— MAKE-UP LECTURES NEXT WEEK
USUAL TIME & PLACE

— Please submit a course evaluation

5.3 Lagrangians for Spinor Fields

Recall: $(0,\frac{1}{2})$ rep of $SO(3,1)$:

$$\xi_R \longrightarrow e^{\frac{1}{2}\vec{\sigma}\cdot(\vec{\beta}-i\vec{\theta})}\xi_R$$

Boost   Rotation

$(\frac{1}{2},0)$ rep

$$\psi_L \longrightarrow e^{-\frac{1}{2}\vec{\sigma}\cdot(\vec{\beta}+i\vec{\theta})}\psi_L .$$

$$\xi_R^\dagger \sigma^\mu \psi_R = (\xi_R^{\alpha})^\dagger \sigma^\mu_{\alpha\dot{\alpha}} \psi_R^{\dot{\alpha}} \qquad \sigma^\mu \equiv (\mathbb{1},\vec{\sigma})^\mu$$

IS A VECTOR.

$\psi_L^\dagger \bar{\sigma}^\mu \xi_L$ is a vector. $\qquad \bar{\sigma}^\mu \equiv (\mathbb{1},-\vec{\sigma})^\mu$

Comments: • ξ_R^{+} transforms like χ_L.

$\Rightarrow$ $\chi_L^{\alpha}\,\sigma^{\mu}_{\alpha\dot\alpha}\,\psi_R^{\dot\alpha}$ is a vector.

• $\xi_R^{+}\psi_R$ is not a scalar but the time component of a vector

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Given ψ_R, find $\mathcal{L}(\psi_R, \psi_R^{+}, \partial_\mu\psi_R, \partial_\mu\psi_R^{+})$

guess? $\psi_R^{+}(\underbrace{\Box + m^2}_{}) \psi_R$?

is the 0 component of a vector.

But $\psi_R^{+}\sigma^{\mu}\psi_R$ is a vector

$$\mathcal{L}_{\text{Weyl}} \equiv \psi_R^{+}\sigma^{\mu} i\partial_\mu \psi_R = \psi_R^{+} i\partial_t \psi_R + \psi_R^{+}\vec{\sigma}\cdot i\vec{\nabla}\psi_R$$

($i\partial_\mu$ is hermitian)

- $\mathcal{L}_{weyl}^{\dagger} = -i\,(\partial_\mu \psi_R^\dagger)\,(\sigma^\mu)^\dagger \psi_R$

  $\overset{IBP}{=} \psi_R^\dagger \sigma^\mu i\, \partial_\mu \psi_R = \mathcal{L}_{weyl}$

  $(+\ t.d.)$ $(+t.d.)$

  <u>Real!</u>

- $\mathcal{L}_{weyl}(\psi_L) = \psi_L^\dagger i\, \bar{\sigma}^\mu \partial_\mu \psi_L$.

- <u>Mass term?</u> $\mathcal{L}_{Majorana} \equiv \psi_R\, i \sigma^2 \psi_R + h.c.$

  - Lorentz inv't
  - is not inv't under $\underline{\psi_R \to e^{i\theta} \psi_R}$.
  - $\mathcal{L}_{Majorana} = \psi_1 \psi_2 - \psi_2 \psi_1 + h.c.$

    $\neq 0$.

DIRAC SPINORS $\equiv \underset{\psi_L}{\left(\tfrac{1}{2},0\right)} \oplus \underset{\psi_R}{\left(0,\tfrac{1}{2}\right)}$

$$\ni \quad \Psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$\bar{\Psi} \equiv \left(\psi_R^\dagger, \psi_L^\dagger\right) = \Psi^\dagger \begin{pmatrix} 0 & \mathbb{1}_{2\times 2} \\ \mathbb{1}_{2\times 2} & 0 \end{pmatrix}$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad\qquad \equiv \Psi^\dagger \gamma^0$$

$$\mathcal{L}_{\text{DIRAC}} = \psi_R^\dagger i \sigma^\mu \partial_\mu \psi_R + \psi_L^\dagger i \bar{\sigma}^\mu \partial_\mu \psi_L$$

$$- m\left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L\right)$$

$$= \bar{\Psi}\left(i \gamma^\mu \partial_\mu - m\right)\Psi$$

$$\text{w/} \quad \gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$= \bar{\Psi}\left(i \not\partial - m\right)\Psi. \qquad \underline{\gamma^\mu \partial_\mu \equiv \not\partial.}$$

is
- Lorentz inv't
- inv't under $\Psi \to e^{i\theta}\Psi$.

<u>eom</u>: $$0 = \frac{\delta S_{DIRAC}}{\delta \bar{\Psi}} = (i \gamma^\mu \partial_\mu - m)\Psi$$

$$= (i \not{\partial} - m)\Psi .$$

$$0 = \left(i\, \gamma^\mu_{ab} \partial_\mu - m\, \delta_{ab}\right) \Psi_b$$

$a, b = 1 .. 4$

$$0 = [S_{kin}] = 2[\Psi] + 1 - D$$

$$\Rightarrow [\Psi] = \frac{D-1}{2} \overset{D=3+1}{=} \frac{3}{2} .$$

$$\underline{[m] = 1 .} \quad \checkmark$$

- $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. (Clifford)

<u>CLAIM</u>: Given D matrices ($k \times k$) satisfying (Clifford)

we can build a k-dim'l Rep of <u>so(1,D-1)</u>

By: $$J^{\mu\nu}_{DIRAC} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] .$$

i.e. $[J^{\mu\nu}_{\text{Dirac}}, J^{\rho\sigma}_{\text{Dirac}}] = i\left(\eta^{\rho\nu} J^{\mu\sigma}_{\text{Dirac}} + \eta^{\mu\sigma} J^{\rho\nu}_{\text{Dirac}} - (\mu \leftrightarrow \nu)\right)$

($so(1, D-1)$.)

D	k (minimum)
1	1
2	2
3	2
4	4
5	4
6	8
7	8
⋮	⋮

- reducible in even dimensions.

eg in $D=4$

Dirac $= (0, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, 0)$.

$$J^{\mu\nu}_{\text{Dirac}} \overset{\text{Weyl Basis}}{=} \frac{i}{4}\left[\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^\nu \\ \bar\sigma^\nu & 0 \end{pmatrix}\right]$$

$$= \ldots = \frac{i}{4}\begin{cases} \begin{pmatrix} -2\sigma^i & \\ & -2\sigma^i \end{pmatrix} & \mu\nu = 0i \\ \begin{pmatrix} -2i\epsilon^{ijk}\sigma^k & \\ & -2i\epsilon^{ijk}\sigma^k \end{pmatrix} & \mu\nu = ij \end{cases}$$

Notice: Clifford is basis-independent.

• 4d Dirac spinor $\neq$ 4d vector rep

$$\Lambda_{\text{Dirac}}(\theta = 2\pi\hat{z}, \beta = 0) = e^{-i 2\pi J^{12}}$$

$$= e^{i\pi\sigma^3 \otimes \mathbb{1}_2}$$

$$= \cos\pi\, \mathbb{1}_4 + \sin\pi\, \sigma^3 \otimes \mathbb{1}_2 = -\mathbb{1}_{4\times 4}.$$

• $\begin{cases} \gamma^\mu \longrightarrow \tilde{\gamma}^\mu \equiv U \gamma^\mu U^\dagger \\ \Psi \longrightarrow \tilde{\Psi} = U\Psi \end{cases}$ preserves physics.

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}$$

$$\tilde{J}^{\mu\nu}_{\text{Dirac}} = U J^{\mu\nu}_{\text{Dirac}} U^\dagger$$ satisfy $SO(3,1)$

__eg__: $\gamma^0_m = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$ $\gamma^1_m = \begin{pmatrix} i\sigma^1 & \\ & i\sigma^1 \end{pmatrix}$

$\gamma^2_m = \begin{pmatrix} 0 & -\sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$ $\gamma^3_m = \begin{pmatrix} i\sigma^3 & \\ & i\sigma^3 \end{pmatrix}$ $\Bigg\}$ all imaginary

In this Basis $J^{\mu\nu}_{Dirac} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ are also imaginary

$$\Rightarrow \Lambda_{Dirac}(\theta, \beta) = e^{-i\theta_{\mu\nu} J^{\mu\nu}_{Dirac}}$$

are real.

$\Rightarrow$ can choose $\Psi \to \Lambda_{Dirac}\Psi$.

to be real.

"Majorana spinor".

if $\gamma^\mu_m = U \tilde{\gamma}^\mu U^\dagger$

$$\Psi = \Psi^* \Longleftrightarrow \tilde{\Psi}^* = (U^*)^{-1} U \tilde{\Psi}$$

real scalar field : complex scalar ::

Majorana spin : Dirac spinor

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \mathbb{1}_{4\times 4}.$$

- $(i\partial\!\!\!/ - m)\Psi = 0$ Dirac eqn

$$\Longrightarrow \quad (\partial^2 + m^2)\Psi = 0.$$

$$(i\partial\!\!\!/ + m)\Big((i\partial\!\!\!/ - m)\Psi = 0\Big) \Longrightarrow$$

$$0 = (i\partial\!\!\!/ + m)(i\partial\!\!\!/ - m)\Psi$$

$$= (-\gamma^\mu\gamma^\nu \partial_\mu\partial_\nu - m^2)\Psi$$

$$\gamma^\mu\gamma^\nu = \underbrace{\tfrac{1}{2}[\gamma^\mu, \gamma^\nu]}_{A.S.} + \underbrace{\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\}}_{= \eta^{\mu\nu}\mathbb{1}}$$

0

$$= -(\partial^2 + m^2)\Psi.$$

- $$0 = \frac{\delta S_{Dirac}}{\delta \Psi} = \underline{\bar{\Psi}(-i\overleftarrow{\partial}_\mu \gamma^\mu - m)}$$

<u>Note:</u> $\gamma^0 = (\gamma^0)^\dagger \quad \vec{\gamma}^\dagger = -\vec{\gamma}.$ $\quad \bar{\Psi} \equiv \Psi^\dagger\gamma^0.$

- Lorentz tranf. of Dirac Spinor.

$$\Psi \to e^{-i\theta_{\mu\nu} J^{\mu\nu}_{Dirac}} \Psi \equiv \Lambda_{\frac{1}{2}} \Psi$$

$$\Lambda_{\frac{1}{2}} = \begin{pmatrix} M & \\ & \sigma^2 M^* \sigma^2 \end{pmatrix}$$

$$\left(M = e^{-\frac{1}{2}\vec{\sigma}\cdot(\vec{\beta}+i\vec{\theta})} \right)$$

$$\bar{\Psi} \to \Psi^\dagger e^{+i\theta_{\mu\nu}(J^{\mu\nu}_{Dirac})^\dagger} \gamma^0$$

$$= \Psi^\dagger \gamma^0 \Lambda_{\frac{1}{2}}^{-1} = \bar{\Psi} \Lambda_{1/2}^{-1}$$

using $(\gamma^\mu)^\dagger \gamma^0 = \gamma^0 \gamma^\mu$.

$\Rightarrow \bar{\Psi}\Psi$ to invt.

CLAIM: $\Lambda_{\frac{1}{2}}^{-1}(\theta)\, \gamma^\mu \, \Lambda_{\frac{1}{2}}(\theta) = \Lambda^\mu{}_\nu(\theta)\, \gamma^\nu$.

$$\Rightarrow \quad V^{\mu_1 \dots \mu_n} \equiv \bar{\Psi} \gamma^{\mu_1} \dots \gamma^{\mu_n} \Psi$$

is a <u>tensor</u>

i.e. $V^{\mu_1 \dots \mu_n} \longrightarrow \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_n}{}_{\nu_n} V^{\nu_1 \dots \nu_n}.$

Any bit of $A_{\mu_1 \dots \mu_n} \bar{\Psi}^{\mu_1 \dots \mu_n} \Psi$

which is symmetric under $\mu_i \leftrightarrow \mu_j$ is a lower-rank tensor.

$$\{\gamma^{\mu_i}, \gamma^{\mu_j}\} = 2\eta^{\mu_i \mu_j}$$

$$\bar{\Psi}_a \Psi_b \underbrace{\Gamma^{ab}}_{4\times 4} = \sum_{n=0}^{D} A_{\mu_1 \dots \mu_n} \bar{\Psi} \gamma^{\mu_1 \dots \mu_n} \Psi$$

$$4 \times 4 = 1 + 4 + 6 + 4 + 1$$

$$\left\{ \begin{array}{l} \gamma^{\mu_1 \dots \mu_n} \\ \equiv \frac{1}{n!}\left(\gamma^{\mu_1} \cdots \gamma^{\mu_n} \pm \text{perms}\right) \end{array} \right.$$

why care abt Bispinors? • e.g.

$$\mathcal{L}_2 \equiv \bar{\Psi}\gamma^{\mu\nu}\Psi \cdot \bar{\Psi}\gamma_{\mu\nu}\Psi$$

is Lorentz inv't.

• $$j^\mu \equiv \bar{\Psi}\gamma^\mu\Psi = \psi_R^\dagger\sigma^\mu\psi_R + \psi_L^\dagger\bar{\sigma}^\mu\psi_L$$

is the current ass. to $\Psi \to e^{-i\theta}\Psi$.

$$\partial_\mu j^\mu = 0 \Longleftarrow \text{Dirac eqn.}$$

$$j_5^\mu \equiv \bar{\Psi}\gamma^\mu\gamma^5\Psi = \psi_R^\dagger\sigma^\mu\psi_R - \psi_L^\dagger\sigma^\mu\psi_L$$

$$\gamma^5 \overset{\text{Weyl Basis}}{\equiv} \begin{pmatrix} -\mathbb{1} & \\ & \mathbb{1} \end{pmatrix}$$

'axial current'!

$$\partial_\mu j_5^\mu = 2im\bar{\Psi}\Psi$$

claim: $(\gamma^5)^2 = \mathbb{1}$. $\gamma^{5\dagger} = \gamma^5$.

$$\{\gamma^5, \gamma^\mu\} = 0. \quad \mu = 0..3.$$

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$

$$[\gamma^5, J^{\mu\nu}_{\text{Dirac}}] = 0. \quad \underline{\text{Casimir!}}$$

$$\left(\Longrightarrow \gamma^5 \overset{\text{Weyl basis}}{=} \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\right)$$

$$P_{R/L} = \frac{1 \pm \gamma^5}{2}$$ projects onto R/L bits.

$$\begin{cases} P_R\gamma^\mu = \gamma^\mu P_L \\ P_L\gamma^\mu = \gamma^\mu P_R. \end{cases}$$

$$\gamma^{\mu\nu\rho\sigma} = -i\epsilon^{\mu\nu\rho\sigma}\gamma^5$$

$$\gamma^{\mu\nu\rho} = +i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5 .$$

Bispinors	#	rep
$\bar{\Psi}\mathbb{1}\Psi$	1	scalar
$\bar{\Psi}\gamma^\mu\Psi$	4	vector
$\bar{\Psi}\gamma^{\mu\nu}\Psi$	6	A-S tensor
$i\bar{\Psi}\gamma^\mu\gamma^5\Psi$	4	pseudovector
$i\bar{\Psi}\gamma^5\Psi$	1	pseudoscalar.

$$= i(\psi_L^\dagger\psi_R - \psi_R^\dagger\psi_L)$$

$$P: \psi_L \leftrightarrow \psi_R .$$

Coupling to EM field: A_μ.

$$\mathcal{L}_{EM} = -e\, j^\mu A_\mu$$

$$\mathcal{L} = \bar{\Psi}\left[i\underbrace{(\partial_\mu + ieA_\mu)}_{D_\mu}\gamma^\mu - m\right]\Psi$$

is invariant under $\begin{cases} \Psi(x) \to \underline{e^{i\alpha(x)}}\,\Psi(x) \\ A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha \end{cases}$

$$D_\mu \Psi \longrightarrow e^{i\alpha(x)}(D_\mu\Psi).$$

[i.e. Replace $\partial_\mu \rightsquigarrow D_\mu$.]

$$0 = (i\cancel{D}+m)\underbrace{(i\cancel{D}-m)\Psi}\; = (iD_\mu iD_\nu \gamma^\mu\gamma^\nu - m^2)\Psi$$

$$[D_\mu, D_\nu] = ei(\partial_\mu A_\nu - \partial_\nu A_\mu) = ei\,F_{\mu\nu}. \neq 0.$$

$$0 = \left((\partial_\mu + ieA_\mu)^2 + \frac{ei}{4}[\gamma^\mu, \gamma^\nu] F_{\mu\nu} + m^2 \right) \Psi$$

Weyl Basis

$$= \left[(\partial_\mu + ieA_\mu)^2 - e \begin{pmatrix} (\vec{B} + i\vec{E})\cdot\vec{\sigma} & \\ & (\vec{B} - i\vec{E})\cdot\vec{\sigma} \end{pmatrix} + m^2 \right] \Psi$$

intrinsic magnetic Dipole moment

5.4 Free-particle sol'ns of Dirac eq'n

Dirac $\Rightarrow$ KG $\Rightarrow$ $\Psi_p(x) = e^{-ipx} u(p)$

w/ $0 \overset{!}{=} (\gamma_\mu p^\mu - m) u(p)$

$\underline{m \neq 0}$. Rest frame $p_0^\mu = (m, 0)^\mu$.

$$u(p) = \Lambda_{\frac{1}{2}} u(p_0) \text{ w/ } \Lambda_\mu{}^\nu (p_0)_\nu = p_\nu.$$

$$\to \quad 0 = (m\gamma^0 - m)\, u(p_0)$$

$$\overset{\text{Weyl Basis}}{=} m \begin{pmatrix} -\mathbb{1}_2 & \mathbb{1}_2 \\ \mathbb{1}_2 & -\mathbb{1}_2 \end{pmatrix} u(p_0)$$

solved by $\quad u(p_0) \propto \begin{pmatrix} \xi \\ \xi \end{pmatrix}$.

2 sol'ns for each p w/ $p^0 > 0$
are from spin $1/2$.

Convention: $\quad u(p_0) = \sqrt{2m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \quad \xi^\dagger \xi = 1$

$$p_0 \to \begin{pmatrix} E \\ p^3 \end{pmatrix} = \underline{\underline{\exp\left(\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)}} \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix}$$

$$u(p_0) \longmapsto \underline{\underline{\Lambda_{\frac{1}{2}}(\eta)}}\, u(p_0) = \underbrace{\exp\left(-\tfrac{1}{2}\eta \begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix}\right)} \times \sqrt{2m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \cosh\eta/2\, \mathbb{1} + \sinh\eta/2 \begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix}$$

$$\Rightarrow \boxed{u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar\sigma}\ \xi \end{pmatrix}} \qquad p\cdot\sigma = p^\mu \sigma_\mu .$$

$$= \begin{pmatrix} \left(\sqrt{E+p^3\sigma^3}\, P_+ + \sqrt{E+p^3\sigma^3}\, P_-\right)\xi \\ \left(\sqrt{E-p^3\sigma^3}\, P_+ + \sqrt{E-p^3\sigma^3}\, P_-\right)\xi \end{pmatrix}$$

$$\left(P_\pm = \frac{1\pm\sigma^3}{2} \right)$$

claim: $(p\cdot\sigma)(p\cdot\bar\sigma) = p^2$

Negative energy sol'ns: $\Psi = e^{+ipx}\, v$

$$\longrightarrow \quad v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \eta^s \\ -\sqrt{p\cdot\bar\sigma}\ \eta^s \end{pmatrix} \qquad s=1,2.$$

Correction:

Demand $[J^i, J^j] = +i\epsilon^{ijk} J^k$.

then $\underline{\underline{(J^i)^j{}_k}} = -\underline{i}\epsilon^{ijk}$ w/ $\underline{\epsilon^{123} = 1}$.

$$\Lambda = e^{-\underline{i}\theta\cdot J}$$

$$\underline{\underline{\delta_{ij}}}$$

Peskin write:

$$(J^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)$$