Lie Groups & Lie Algebras & their representations, briefly (cont'd)

$$G \ni O = e^{-i\sum_{a=1}^{\dim G} \beta^a T^a}$$

Lie group

def'd by Taylor expansion

β^a: coords on G.

T^a generators of $\mathfrak{g}$ — Lie alg.

$$G \ni \mathbb{1} = e^{-i\sum 0 \cdot T}$$

eg: $O(n) = \{ n\times n$ real matrices O s.t. $\underline{O^t O = \mathbb{1}} \}$

$\supset SO(n) = \{ \ \dots \quad \det O = 1 \}$.

$\Rightarrow O = e^{-i\beta^a T^a} \in \mathbb{R} \iff \beta$ is real and T is pure imaginary

$$O^{-1} \overset{!}{=} O^t = e^{-i\beta^a (T^a)^t} \iff T^t = -T \ (A.S.)$$

$\Rightarrow$ so(n) is generated by all possible $n\times n$ pure imaginary antisym. matrices:

A basis is: $(\underline{J^{ij}})^k{}_\ell = i\left(\delta^{ik}\delta^j_\ell - \delta^{jk}\delta^i_\ell\right)$

These satisfy

$$[J^{ij}, J^{k\ell}] = i\left(\delta^{jk}J^{i\ell} + \delta^{i\ell}J^{jk} - (i \leftrightarrow j)\right)$$

$\equiv$ so(n) Lie alg.

$U(N) \equiv \{ N\times N \text{ complex matrices s.t. } U^\dagger U = \mathbb{1} \}$

$$U = e^{-i\beta^a T^a}$$ is solved by $(T^a)^\dagger = T^a$.

A basis:

$$\left\{ \begin{array}{l} T^1 = \frac{1}{2}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{12}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & -2 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \\ T^3 = \frac{1}{\sqrt{24}}\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3 & \\ & & & & 0 \end{pmatrix} \end{array} \right\}_{N-1}$$

$$\left(\operatorname{tr} T^a T^b = \frac{1}{2}\delta^{ab} \right)$$

$$\left(\text{for } i \neq j : \ (T_x^{ij})^k{}_\ell = \frac{1}{2}\left(\delta^{ik}\delta^j_\ell + \delta^{jk}\delta^i_\ell\right) \right)_{\frac{N(N-1)}{2}}$$

like σ^3

like σ^1

$$\left(\text{for } i \neq j : \ (T_y^{ij})^k{}_\ell = \frac{i}{2}\left(\delta^{ik}\delta^j_\ell - \delta^{jk}\delta^i_\ell\right) \right)_{\frac{N(N-1)}{2}}$$

like σ^y

$$1: \quad T^{N^2} = \frac{1}{\sqrt{2N}} \mathbb{1}_{N\times N}.$$

$N-1 + \frac{N(N-1)}{2} \cdot 2 + 1 = N^2$ of these.

$$U = e^{-i\beta^a T^a} = \underbrace{e^{-i\sum_{a=1}^{N^2-1}\beta^a T^a}}\ \underline{e^{-i\beta^{N^2} T^{N^2}}}$$

$$\log \det U = \operatorname{tr} \log U = -i \cancel{\Sigma}\, \beta^{N^2} \operatorname{tr}(T^{N^2}) = -i\beta^{N^2}.$$

$$U(N) \supset SU(N) = \{ U \in U(N) \mid \det U = 1 \}$$
$$= e^{-i \sum_{i=1}^{N^2-1} \beta^a T^a}$$

<u>Note</u>: $su(2) = so(3)$.

But $SU(2) \neq SO(3)$ ← a 2π rot. is $\mathbb{1}$.

a 2π rotation

$$U = e^{-i 2\pi \frac{\sigma^3}{2}} = -\mathbb{1} \neq \mathbb{1}$$

is not $= \mathbb{1}$

$\frac{1}{2}$-integer spin reps are <u>projective</u> reps of $SO(3)$.

$$U(g_1) U(g_2) = \underbrace{e^{i\omega(g_1, g_2)}}_{= \pm 1} U(g_1 g_2)$$

Lorentz Group:

$$O(d,1) = \left\{ \text{real } d{+}1 \times d{+}1\text{-dim'l matrices } \Lambda_\mu{}^\nu \ \text{ s.t. } \ \underline{\eta = \Lambda^t \eta \Lambda} \right\}$$

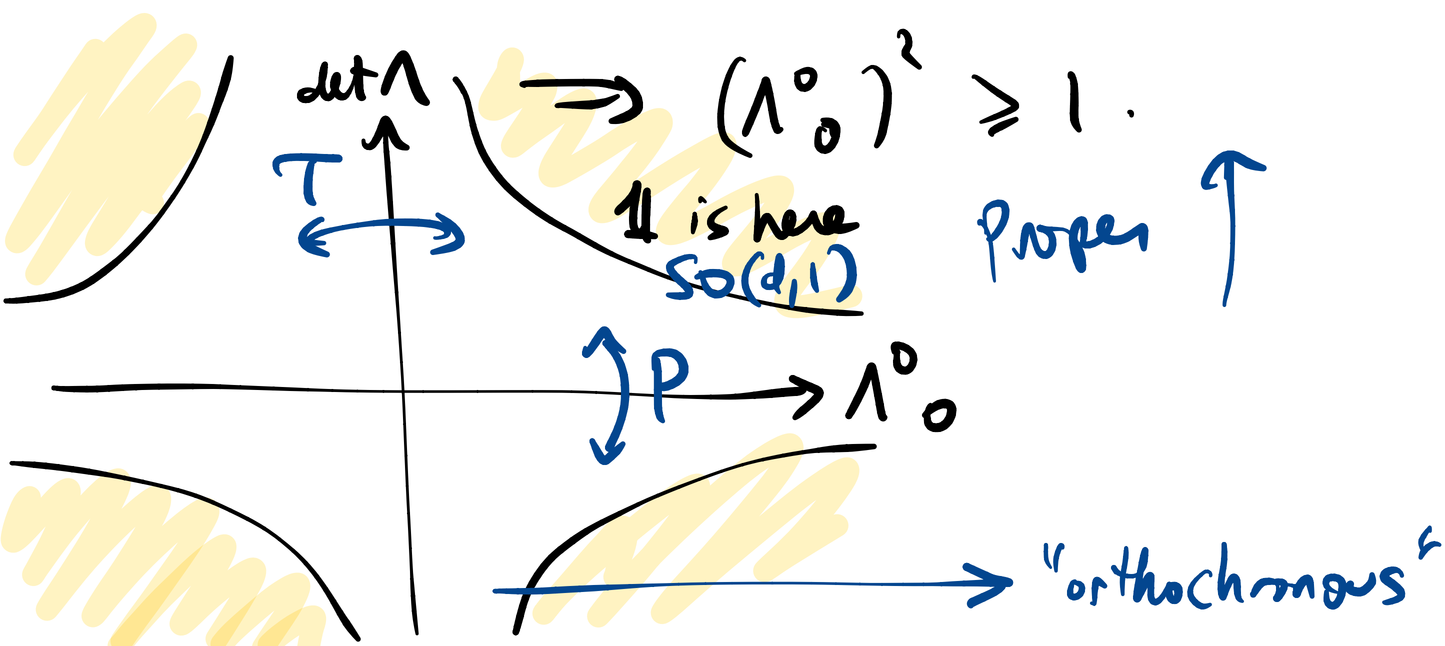
i.e. $\underline{\eta_{\mu\nu} = (\Lambda^t)_\mu{}^\rho\, \eta_{\rho\sigma}\, \Lambda^\sigma{}_\nu}$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & \\ & -\mathbb{1}_{d\times d} \end{pmatrix}$$

$\Rightarrow \det \Lambda = \pm 1.$ $\leftarrow$ proper / $\leftarrow$ improper transfo

$\underline{\mu\nu = 00}$: $\quad 1 = (\Lambda^0{}_0)^2 - \sum_i (\Lambda^i{}_0)^2$

$\Rightarrow (\Lambda^0{}_0)^2 \geq 1.$



$$P = \begin{pmatrix} 1 & \\ & -\mathbb{1}_{3\times 3} \end{pmatrix}$$ has $\det P = -1$.

$$T = \begin{pmatrix} -1 & \\ & \mathbb{1} \end{pmatrix} \otimes K$$

want T to preserve e^{-iHt}

$K : i \longrightarrow -i$. "antilinear transformation"

More generally: $$\eta^{(m,n)}_{\mu\nu} = \left(\begin{array}{c|c} +\mathbb{1}_m & \\ \hline & -\mathbb{1}_n \end{array}\right)$$

$$O(m,n) \equiv \left\{ \Lambda \,\middle|\, \Lambda^t \eta^{(m,n)} \Lambda = \eta^{(m,n)} \right\}$$

$$\overset{\dots}{\Longrightarrow} [J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\nu\rho} J^{\mu\sigma} + \eta^{\mu\sigma} J^{\nu\rho} - (\mu \leftrightarrow \nu)\right)$$

$\rho,\sigma \leftrightarrow \rho,\sigma$

is $o(m,n)$ Lie algebra.

ie replace $\delta^{ij} \to \eta^{\mu\nu}$
$\delta^i_j \to \delta^\mu_\nu$

5.2 Reps. of $so(d,1)$ on fields

Collection of fields $\phi_r \equiv (\phi_1 \cdots \phi_n, \psi_\alpha, A_\mu \cdots)_r$ transforms in some rep of Lorentz:

$$\phi_r(x) \longmapsto \underline{\underline{D_{rs}(\Lambda)}}\, \phi_s(\Lambda x)$$

<u>eg</u>: • $D(\Lambda) = 1$. (scalar)

• $D(\Lambda)_\mu^{\ \nu} = \Lambda_\mu^{\ \nu}$ (vector)

$$V^\mu \to V'^\mu = \Lambda^\mu{}_\nu V^\nu$$

$$\Lambda(\theta^a, \beta^a) = \exp\Big(-i\,\theta^a \underbrace{T^a_{rot}}_{\equiv J^a} - i\beta^a \underbrace{T^a_{boost}}_{\equiv K^a}\Big)$$

$$J^i = \left(\begin{array}{c|c} 0 & \\ \hline & J^i \end{array}\right)$$ is the 3x3 rot. generator

$(J^i)_{jk} = i\epsilon^{ijk}$.

$$(K^i)^j{}_0 = i\delta^i{}_j = (K^i)^0{}_j$$ other entries zero.

$$e^{-i\beta K^1} = \mathbb{1} - i\beta K^1 + O(\beta^2)$$

$$= \begin{pmatrix} 1 & +\beta & \\ +\beta & 1 & \\ & & \mathbb{1}_{2\times 2} \end{pmatrix} + O(\beta^2)$$

$$= \begin{pmatrix} \gamma & \beta\gamma & \\ \beta\gamma & \gamma & \\ & & \mathbb{1} \end{pmatrix} + O(\beta^2)$$

i.e. $\delta V^0 = \beta V^1$ $\quad \delta V^{2,3} = 0.$
$\delta V^1 = \beta V^0$

$$\begin{cases} \partial_\beta \Lambda(\beta) = -iK\Lambda \\ \Lambda(0) = \mathbb{1} \end{cases}$$ is an ODE w/ a unique sol'n.

$$e^{-i\beta K^1} = \begin{pmatrix} \cosh\beta & \sinh\beta & \\ \sinh\beta & \cosh\beta & \\ & & \mathbb{1} \end{pmatrix}$$

β = rapidity
(adds under successive boosts.)

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = 1 + \mathcal{O}(\beta^2) \qquad \boxed{\beta \equiv v/c}$$

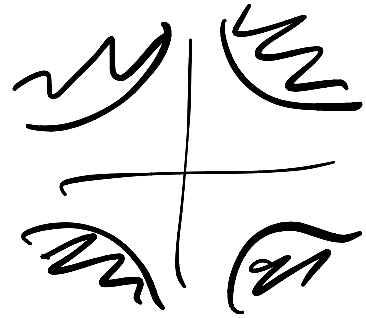
<u>Scary fact:</u> unlike $J^\dagger = J$.

$(K^i)^j_{\ 0} = i\delta^{ij}$ $\qquad K^\dagger \neq K$.

$e^{i\beta K}$ is not <u>unitary</u>.

It's ok because <u>fields</u> $\neq$ <u>wavef'ns</u>.

Lorentz group is <u>not</u> <u>compact</u>.



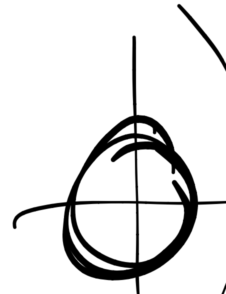
$\Rightarrow$ tension between

<u>reps are unitary</u>

vs.

<u>reps are finite-dim'l</u>.

(vs rots:



)

$D=3+1$:

$so(3,1)$: $\left\{\begin{array}{l} [J^i, J^j] = i\epsilon^{ijk} J^k \\ [J^i, K^j] = i\epsilon^{ijk} K^k \\ [K^i, K^j] = -i\epsilon^{ijk} J^k \end{array}\right\}$ "J and K are vectors under rotations".

preserved by $\left\{\begin{array}{l} J \to J \\ K \to -K \end{array}\right.$. parity.

let $\vec{J}^{\pm} = \frac{1}{2}(\vec{J} \pm i\vec{K})$

satisfy $[J^i_+, J^j_-] = 0 \quad \forall\, i,j$

$$[J^i_\pm, J^j_\pm] = i\epsilon^{ijk} J^k_\pm$$

$$\Rightarrow \boxed{so(3,1) = su(2)_L \times su(2)_R .}$$

$su(2)_L$ gen. by $\vec{J}_+$ $\quad$ $su(2)_R$ gen. by $\vec{J}_-$.

$\Rightarrow$ IRRep of $SO(3,1)$ $= (j_L, j_R) = \left\{ \begin{array}{l} (m_L, m_R) \\ m_L \in \{-j_L \cdots j_L\},\ m_R \cdots \end{array} \right\}$

has dim $(2j_L+1)(2j_R+1)$.

(j_+, j_-)	dim	physics preview
$(0,0)$	1	scalar
$(\frac{1}{2},0)$	2	left-handed Weyl spinor
$(0,\frac{1}{2})$	2	right- " " "
$(\frac{1}{2},0)\otimes(0,\frac{1}{2})=(\frac{1}{2},\frac{1}{2})$	$2\times 2=4$	4-vector.
$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ (reducible)	$2+2=4$	Dirac spinor
$(1,0)\oplus(0,1)$	$3+3=6$	$V^{\mu\nu}=\pm\epsilon^{\mu\nu}{}_{\rho\sigma}V^{\rho\sigma}$ $V^{\mu\nu}=-V^{\nu\mu}$ AS. tensor.
$\vdots$		

(photon).

Weyl Spinors

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\to\left(D_{(\frac{1}{2},0)}(\theta,\underbrace{\beta}_{\text{rapidity}})\right)_{\alpha\beta}\psi_\beta \qquad \alpha,\beta=1,2$$

$$D_{(\frac{1}{2},0)}(\theta,\beta)=e^{-i(\theta^i J^i+\beta^i K^i)}$$

J^i, K^i are 2x2 matrices.

who are J^i, K^i 2x2?

is a singlet of $SU(2)_R$ $\Longrightarrow$

$$0 \overset{!}{=} J_-^i \psi = \frac{1}{2}(J^i - K^i)\psi$$

ie $J = iK$ acting ψ

$$\Rightarrow J_+^i \psi = \frac{1}{2}(J^i + iK^i)\psi = \frac{1}{2}(J^i + J^i)\psi$$

$$= J^i \psi .$$

$$\Longrightarrow \vec{J} = \vec{J}_{(\frac{1}{2})} = \frac{1}{2}\vec{\sigma} \Rightarrow \vec{K} = -i\frac{\vec{\sigma}}{2}.$$

$$\psi_\alpha \mapsto \left(e^{-i\frac{1}{2}\theta\cdot\sigma - \frac{1}{2}\beta\cdot\sigma}\right)_\alpha{}^\beta \psi_\beta$$

$$= \left(e^{-\frac{1}{2}\vec{\sigma}\cdot(\vec{\beta}+i\vec{\theta})}\right)_\alpha{}^\beta \psi_\beta \equiv M_\alpha{}^\beta \psi_\beta .$$

M is a rot. in a complex angle

$\in SL(2,\mathbb{C}) \leftarrow$ 2×2 cx matrices

w/ $\det M = 1$.

$\underline{(0, \tfrac{1}{2})}$: $\quad J = -iK$.

$$\chi_{\dot\alpha} \mapsto \left(e^{+\frac{1}{2}\sigma\cdot(\beta - i\theta)}\right)_{\dot\alpha}{}^{\dot\beta} \chi_{\dot\beta}$$

$$= \underline{\underline{(\sigma^2 M^* \sigma^2)}}_{\dot\alpha}{}^{\dot\beta} \chi_{\dot\beta}$$

$\underline{\text{fact}}$:
$$\left\{\begin{array}{l} \sigma^2 \vec\sigma^* \sigma^2 = -\vec\sigma \\ = \sigma^2 \vec\sigma^T \sigma^2 \end{array}\right. \qquad (\vec\sigma^\dagger = \vec\sigma)$$

if ψ is $(\tfrac{1}{2}, 0)$ $\underline{(L)}$: $\underline{\underline{\sigma^2 \psi^*}}$ is $(0, \tfrac{1}{2})$.
(R) $\qquad (\sigma^2)^2 = 1$

$$\sigma^2 \psi^* \to \sigma^2 M^* \psi^* = (\sigma^2 M^* \sigma^2) \sigma^2 \psi^*$$

Invariants:

$V^\mu V_\mu = V^\mu V^\nu \eta_{\mu\nu}$ is an invariant.

Q: Can I make a singlet out of $\underbrace{(\tfrac{1}{2},0)}_{\psi_\alpha} \otimes \underbrace{(\tfrac{1}{2},0)}_{\xi_\alpha}$?

(two L Weyl spinors)

Yes.

$$\frac{1}{2}\otimes\frac{1}{2} = \underset{\text{antisymmetric}}{0} \oplus \underset{\text{symmetric}}{1}.$$

$$\boxed{\begin{aligned} \epsilon^{\alpha\beta} &= (i\sigma^2)^{\alpha\beta} \\ &= \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}^{\alpha\beta} \end{aligned}}$$

CLAIM: $\epsilon^{\alpha\beta}\psi_\alpha \xi_\beta$ is a singlet

$$i\sigma^2\psi \longmapsto i\sigma^2 e^{-\frac{1}{2}(\beta+i\theta)\cdot\sigma}\, \underset{\sigma^2\cdot\sigma^2}{1}\,\psi$$

$$= \exp\Big(-\tfrac{1}{2}(\beta+i\theta)\, \underbrace{\sigma^2\vec{\sigma}\sigma^2}_{=-\vec{\sigma}^t}\Big)\, i\sigma^2\psi$$

$$= \left(e^{+\frac{1}{2}(\beta+i\theta)\cdot\sigma^t}\right) i\sigma^2\psi.$$

$$\underline{\underline{\psi^\alpha}} \equiv (i\sigma^2\psi)^\dagger_\alpha$$

$$\mapsto \psi^\beta \left(e^{+\frac{1}{2}(\beta+i\theta)\cdot\tilde{\sigma}}\right)_\beta{}^\alpha$$

$$= \psi^\beta (M^{-1})_\beta{}^\alpha .$$

$$\Rightarrow \psi^\alpha \xi_\alpha \equiv (i\sigma^2\psi)\xi \quad \text{is invariant.}$$

<u>CLAIM:</u> $(j_L, j_R) = (\frac{1}{2}, \frac{1}{2})$ is a vector.

$$= (\tfrac{1}{2}, 0) \otimes (0, \tfrac{1}{2})$$

L spin R spin.

<u>intertwiners</u>

$$\begin{cases} \sigma^\mu_{\alpha\dot{\alpha}} \equiv (\mathbb{1}_{\alpha\dot{\alpha}}, \tilde{\sigma}_{\alpha\dot{\alpha}})^\mu \\ \bar{\sigma}^\mu_{\dot{\alpha}\alpha} \equiv (\mathbb{1}_{\dot{\alpha}\alpha}, -\tilde{\sigma}_{\dot{\alpha}\alpha})^\mu . \end{cases}$$

if ψ, χ are L & R weyl spinors

claim: ① $\psi^{\alpha} \sigma^{\mu}_{\alpha\dot\alpha} \chi^{\dot\alpha}$ is a vector.

② given ξ_R, ψ_R

then $\xi_R^{\dagger} \sigma^{\mu} \psi_R$ is a vector.

pf of 2: $\xi_R^{\dagger} \longmapsto \xi_R^{\dagger} e^{+\frac{1}{2}(i\theta+\beta)\cdot\sigma}$

$\xi_R^{\dagger} \sigma^{\mu} \psi_R \longmapsto$

$$\xi_R^{\dagger} \underbrace{e^{\frac{1}{2}(i\theta+\beta)\cdot\sigma} \sigma^{\mu} e^{\frac{1}{2}(i\theta+\beta)\cdot\sigma}}_{\overset{?}{=} \Lambda(\theta,\beta)^{\mu}{}_{\nu} \sigma^{\nu}} \psi_R$$

$$\overset{?}{=} \underline{\Lambda(\theta,\beta)^{\mu}{}_{\nu} \sigma^{\nu}}.$$

$$\Lambda(\theta,\beta)^{\mu}{}_{\nu} \equiv \left(e^{-i(\theta\cdot J + \beta K)}\right)^{\mu}{}_{\nu}$$

4x4 rep:

$$\delta(\xi_R^\dagger \sigma^\mu \psi_R) = \delta\xi_R^\dagger \sigma^\mu \psi_R + \xi_R^\dagger \sigma^\mu \delta\psi_R$$

$$= \xi_R^\dagger \left(\tfrac{1}{2}(i\theta+\beta)^j \sigma^j \sigma^\mu + \sigma^\mu \tfrac{1}{2}(-i\theta+\beta)^j \sigma^j \right) \psi_R$$

$$= \begin{cases} \xi_R^\dagger \, \tfrac{1}{2} \cdot 2\beta_j \sigma^j \psi_R & \mu = 0 \\ \xi_R^\dagger \Big(\beta_j \big(\underbrace{\sigma^j \sigma^i + \sigma^i \sigma^j}_{=2\delta^{ij}} \big) + i\theta_j \big(\underbrace{\sigma^j \sigma^i - \sigma^i \sigma^j}_{=-2i\epsilon^{ijk}\sigma^k} \big) \Big) \psi_R & \mu = i \end{cases}$$

<u>RHS:</u>

$$\delta V^\mu = -\left(i\beta_j (K^j)^\mu{}_\nu + i\theta_j (J^j)^\mu{}_\nu \right) V^\nu$$

$$= \begin{cases} \beta_j V^j & \text{if } \mu = 0 \\ \beta_i V^0 - \theta_j \epsilon_{jim} V^m & \text{if } \mu = i. \end{cases}$$