

Physics 215A QFT Fall 2022 Assignment 5

Due 11:00am Thursday, October 27, 2022

1. **Brain-warmer: the identity does nothing twice.** Check our relativistic state normalization by squaring the expression for the identity in the 1-particle sector:

$$\mathbb{1}_1^2 \stackrel{!}{=} \mathbb{1}_1 = \int \frac{d^d p}{2\omega_{\vec{p}}} |\vec{p}\rangle \langle \vec{p}|.$$

2. **Even more about QFT in 0 + 0 dimensions.**

In this problem we return to the simplest scalar QFT, namely a single integral with quartic action.

- (a) By a change of integration variable show that

$$Z = \int_{-\infty}^{\infty} dq e^{-S(q)}$$

with $S(q) = \frac{1}{2}m^2q^2 + gq^4$ is of the form

$$Z = \frac{1}{g^{1/4}} \mathcal{Z}(m^2/\sqrt{g}).$$

This means you can make your life easier by setting $g = 1$, without loss of generality.

- (b) Convince yourself (*e.g.* with Mathematica) that the integral really is expressible as a Bessel function.
- (c) It would be nice to find a better understanding for why the partition function of (0+0)-dimensional ϕ^4 theory is a Bessel function. Find a Schwinger-Dyson equation for this system which has the form of Bessel's equation for

$$K(x^2/a) \equiv e^{-x^2/a}(x^2)^{-1/4} \mathcal{Z}(x)$$

for some constant a . (If you get stuck I can tell you what function to choose for the 'anything' in the S-D equation.)

- (d) Make a plot of the perturbative approximations to the 'Green function' $G \equiv \langle q^2 \rangle$ as a function of g , truncated at orders 1 through 6 or so. Plot them against the exact answer.

- (e) (Bonus problem) Show that $c_{n+1} \sim -\frac{2}{3}nc_n$ at large n , where c_n is the coefficient of g^n in either G or Z (by brute force or by cleverness).

3. **The vacuum is a fluid with $p = -\rho$.** [Bonus problem]

We said in lecture that the vacuum energy density ρ gravitates and that, when positive, its effect is to cause space to inflate – to expand exponentially in time. An important aspect of this phenomenon is that the vacuum fluctuations produce not only an energy density, but a *pressure*, $p = T_i^i$ (no sum on i), of the form $p = -\rho$, which is negative for $\rho > 0$. The vacuum therefore acts as a perfect fluid with $P = -\rho$. (The stress tensor for a perfect fluid in terms of its velocity field u^μ takes the form $T^{\mu\nu} = (p + \rho)u^\mu u^\nu + pg^{\mu\nu}$, so in a frame with $u^\mu = (1, \vec{0}^\mu)$, $T_0^0 = \rho, T_i^i = P$.) Solving Einstein's equations with such a source produces an inflating universe. In this problem we show that this is the case.

- (a) Show that the energy-momentum tensor for a free relativistic scalar field ($S[\phi] = \int d^D x \sqrt{g} \mathcal{L}, \mathcal{L} = \frac{1}{2}g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{m^2}{2} \phi^2$) takes the form

$$T_{\mu\nu} = a \partial_\mu \phi \partial_\nu \phi - b g_{\mu\nu} \mathcal{L}$$

with some constants a, b .

- (b) Reproduce the formal expression for the vacuum energy

$$\langle 0 | \mathbf{H} | 0 \rangle = V \int \bar{d}^d k \frac{1}{2} \hbar \omega_{\vec{k}}$$

using the two point function

$$\langle 0 | \phi(x)^2 | 0 \rangle = \langle 0 | \phi(0) \phi(0) | 0 \rangle = \lim_{\vec{x}, t \rightarrow 0} \langle 0 | \phi(x) \phi(0) | 0 \rangle$$

and its derivatives. (V is the volume of space.)

(We will learn to draw this amplitude as a Feynman diagram which is a circle (a line connecting a point to itself).)

- (c) Show that the vacuum expectation value of the pressure

$$\langle 0 | T_{ii} | 0 \rangle$$

(no sum on i) gives the same answer, up to a sign.

[Hints: You'll find a quite different looking integral from the vacuum energy. Use rotation invariance of the vacuum to simplify the answer. The claim is that however you regulate the integral vacuum pressure and $\frac{1}{2} \int \bar{d}^d k \omega_k$, you'll get the same answer. A convenient regulator is *dimensional regularization*: treat the dimension d as an arbitrary complex number.]

- (d) Show that the resulting vacuum energy momentum tensor ($T_{00} = \rho, T_{ii} = -\rho$ (no sum on i)) is the same as the contribution to the energy-momentum tensor from an action of the form

$$S_{cc} = \int d^D x \sqrt{g} \Lambda$$

where Λ is a constant (the cosmological constant).

- (e) Argue that $p = -\rho$ is required in order that the vacuum energy does not specify a preferred rest frame.

4. **Non-Abelian currents.** On a previous homework, we studied a complex scalar field. Now, we make a big leap to *two* complex scalar fields, $\Phi_{\alpha=1,2}$, with

$$S[\Phi_\alpha] = \int d^d x dt \left(\frac{1}{2} \partial_\mu \Phi_\alpha^* \partial^\mu \Phi_\alpha - V(\Phi_\alpha^* \Phi_\alpha) \right)$$

Consider the objects

$$Q^i \equiv \frac{1}{2} \int d^d x \mathbf{i} \left(\Pi_\alpha^\dagger \sigma_{\alpha\beta}^i \Phi_\beta^\dagger \right) + h.c.$$

where $\sigma^{i=1,2,3}$ are the three Pauli matrices.

- (a) What symmetries do these charges generate (*i.e.* how do the fields transform)? Show that they are symmetries of S .
- (b) If you want to, show that $[Q^i, H] = 0$, where H is the Hamiltonian.
- (c) Evaluate $[Q^i, Q^j]$. Hence, non-Abelian.
- (d) To complete the circle, find the Noether currents J_μ^i associated to the symmetry transformations you found in part 4a. Show that the resulting charge $Q^i = \int d^d x J_0^i$ agrees with our starting point.
- (e) Generalize to the case of N scalar fields.
5. **Combinatorics from 0-dimensional QFT.** [This is a more sophisticated bonus problem. I will not post the solutions of this problem until later. If you have a hard time with it now, please try again in a week.]

Catalan numbers $C_n = \frac{(2n)!}{n!(n+1)!}$ arise as the answer to many combinatorics problems (beware: there is some disagreement in the literature about whether this is C_n or C_{n+1}).

One such problem is: count random walks on a 1d chain with $2n$ steps that start at 0 and end at 0 without crossing 0 in between.



Another such problem is: in how many ways can $2n$ (distinguishable) points on a circle be connected by chords that do not intersect within the circle.



Consider a zero-dimensional QFT with the following Feynman rules:

- There are two fields h and l .
- There is an $\sqrt{th^2l}$ vertex in terms of a coupling t .
- The bare l propagator is 1.
- The bare h propagator is 1.
- All diagrams can be drawn on a piece of paper without crossing.
- An annoying extra rule: All the l propagators must be on one side of the h propagators¹.
- There are no loops of h .

The last two rules can be realized from a lagrangian by introducing a large N (below).

(a) Show that the full two-point green's function for h is

$$G(t) = \sum_n t^n C_n$$

the generating function of Catalan numbers.

- (b) Let $\Sigma(t)$ be the sum of diagrams with two h lines sticking out that may not be divided into two parts by cutting a single intermediate line. (This property is called 1PI (one-particle irreducible), and Σ is called the "1PI self-energy of h ". We'll use this manipulation all the time later on.) Show that $G(t) = \frac{1}{1-\Sigma(t)}$.
- (c) Argue by diagrams for the equation (sometimes this is also called a Schwinger-Dyson equation)






¹You'll see in part 5f how to justify this.

where Σ is the 1PI self-energy of h .

- (d) Solve this equation for the generating function $G(t)$.
- (e) If you are feeling ambitious, add another coupling N^{-1} which counts the crossings of the l propagators. The resulting numbers can be called Touchard-Riordan numbers.
- (f) How to realize the no-crossings rule? Consider

$$L = \frac{\sqrt{t}}{\sqrt{N}} l_{\alpha\beta} h_{\alpha} h_{\beta} + \sum_{\alpha,\beta} l_{\alpha\beta}^2 + \sum_{\alpha} h_{\alpha}^2$$

where $\alpha, \beta = 1 \cdots N$. By counting index loops, show that the dominant diagrams at large N are the ones we kept above. Hint: to keep track of the factors of N , introduce ('t Hooft's) double-line notation: since l is a matrix, its propagator looks like: $\begin{matrix} \alpha & - & - & - & - & - & - & \alpha \\ \beta & - & - & - & - & - & - & \beta \end{matrix}$, while the h propagator is just one index line $\alpha \text{-----} \alpha$, and the vertex is $\text{---}!!\text{---}$. If you don't like my ascii diagrams, here are the respective pictures: $\langle l_{\alpha\beta} l_{\alpha\beta} \rangle =$  ,

$\langle h_{\alpha} h_{\alpha} \rangle =$  and the hhl vertex is:  . A

closed index loop gives a factor of N .

- (g) Use properties of Catalan numbers to estimate the size of non-perturbative effects in this field theory.
- (h) There are many other examples like this. Another similar one is the relationship between symmetric functions and homogeneous products. A more different one is the enumeration of planar graphs. For that, see [BIPZ](#).