What's a virtual particle? (continued)

$$T = V + V\Pi V + V\Pi V\Pi V + \dots$$

$$\Pi = \frac{1}{E - H_0 + i\epsilon}$$

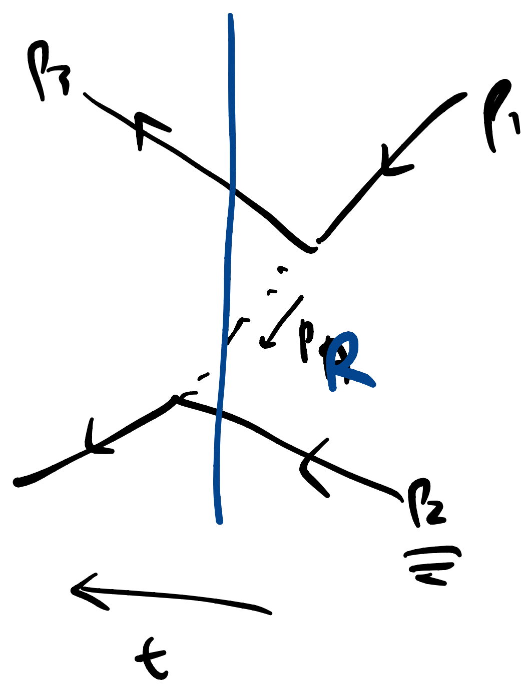
eg: $V = g \int d^d x \; \Phi^*(x)\,\Phi(x)\,\phi(x)$

$|i\rangle = |\vec{p}_1, \vec{p}_2\rangle$ 2 nucleons

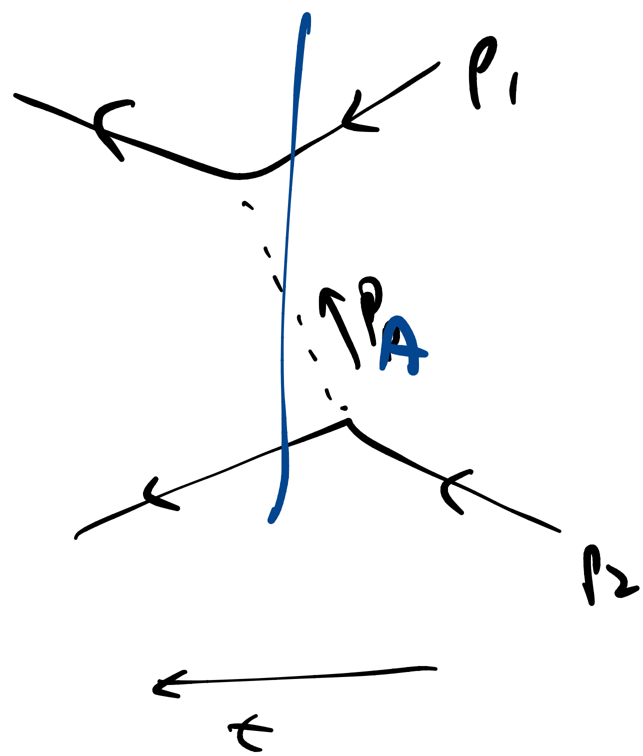
$|f\rangle = |\vec{p}_3, \vec{p}_4\rangle$ " "

$$T_{if} = \sum_{n=R,A} V_{fn} \frac{1}{E_i - E_n + i\epsilon} V_{ni} + \dots$$

$\boxed{R:}$



$\boxed{A:}$



$$|R\rangle = |p^3, p_\phi, p^2\rangle \qquad |A\rangle = |p^4, p_\phi, p_1\rangle$$

$$V_{ni}^R = \langle R|\, V \,|\vec{p}_1 \vec{p}_2\rangle$$

$$= \langle p^3, p_\phi |V| p_1\rangle \cancelto{1}{\langle p_2|p_2\rangle}$$

$$= g \int d^d x\, \underbrace{\langle p_\phi|\, \phi(x)\,|0\rangle_\phi}_{e^{-i\vec{p}_\phi \vec{x}}} \underbrace{\langle p^3|\, \Phi^\dagger \Phi\,|p^1\rangle}_{e^{-i(\vec{p}_3-\vec{p}_1)x}}$$

$$= g\, \delta^d(\vec{p}_1 - \vec{p}_3 - \vec{p}_\phi).$$

$$\Rightarrow T_{fi} = \sum_{n=R,A} \int d^d p_\phi\, \delta^d(p_1 - p_3 - p_\phi)\, \delta^d(p_2 - p_4 - p_\phi)\, \frac{g^2}{E_i - E_n + i\epsilon}$$

on-shell ϕ particles:

$$E_\phi = \sqrt{|p_\phi|^2 + m_\phi^2}$$

$$E_{n=R} = E_3 + E_\phi^R + E_2 = E_3 + \sqrt{|\vec{p}_1 - \vec{p}_3|^2 + m_\phi^2} + E_3$$

$$\Rightarrow E_i - E_{n=R} = E_1 - E_3 - E_\phi^R = -\Delta E - E_\phi$$

$$\Delta E = E_1 - E_3 = E_2 - E_4.$$

$\underline{A}$: $\tilde{p}_\phi = p_2 - p_4 = p_3 - p_1$

$$\Rightarrow E_\phi^A = \sqrt{(p_1-p_3)^2 + m_\phi^2} = \sqrt{(p_2-p_4)^2 + m_\phi^2} = E_\phi^R$$

$$= E_\phi .$$

$$\Rightarrow E_i - E_{n=A} = +\Delta E - E_\phi$$

$$\rightarrow \sum_{n=R,A} \frac{g^2}{E_i - E_n} = \frac{g^2}{-\Delta E - E_\phi} + \frac{g^2}{+\Delta E - E_\phi}$$

$$= 2E_\phi \frac{g^2}{\Delta E^2 - E_\phi^2} = 2E_\phi \cdot \underbrace{\frac{g^2}{k^2 - m_\phi^2 + i\epsilon}}$$

$$k^\mu = p_1^\mu - p_3^\mu = \underbrace{(\Delta E, \vec{p}_\phi)}^\mu .$$

$$k^2 = (\Delta E)^2 - p_\phi^2 \neq m_\phi^2 .$$

4.9 Observables from G_R

$$G^R_{AB}(x) = -i\,\theta(t)\langle [A(t,\vec{x}), B(0,\vec{0})]\rangle.$$

<u>consider</u>: $\overset{\text{perturb.}}{\delta H(t)} = \int d^dx\, \underset{\uparrow \text{ we pick.}}{f(t,\vec{x})}\, B(\vec{x})$

Let ρ_0 be some nice state.

eg: $\rho_0 = \frac{e^{-\beta H_0}}{Z} \xrightarrow{\beta\to\infty} \underset{\uparrow \text{ g.s. of } H_0}{|0\rangle\langle 0|}$.

$$\rho(t) = U(t)\,\rho_0\,U(t)^\dagger$$

<u>observe</u>:

$$\langle A(\vec{x})\rangle(t) = \mathrm{tr}\, \underline{\rho(t)}\, A(\vec{x})$$

$$= \mathrm{tr}\,\rho_0\, \underline{U^\dagger(t)}\, A(x)\, U(t)$$

$$= \mathrm{tr}\,\rho_0\, U_I^\dagger\, \underline{A(x,t)}\, U_I(t)$$

<u>interaction picture</u>:

$$\begin{cases} U_I(t) = \mathcal{T} e^{-i\int^t \delta H(t')dt'} \\ A(x,t) = e^{iH_0t} A e^{-iH_0t}. \end{cases}$$

Linear Response: assume f small.

$$\delta \langle A \rangle (t, \tilde{x}) = -i \,\mathrm{tr}\, \rho_0 \int^t dt' \, [A(t,\vec{x}), \delta H(t')]$$

$$= -i \int^t d^d x' dt' \langle [A(t,\vec{x}), B(t',\vec{x}')] \rangle f(t', \tilde{x}')$$

$$= -i \int_{-\infty}^{\infty} d^D x' \, \theta(t') \langle [A(x), B(x')] \rangle f(x')$$

$$= \int d^D x' \; G^R_{AB}(x, x') f(x').$$

$$\left[\delta \langle A \rangle (\omega, \vec{k}) = \underbrace{G^R_{AB}(\omega, \tilde{k})}_{\text{"susceptibility"}} \cdot f(\omega, \vec{k}). \right]$$

suppose conserved charge.

$\underline{g}$: $\underline{J_\mu(x)}$.

$$E_x = i\omega A_x \qquad (A_0 = 0)$$

$$\delta H = A_x J^x \qquad \text{i.e.} \begin{cases} \hat{B} = J_x \\ f = A_x \end{cases}$$

measure: $\langle J_x \rangle$. i.e. $\hat{A} = J_x$.

Fact: $\langle \vec{J} \rangle_{E=0} = 0$ [Bloch].

$$\langle J(\omega, \vec{k}) \rangle = G^R_{JJ}(\omega, \vec{k})\, A_x(\omega, \vec{k})$$

$$= G^R_{JJ}(\omega, \vec{k}) \cdot \frac{E_x(\omega, \vec{k})}{i\omega}$$

Ohm's Law: $J = \sigma E$

σ ← conductivity

$$\Rightarrow \boxed{\sigma(\omega, \vec{k}) = \frac{G^R_{JJ}(\omega, \vec{k})}{i\omega}}$$

Kubo formula.

5 Spinor fields & fermions

5.1 Lightning summary of group theory.

Group: $G = \{ g_i \}$

s.t. ① $g_1 g_2 \in G$

② $(g_1 g_2) g_3 = g_1 (g_2 g_3) \in G$

③ $\exists\, g_0$ s.t. $g_0 g_i = g_i \quad \forall g_i$

④ $\forall i \quad \exists\, g_i^{-1}$ s.t. $g_i g_i^{-1} = g_0$.
$= g_i^{-1} g_i$.

[If $g_i g_j = g_j g_i \quad \forall i, j$
G is abelian.]

order of $G \quad |G| \equiv$ # of elements.

A Lie group has elts labelled by continuous parameters.

i.e. a Lie group $G = \{ g(\theta_1 \dots \theta_n) \}$

$n = \dim G$ = # of coords.

$|G| = \infty$.

A <u>Representation</u> R of G

is a map $G \longrightarrow$ linear ops on some $\mathcal{H}$

$$g \longmapsto \hat{D}_R(g) : \mathcal{H} \to \mathcal{H}$$

respecting the group law (a homomorphism)

$$\underbrace{\hat{D}_R(g_1)\, \hat{D}_R(g_2)}\ \overset{\downarrow}{=}\ \hat{D}_R(g_1 g_2)$$

If: $\quad \wedge\ e^{i\phi(g_1, g_2)}$ "projective representation"

Regard $R \simeq R'$ if $\exists\, S$ s.t. $D_R(g) = S^{-1} D_{R'}(g)\, S \quad \forall g$

R is <u>reducible</u> if $D_R \simeq \left(\begin{array}{c|c|c} D_{R_1} & 0 & \\ \hline 0 & D_{R_2} & 0 \\ \hline & & \dots \end{array}\right)$

$$R \simeq R_1 \oplus R_2 \oplus \cdots$$

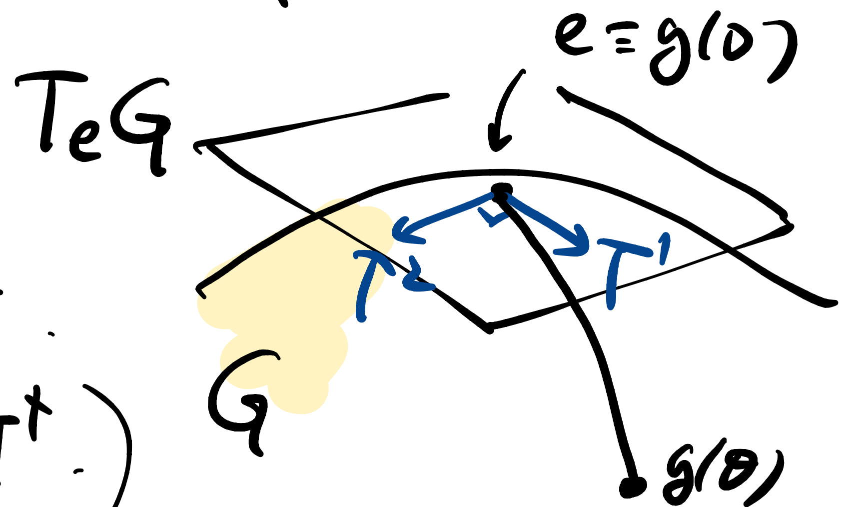
$$\dim R = \dim \mathcal{H} .$$

Lie algebra: Let $g(\theta_a=0)$ be the identity

$$D_R(g(\theta_a \sim 0)) = \mathbb{1} + i\,\theta_a T_R^a + O(\theta^2)$$

$$\underline{T_R^a} \equiv -i\,\partial_{\theta_a} D_R(g(\theta))\Big|_{\theta=0}.$$

generators of G in rep R.



$$D_R(\overbrace{g(\theta)}^{\equiv g_{1,2}}{}_{1,2}) = e^{-i\theta_a^{1,2} T_R^a}$$

(is unitary if $T = T^\dagger$.)

$$D_R(g_1)\cdot D_R(g_2) \overset{\text{def of } R}{=} D_R(g_1 g_2) \overset{\exists \theta^3 \text{ s.t.}}{=} e^{-i\theta_a^3 T_R^a}$$

<u>θ's small</u>: $\theta^3_a = \theta^1_a + \theta^2_a - \frac{1}{2}\theta^1_b\theta^2_b f^{ab}_c + \mathcal{O}(\theta)^3$

f^{ab}_c defined by

$$\boxed{[T^a, T^b] = i f^{ab}_c T^c}$$

(Lie algebra .)

Structure constants

[same for every R .

<u>Convention</u>: $\mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$.

Lie algebra

Lie group

$G_0 \sim \exp(\mathfrak{g})$

component of G contains $\mathbb{1}$.

Casimir $\equiv$ op that commutes w/ generators

$\propto \mathbb{1}$ on an irreducible rep
$\equiv$ irrep.

---

eg: rotation group in 3d.

generators of rotation about x, y, z axes

$J^{x,y,z}$ satisfy $(i,j,k = x,y,z)$

$$\boxed{[J^i, J^j] = i\epsilon^{ijk} J^k}$$

$\leftarrow$ SO(3) = SU(2).

$$f^{ij}_{\ \ k} = \epsilon^{ijl} \delta_{lk}$$

A Casimir is $J^2 \equiv \sum_i (J^i)^2$

$$[J^2, J^i] = 0 \quad \forall i.$$

$J^2 = j(j+1)\,\mathbb{1}$ on the spin j rep.

$$\dim R_j = 2j+1.$$

A finite rot. on $\mathcal{H}_j$

$$D(\hat{n},\theta) = e^{-i\theta\hat{n}\cdot\vec{J}}$$

3 parameters

$\hat{n}\cdot\hat{n}=1.$

$\dim SO(3)$

$= \dim SU(2) = 3.$

eg: j=1: $(J^i_{(j=1)})_{jk} = i\epsilon^{ijk}$

eg: $(J^3_{(j=1)}) = i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$j=\frac{1}{2}$: $(J^i_{j=1/2}) = \frac{1}{2}\sigma^i$.

j=0: $(J^i_{j=0}) = 0$.

i.e. $e^{i\theta J_{0=0}} = \mathbb{1}.$

spin $j\in\mathbb{Z}/2$: $(J^3)_{mm'} = \delta_{mm'}m$ $\quad (J^\pm)_{mm'} = (J^1 \pm iJ^2)_{mm'}$

$m, m' \in \{-j, -j+1 \dots j-1, j\}.$ $\quad = \delta_{m', m\mp 1}\sqrt{(j\mp m)(j\pm m+1)}.$

$\circledast$ $[J^i, J^j] = i\epsilon^{ijk} J^k$ $\equiv$ "J is a vector"

$|\psi\rangle \to D_R |\psi\rangle$

$\Rightarrow \mathcal{O} \to D_R \mathcal{O} D_R^\dagger$.

i.e.: $D_{(j=1)}(g)^k{}_j J^j = D_R(g) J^k D_R(g)^\dagger$.

inf'l of this is $\circledast$

$[J^i, K^j] = i\epsilon^{ijk} K^j$ $\equiv$ "K is a vector".

General d: let $J^i \equiv \epsilon^{ijk} \underset{\sim}{J^{jk}}$

not about i — not in jk plane

$$\boxed{[J^{ij}, J^{kl}] = i\left(\delta^{jk} J^{il} + \delta^{il} J^{jk} - \delta^{ik} J^{jl} - \delta^{jl} J^{ik}\right).}$$

so(d)

$$(J^{ij}_{(j=1)})^k{}_l = i\left(\delta^{ik}\delta_l^j - \delta^{jk}\delta_l^i\right).$$

$$J^{ij}_{(j=1/2)} = \epsilon^{ijk} \tfrac{1}{2}\sigma^k = \frac{i}{4}[\sigma^i, \sigma^j].$$

for general d: given d matrices $k\times k$ satisfying $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$

we can make $J^{ij} = \frac{i}{4}[\gamma^i, \gamma^j]$

satisfying $so(d)$

Define the Lie group

$$O(d) \equiv \{ d\times d \text{ real matrices } O_i{}^j \text{ s.t. } |On| = |n| \}$$

$$\begin{cases} |n| \equiv n_i n_j \delta^{ij} \\ On \equiv O_i{}^j n_j . \end{cases}$$

i.e. $O^t O = \mathbb{1}$

i.e. $(O^t)_i{}^j \delta_{jk} O^k{}_l = \delta_{il}.$

$$O = e^{-i\theta^{ij} J^{ij}}$$

$$O^t O = \mathbb{1} \implies 1 = \det O^t O$$
$$= (\det O)^2$$
$$\det O = \pm 1 .$$

$\implies O(d)$ has 2 components

$$= \underset{e^{\epsilon}}{SO(d)} \cup P \cdot \underline{SO(d)}$$

$\uparrow$ $\det O = 1$

$$\underline{P = \begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}}$$

'parity.

$\implies$ O is real, $O^t = O^{-1}$

$\implies$ J^{ij} is $\underline{\text{antisymmetric}}$ & imaginary.

there are $\frac{d(d-1)}{2}$ of them.

a basis $\left(J^{ij}_{(j=1)}\right)^k{}_l$.

$$\left(J^{ij}_{(j=1)}\right)^k{}_l = i\left(\delta^{ik}\delta_l{}^j - \delta^{jk}\delta_l{}^i\right).$$

eg $d=2$ $\quad SO(2)$:

$$T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^2$$

$$e^{i\beta T} \underset{\uparrow}{=} \mathbb{1}\cos\beta + i\sigma^2 \sin\beta$$

$T^2 = \mathbb{1}$

$$= \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}.$$

$$\underline{\underline{D_{j=\frac{1}{2}}(\hat{n}=\hat{z},\theta)}} = e^{-i\theta\sigma^3}$$

$$= \mathbb{1}\cos\theta - i\sigma^3 \sin\theta$$

$$= \begin{pmatrix} \cos\theta - i\sin\theta & \\ & \cos\theta + i\sin\theta \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$

$$D_j(g_1 g_2) = D_j(g_1) D_j(g_2).$$

$$G \longrightarrow \text{linear ops}$$

$$g \longmapsto D(g)$$

$\underline{j=0:}$ $\quad g \longmapsto \mathbb{1}.$

$$\underline{g(\vec{\theta}_1)\, g(\vec{\theta}_2) = g(\vec{\theta}_3(\vec{\theta}_1, \vec{\theta}_2))}$$

$$\left[J^i, J^j \right] = i\epsilon^{ijk} J^k$$

rep of Lie alg.

$$\Longrightarrow D(\theta) = e^{i\theta \hat{n}\cdot\vec{J}}$$

is a rep of Lie group.