University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2022
Assignment 10

Due 11:00am Thursday, December 1, 2022

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1. **Lorentz algebra in $D = 3 + 1$.**

   (a) Check the algebra satisfied by rotations and boosts

   $$[J^i, J^j] = \mathbf{i}\epsilon^{ijk} J^k, \quad [J^i, K^j] = \mathbf{i}\epsilon^{ijk} K^k, \quad [K^i, K^j] = -\mathbf{i}\epsilon^{ijk} J^k \qquad (1)$$

   using the explicit matrices in the vector representation given in lecture, namely

   $$J^i = \begin{pmatrix} 0 & \\ & \mathbf{J}^i \end{pmatrix} \qquad (2)$$

   (where the 3×3 matrix $\mathbf{J^i}$ is $(\mathbf{J}^i)^j_k = -\mathbf{i}\epsilon^{ijk}$) and

   $$\left(K^i\right)^j{}_0 = \mathbf{i}\delta^i_j = \left(K^i\right)^0{}_j \qquad (3)$$

   and all other components zero.

   For this purpose, I think typing them into Mathematica and writing $K.J - J.K$ etc.... is perfectly acceptable.

   (b) Check that in terms of the antisymmetric tensor of operators

   $$J^{\mu\nu} = \begin{cases} \epsilon^{ijk} J^k, & \mu\nu = ij \\ K^i, & \mu\nu = 0i \end{cases},$$

   (1) can be rewritten as

   $$[J^{\mu\nu}, J^{\rho\sigma}] = \mathbf{i}\left(\eta^{\nu\rho} J^{\mu\sigma} + \eta^{\mu\sigma} J^{\nu\rho} - (\mu \leftrightarrow \nu)\right), \qquad (4)$$

   which is the form of the $\mathsf{so}(d, 1)$ Lie algebra for general d. Make sure you take advantage of symmetries to avoid working too hard.

   (c) Show that in $D = 3 + 1$, (1) is equivalent to $\mathsf{su}(2)_L \times \mathsf{su}(2)_R$:

   $$[J^i_+, J^j_-] = 0, \qquad [J^i_\pm, J^j_\pm] = \mathbf{i}\epsilon^{ijk} J^k_\pm$$

   in terms of

   $$J^i_\pm \equiv \frac{1}{2}(J^i \pm \mathbf{i}K^i).$$

2. **A representation of the Clifford algebra gives a representation of Lorentz.** [Bonus problem] Show the following: Given a collection of $k \times k$ matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ (the Clifford algebra, with $\mu, \nu = 0..d$), we can make a k-dimensional representation of $SO(1, d)$ with generators

$$J^{\mu\nu} = \frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu].$$

As an intermediate step, it is helpful to show that

$$[J^{\mu\nu}, \gamma^\rho] \propto (\gamma^\mu \eta^{\nu\rho} - \gamma^\nu \eta^{\rho\mu}) .$$

Convince yourself that this last equation says that γ^ρ transforms as a four-vector, *i.e.*

$$[\gamma^\rho, J^{\mu\nu}_{\text{Dirac}}] = (J^{\mu\nu}_{\text{vector}})^\rho{}_\sigma \gamma^\sigma .$$

The following problems are about discrete symmetries of scalar field theories. They are a useful warmup to the discussion of discrete symmetries of the Dirac fermion.

3. **Brain warmer: $\mathbb{Z}_2$ symmetry of real scalar field theory.**

   What does the operator

$$U \equiv e^{\mathbf{i}\pi \sum_k \mathbf{a}_k^\dagger \mathbf{a}_k}$$

   do to the real scalar field

$$\phi(x) \to \phi'(x) = U\phi(x)U^\dagger$$

   whose ladder operators are $\mathbf{a}, \mathbf{a}^\dagger$?

   For which Lagrangians is this a symmetry?

4. **Charge conjugation in complex scalar field theory.** [Bonus problem]

   Consider again a free complex Klein-Gordon field Φ. Define a discrete symmetry operation (charge conjugation) C, by

$$\Phi(x) \mapsto C\Phi(x)C^{-1} = \eta_c \Phi^\dagger(x)$$

   where C is a unitary operator, and η_c is an arbitrary phase factor. Assume that the vacuum is invariant under charge conjugation: $C|0\rangle = |0\rangle$.

   (a) Show that the free lagrangian is invariant under C, but the particle number current j^μ changes sign.

(b) Show that the annihilation operators satisfy

$$C\mathbf{a}_k C^{-1} = \eta_c \mathbf{b}_k, \quad C\mathbf{b}_k C^{-1} = \eta_c^\star \mathbf{a}_k$$

and hence show that C interchanges particle and antiparticle states, up to a phase.

5. **Parity symmetry of scalar field theory.** [Bonus problem]

Under the parity transformation

$$\vec{x} \mapsto \vec{x}' = -\vec{x}$$

a real Klein-Gordon transforms as

$$\phi(t,\vec{x}) \mapsto P\phi(t,x)P^{-1} = \eta_p \phi(t,-\vec{x}) \tag{5}$$

where P is unitary and $\eta_P = \pm 1$ is the *intrinsic parity* of the field ϕ. Again assume $P|0\rangle = |0\rangle$.

(a) Show that the parity transformation preserves the free Lagrangian (though not the Lagrangian density), for both values of η_P.

(b) Show that an arbitrary n-particle state transforms as

$$P\left|\vec{k}_1, \cdots \vec{k}_n\right\rangle = \eta_P^n \left|-\vec{k}_1, \cdots, -\vec{k}_n\right\rangle.$$

(c) Here we give an explicit realization of the parity operator. Let

$$P_1 \equiv e^{-\mathbf{i}\frac{\pi}{2}\sum_k \mathbf{a}_k^\dagger \mathbf{a}_k}, \quad P_2 \equiv e^{\mathbf{i}\eta_p\frac{\pi}{2}\sum_k \mathbf{a}_k^\dagger \mathbf{a}_{-k}}.$$

Show that

$$P_1 \mathbf{a}_k P_1^{-1} = \mathbf{i}\mathbf{a}_k, \quad P_2 \mathbf{a}_k P_2^{-1} = -\mathbf{i}\eta_p \mathbf{a}_{-k}.$$

Hint: Use the following version of the Campbell-Baker-Hausdorff formula

$$e^{\mathbf{i}\alpha A} B e^{-\mathbf{i}\alpha A} = \sum_{n=0}^{\infty} \frac{(\mathbf{i}\alpha)^n}{n!} B_n$$

where $B_0 \equiv B$ and $B_n = [A, B_{n-1}]$ for $n = 1, 2.....$

Show that $P \equiv P_1 P_2$ is unitary, and satisfies (5).

(d) **Action on the current of a complex scalar field.** Consider now a complex scalar field. Using the results from problems 4 and the preceding parts of 5, find the action of parity on the particle current $j^\mu \mapsto P j^\mu P^{-1}$. (You'll have to extend the action of P from the case of a real field to the complex case.)