University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2022
Assignment 1

Due 11am Thursday, September 29, 2022

- Please hand in your homework electronically via Canvas. The preferred option is to typeset your homework. It is easy to do and you need to learn to do it anyway as a practicing scientist. A LaTeX template file with some relevant examples is provided here. If you need help getting set up or have any other questions please email me. I am happy to give TeX advice.
- To hand in your homework, please submit a pdf file through the course's Canvas website, under the assignment labelled hw01.

Thanks in advance for following these guidelines. Please ask me by email if you have any trouble.

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1. **Survey.** What area(s) of physics are you working on or hoping to work on? (QFT is a big subject and knowing this will help guide some choices I have to make.)

2. **Brain-warmer.** Consider the Fourier transformation $q_n = \frac{1}{\sqrt{N}} \sum_k e^{-\mathbf{i}kna} q_k \equiv \sum_k U_{nk} q_k$. In the sum, k can be taken from the set $\{\frac{2\pi}{Na} j, j = 0..N-1\}$. Check that this transformation from real space (modes labelled by n) to Fourier space (modes labelled by k) is encoded in an $N \times N$ unitary matrix, U (that is, $UU^\dagger = \mathbb{1} = U^\dagger U$).

3. **More balls and springs.**

   (a) In lecture, we studied a one-dimensional lattice of balls and springs. Now consider a system in two dimensions with Lagrangian

$$L = \frac{m}{2} \sum_{x,y=1}^{N} \left(\dot{\phi}_{xy}^2 - c_1^2 \left(\frac{\phi_{xy} - \phi_{x+1,y}}{a} \right)^2 - c_2^2 \left(\frac{\phi_{xy} - \phi_{x,y+1}}{a} \right)^2 \right) \quad (1)$$

   where ϕ_{xy} is a real variable. (This does not quite describe a two-dimensional grid of balls and springs (like a mattress), since ϕ is a scalar quantity.) Assume periodic boundary conditions in both i, j. Find the equations of motion, and find the normal modes (hint: the problem is translation invariant in two dimensions and linear). What is the dispersion relation? What is the continuum limit? Interpret the constants c_1, c_2 and tell me what is special about the case $c_1 = c_2$?

(b) **Dimensional reduction.** Now consider a generalization where we have different numbers of lattice sites in the two directions:

$$L = \frac{m}{2}\sum_{x=1}^{N_x}\sum_{y=1}^{N_y}\left(\dot{\phi}_{xy}^2 - c_1^2\left(\frac{\phi_{xy}-\phi_{x+1,y}}{a}\right)^2 - c_1^2\left(\frac{\phi_{xy}-\phi_{x,y+1}}{a}\right)^2\right). \tag{2}$$

Consider a special case with a small number of sites in the y-direction, such as $N_y = 2$, and interpret this as a chain in the x direction, with more than one kind of excitation that propagates in the x-direction. What is the spectrum?

4. **Heisenberg picture fields.**

Here we will try to understand in what sense the field momentum of a free scalar field is $\pi \sim \dot{\phi}$, and we will explain the factor of $\mathbf{i}\omega$ by which π and ϕ differ.

I usually think in what is called *Schrödinger picture*, where we evolve the states in time

$$|\psi(t)\rangle = \mathbf{U}(t)^\dagger|\psi(0)\rangle = e^{-\mathbf{iH}t/\hbar}|\psi(0)\rangle$$

and leave the operators alone. It is sometimes useful to define time-dependent operators by implementing the change of basis associated with $\mathbf{U}$ on the operators[1]:

$$\mathbf{A}(t) \equiv \mathbf{U}(t)\mathbf{A}\mathbf{U}(t)^\dagger = e^{+\mathbf{iH}t/\hbar}\mathbf{A}e^{-\mathbf{iH}t/\hbar}.$$

First consider a simple harmonic oscillator,

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{q}^2 = \hbar\omega\left(\mathbf{a}^\dagger\mathbf{a} + \frac{1}{2}\right)$$

with

$$\mathbf{q} = \sqrt{\frac{\hbar}{2m\omega}}\left(\mathbf{a}+\mathbf{a}^\dagger\right) = \sqrt{\frac{\hbar}{2m\omega}}2\mathrm{Re}\left(\mathbf{a}\right).$$

$$\mathbf{p} = -\mathbf{i}\sqrt{\frac{\hbar m\omega}{2}}\left(\mathbf{a}-\mathbf{a}^\dagger\right) = \sqrt{\frac{m\hbar\omega}{2}}2\mathrm{Im}\left(\mathbf{a}\right).$$

[1]Recall that the signs are designed so that matrix elements are the same in either picture:

$$\underbrace{\left(\langle\phi(0)|\,\mathbf{U}(t)\right)}_{=\langle\phi(t)|}\mathbf{A}\underbrace{\left(\mathbf{U}(t)^\dagger|\psi(0)\rangle\right)}_{|\psi(t)\rangle} = \langle\phi(0)|\underbrace{\left(\mathbf{U}(t)\mathbf{A}\mathbf{U}(t)^\dagger\right)}_{\mathbf{A}(t)}|\psi(0)\rangle$$

and the two pictures differ just by moving around parentheses.

(a) Using the algebra satisfied by $\mathbf{H}$ and $\mathbf{a}$, show that

$$\mathbf{q}(t) \equiv e^{+\mathbf{i}\mathbf{H}t}\mathbf{q}e^{-\mathbf{i}\mathbf{H}t} = \sqrt{\frac{\hbar}{2m\omega}} 2\mathrm{Re}\left(e^{-\mathbf{i}\omega t}\mathbf{a}\right).$$

[Hint for one approach: $O(t) = e^{\mathbf{i}Ht}Oe^{-\mathbf{i}Ht}$ is the unique solution to the differential equation $\partial_t O(t) = \mathbf{i}[H, O(t)]$ with $O(0) = O$. What is $[\mathbf{H}, \mathbf{a}]$?]

(b) Using the expression above, show that

$$\mathbf{p}(t) \equiv e^{+\mathbf{i}\mathbf{H}t}\mathbf{p}e^{-\mathbf{i}\mathbf{H}t} = m\partial_t \mathbf{q}(t)$$

in agreement with what you would want from the Lagrangian formulation and from classical mechanics.

The above was pretty simple, I hope. Now we consider a scalar quantum field theory, in say $d + 1 = 1 + 1$ dimensions:

$$\mathbf{H} = \int d^d x \left(\frac{\boldsymbol{\pi}(x)^2}{2\mu} + \frac{1}{2}\mu v_s^2 \left(\vec{\nabla}\boldsymbol{\phi} \cdot \vec{\nabla}\boldsymbol{\phi}\right) + \frac{1}{2}\tilde{m}^2\boldsymbol{\phi}^2 \right) = \sum_k \hbar\omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2}\right).$$

$$\begin{aligned}
\boldsymbol{\phi}(x) &= \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k L^d}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k^\dagger\right), \\
\boldsymbol{\pi}(x) &= \frac{1}{\mathbf{i}} \sum_k \sqrt{\frac{\hbar\mu\omega_k}{2L^d}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k - e^{-\mathbf{i}\vec{k}\cdot\vec{x}}\mathbf{a}_k^\dagger\right),
\end{aligned}$$

(c) Find ω_k. (Note that I've added a mass term $\tilde{m}^2\phi^2$, relative to the model we studied in lecture. This is why I use μ instead of m for the object which looks like the inertial mass. Also note that $\tilde{m}$ does not have dimensions of mass.)

(d) Do a Legendre transformation to construct the action, $S[\phi] = \int dt d^d x \mathcal{L}(\phi, \dot{\phi})$.

(e) Show that

$$\boldsymbol{\phi}(t, x) \equiv e^{+\mathbf{i}\mathbf{H}t}\boldsymbol{\phi}(x)e^{-\mathbf{i}\mathbf{H}t} = \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k L^d}} \left(e^{\mathbf{i}\vec{k}\cdot\vec{x} - \mathbf{i}\omega_k t}\mathbf{a}_k + e^{-\mathbf{i}\vec{k}\cdot\vec{x} + \mathbf{i}\omega_k t}\mathbf{a}_k^\dagger\right)$$

(f) Using the previous result, show that

$$\boldsymbol{\pi}(t, x) \equiv e^{+\mathbf{i}\mathbf{H}t}\boldsymbol{\pi}(x)e^{-\mathbf{i}\mathbf{H}t} = \mu\partial_t \boldsymbol{\phi}(t, x)$$

so that all is right with the world.