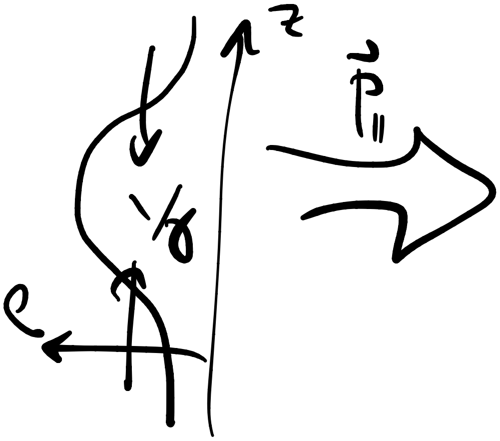
$p_1p_2 \neq p_3p_4$

t

who are n?



$|n^R\rangle = |\vec{p}_2;\ \vec{p}_3, \vec{p}_\phi\rangle$ $|n^A\rangle = |\vec{p}_1;\ \vec{p}_4, \vec{p}_\phi'\rangle$



$$|i\rangle = \int d^2p_\perp \, \Psi(p_\perp) \, |\vec{p}\rangle$$

$$\Psi(p_\perp) \sim e^{i\, p_\perp (z - z_0) + \gamma p_\perp^2}$$

$$G(p) \overset{p^2 \to m^2}{\simeq} \frac{iZ}{p^2 - m^2 + i\epsilon}$$

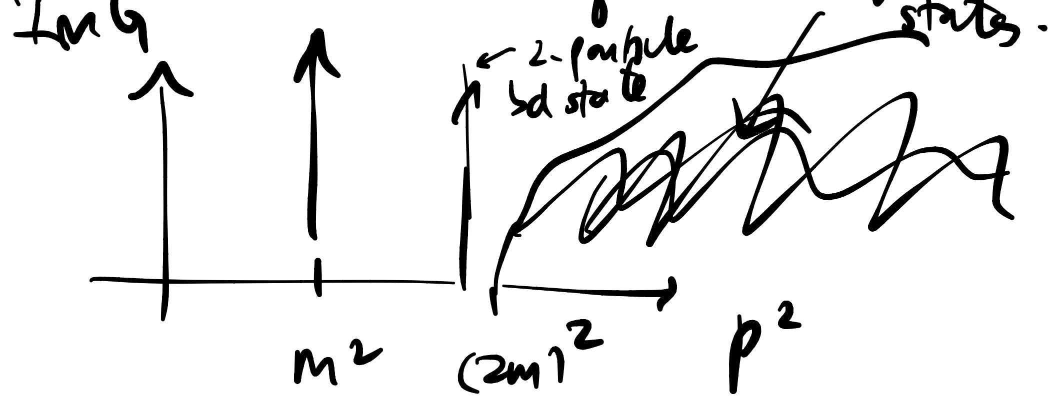
$$\mathrm{Im}\, G(p) = Z\,\delta(p^2 - m^2) + \dots$$

$$= \langle 0 |\, \phi \;\;\; \phi \,| 0 \rangle$$

$$\mathbb{1} = \sum |n\rangle\langle n|$$

eigenstates of H

"spectral weight" of ϕ



$\mathrm{Im}\, G \propto$ density of eigenstates created by ϕ $\underline{\underline{|\langle 0|\phi|n\rangle|^2}}$

$\underline{\geq 0.}$