University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2022
Assignment 4

Due 11:00am Thursday, October 20, 2021

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1. **Zero-phonon processes.**

   We wish to understand the probability for a photon to hit (our crude model of) a crystalline solid without exciting any vibrational excitations. It is a nice simultaneous application of *many* of the things we've learned so far (quantization of the radiation field, solution of the harmonic chain, gaussian integrals, the relationship between path integrals and transition amplitudes).

   When a photon hits a lattice of atoms, Fermi's golden rule says that the (leading approximation to the) probability for a transition from one state of the lattice $|L_i\rangle$ to another $|L_f\rangle$ is proportional to

   $$W(L_i \to L_f) = |\langle L_f| \mathbf{H}_\mathrm{L} |L_i\rangle|^2.$$

   Here $\mathbf{H}_\mathrm{L}$ is the hamiltonian describing the interaction between the photon and an atom in the lattice. For the first parts of the problem, use the following form (to be justified in the last part of the problem):

   $$\mathbf{H}_\mathrm{L} = Ae^{\mathbf{i}K\mathbf{x}} + h.c. \tag{1}$$

   where $\mathbf{x}$ is the (center of mass) position operator of the atom being struck; K is a constant (the photon wavenumber), and (for the purposes of the first parts of the problem) A is a constant. $+h.c.$ means 'plus the hermitian conjugate of the preceding stuff'.

   (a) Recalling that $\mathbf{x}$ (up to an additive constant) is part of a collection of coupled harmonic oscillators:

   $$\mathbf{x} = nx + \mathbf{q}_n$$

   evaluate the "vacuum persistence amplitude" $\langle 0| \mathbf{H}_\mathrm{L} |0\rangle$. You will find the results of the previous problem set useful.

   (b) From the previous calculation, you will find an expression that requires you to sum over wavenumbers. Show that in one spatial dimension, the probability for a zero-phonon transition is of the form

   $$P_{\text{Mössbauer}} \propto e^{-\Gamma \ln L}$$

where L is the length of the chain and Γ is a function of other variables. Show that this infrared divergence is missing for the analogous model of crystalline solids with more than one spatial dimension. (Cultural remark: these amplitudes are called 'Debye-Waller factors').

(c) Convince yourself that a coupling $\mathbf{H}_\mathrm{L}$ of the form (1) arises from the minimal coupling of the electromagnetic field to the constituent charges of the atom, after accounting for the transition made by the radiation field when the photon is absorbed by the atom. 'Minimal coupling' means replacing the momentum operator of the atom $\mathbf{p}$, with the gauge-invariant combination $\mathbf{p} \to \mathbf{p} + \mathbf{A}$. You will also need to recall the form of the quantized electromagnetic field in terms creation and annihilation operators for a photon of definite momentum K.

2. **The iϵ prescription produces time-ordered correlators.** [Bonus problem]

Check that the four-point function of a free scalar field of mass m

$$Z^{-1} \int [D\phi] e^{\mathbf{i}S[\phi]} \prod_{i=1}^{4} \phi(x_i)|_{m^2 \to m^2 - \mathbf{i}\epsilon},$$

defined by the $\mathbf{i}\epsilon$ prescription, is the *time-ordered* vacuum expectation value

$$= \langle 0| \mathcal{T} \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) |0\rangle .$$

One way to do this is to show that they are both equal to

$$D_T(12)D_T(34) + D_T(13)D_T(24) + D_T(14)D_T(23), \tag{2}$$

where $D_T(ij) \equiv \langle 0| \mathcal{T} \phi(x_i)\phi(x_j) |0\rangle$.

3. **The propagator is a Green's function.**

(a) Consider the *retarded* propagator for a real, free, massive scalar field:

$$D_R(x-y) \equiv \theta(x^0 - y^0) \langle 0| [\phi(x), \phi(y)] |0\rangle .$$

Show that it is a Green's function for the Klein-Gordon operator, in the sense that

$$\left(\Box_x + m^2\right) D_R(x-y) = a\delta^{d+1}(x-y), \qquad \Box_x \equiv \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}.$$

Find a.

(b) Use the previous result to generalize the mode expansion of the scalar field to the situation with an external *source*, *i.e.* when we add to the lagrangian density $\phi(x)j(x)$ for some fixed c-number function $j(x)$.

(c) Show that the *time-ordered* propagator for a real, free, massive scalar field

$$D(x-y) \equiv \langle 0| \mathcal{T}\phi(x)\phi(y) |0\rangle \equiv \theta(x^0-y^0) \langle 0| \phi(x)\phi(y) |0\rangle + \theta(y^0-x^0) \langle 0| \phi(y)\phi(x) |0\rangle$$

is *also* a Green's function for the Klein-Gordon operator. That is, consider what happens when you act with the wave operator $+\Box_x + m^2$ on the time-ordered two-point function.

[Hints: Use the canonical equal-time commutation relations:

$$[\phi(\vec{x}), \phi(\vec{y})] = 0, \quad [\partial_{x^0}\phi(\vec{x}), \phi(\vec{y})] = -\mathbf{i}\delta^{D-1}(\vec{x}-\vec{y}).$$

Do not neglect the fact that $\partial_t\theta(t) = \delta(t)$: the time derivatives act on the time-ordering symbol!]

(d) This is an example of something that is easier to understand from the path integral. In the next problem, we'll understand why the correlation functions of ϕ should solve the equations of motion, up to 'contact terms'.

4. **Schwinger-Dyson equations.**

Consider the path integral

$$\int [D\phi] e^{\mathbf{i}S[\phi]}.$$

Using the fact that the integration measure is independent of the choice of field variable, we have

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} \text{(anything)}$$

(as long as 'anything' doesn't grow too fast at large ϕ). So this equation says that we can integrate by parts in the functional integral.

(Why is this true? As always when questions about functional calculus arise, you should think of spacetime as discrete and hence the path integral measure as simply the product of integrals of the field value at each spacetime point, $\int[D\phi] \equiv \int \prod_x d\phi(x)$, this is just the statement that

$$0 = \int d\phi_x \frac{\partial}{\partial\phi_x} \text{(anything)}$$

with $\phi_x \equiv \phi(x)$, *i.e.* that we can integrate by parts in an ordinary integral if there is no boundary of the integration region.)

This trivial-seeming set of equations (we get to pick the 'anything') can be quite useful and they are called Schwinger-Dyson equations (or sometimes Ward identities). Unlike many of the other things we'll discuss, they are true non-perturbatively, *i.e.* are really true, even at finite coupling. They provide a quantum implementation of the equations of motion.

(a) Evaluate the RHS of

$$0 = \int [D\phi] \frac{\delta}{\delta \phi(x)} \left(\phi(y) e^{\mathbf{i}S[\phi]} \right)$$

(defined by the $\mathbf{i}\epsilon$ prescription) to conclude that

$$\frac{1}{Z} \int [D\phi] e^{\mathbf{i}S} \frac{\delta S}{\delta \phi(x)} \phi(y) = +\mathbf{i}\delta(x-y). \tag{3}$$

(b) These Schwinger-Dyson equations are true in interacting field theories; to get some practice with them we consider here a free theory. Evaluate (3) for the case of a free massive real scalar field to show that the (two-point) time-ordered correlation functions of ϕ satisfy the equations of motion, most of the time. That is: the equations of motion are satisfied away from other operator insertions:

$$\left(+\Box_x + m^2\right) \langle \mathcal{T} \phi(x) \phi(y) \rangle = -\mathbf{i}\delta(x-y), \tag{4}$$

with $\Box_x \equiv \partial_{x^\mu} \partial^{x^\mu}$.

(c) Find the generalization of (4) satisfied by (time-ordered) three-point functions of the free field ϕ.

5. **More about 0+0d field theory.** [Bonus problem] Here we will study a bit more some field theories with no dimensions at all, that is, integrals.

Consider the case where we put a label on the field: $q \to q_a, a = 1..N$. So we are studying

$$Z = \int \int_{-\infty}^{\infty} \prod_a dq_a \; e^{-S(q)}.$$

Let

$$S(q) = \frac{1}{2} q_a K_{ab} q_b + T_{abcd} q_a q_b q_c q_d$$

where T_{abcd} is a collection of couplings. Assume K_{ab} is a real symmetric matrix.

(a) Show that the propagator has the form:

$$a - - - - - - b \equiv \langle q_a q_b \rangle_{T=0} = \left(K^{-1}\right)_{ab} = \sum_k \phi_a(k)^\star \frac{1}{k} \phi_b(k)$$

where $\{k\}$ are the eigenvalues of the matrix K and $\phi_a(k)$ are the eigenvectors in the a-basis.

(b) Develop a diagrammatic expansion for the propagator $\langle q_a q_b \rangle$. Show that in a diagram with a loop, we must sum over the eigenvalue label k. (For definiteness, consider the order-T correction to the propagator $\langle q_k q_{k'} \rangle$, where k, k' label eigenvectors of K, and $q_k \equiv \sum_a \phi_a(k)^\star q_a$).

(c) Consider the case where $K_{ab} = t\,(\delta_{a,b+1} + \delta_{a+1,b})$, with periodic boundary conditions: $a + N \equiv a$. Find the eigenvalues. Show that in this case if

$$T_{abcd} q_a q_b q_c q_d = \sum_a g q_a^4$$

the k-label is conserved at vertices, *i.e.* the vertex is accompanied by a delta function on the sum of the incoming eigenvalues.

(d) (Bonus question) What is the more general condition on T_{abcd} in order that the k-label is conserved at vertices?

(e) (Bonus question) Study the physics of the model described in 5c.