

# Physics 215A QFT Fall 2022

## Assignment 11 (“Final Exam”)

Due 11:59pm Thursday, December 8, 2022

---

### 1. Brain-warmers.

(a) Check that

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2.$$

(b) Use the previous part to show that if

$$u_r(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \bar{\sigma}} \xi_r \\ \sqrt{p \cdot \sigma} \xi_r \end{pmatrix} \quad \text{and} \quad v_r(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \bar{\sigma}} \eta_r \\ -\sqrt{p \cdot \sigma} \eta_r \end{pmatrix}$$

with  $p^2 = m^2$  (solutions of the Dirac equation with mass  $m$ ), then

$$\bar{u}_r(\vec{p})u_s(\vec{p}) = 2m\xi_r^\dagger\xi_s \quad \text{and} \quad \bar{v}_r(\vec{p})v_s(\vec{p}) = -2m\eta_r^\dagger\eta_s$$

(where  $\bar{u} \equiv u^\dagger\gamma^0$  as usual).

(c) Show that  $\bar{u}_r(\vec{p})v_s(\vec{p}) = 0$  and  $u_r(\vec{p})^\dagger v_s(-\vec{p}) = 0$  but  $u_r(\vec{p})^\dagger v_s(\vec{p}) \neq 0$ .

### 2. Other bases for gamma matrices. [Bonus problem]

Many different bases of gamma matrices are frequently used by humans. You may read on the internet someone telling you that the gamma matrices are

$$\tilde{\gamma}^0 = \begin{pmatrix} \mathbb{1}_{2 \times 2} & 0 \\ 0 & -\mathbb{1}_{2 \times 2} \end{pmatrix}, \quad \tilde{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

and think that I have lied to you. This basis is useful for studying the non-relativistic limit. The Weyl basis which we introduced in lecture instead makes manifest the reducibility of the Dirac spinor into L plus R Weyl spinors. Find the unitary matrix  $U$  which relates them  $\tilde{\gamma}^\mu = U\gamma^\mu U^\dagger$ .

### 3. Symmetries of the Dirac lagrangian.

(a) Find the Noether currents  $j^\mu$  and  $j_5^\mu$  associated with the transformations  $\Psi(x) \rightarrow e^{-i\alpha}\Psi(x)$  and  $\Psi(x) \rightarrow e^{-i\alpha\gamma^5}\Psi(x)$  of a free Dirac field. Show by explicit calculation that the former is conserved and the latter is conserved if  $m = 0$ .

- (b) Find the conserved currents associated with the Lorentz symmetry  $\Psi(x) \mapsto \Lambda_{\frac{1}{2}}(\theta, \beta)\Psi(\Lambda^{-1}x)$  of the Dirac Lagrangian. Show that the conserved charge takes the form

$$J^{\mu\nu} = \int_{\text{space}} (\mathcal{J}_{\text{orbital}}^{\mu\nu} + \Psi^\dagger J_{\text{Dirac}}^{\mu\nu} \Psi)$$

where  $\mathcal{J}_{\text{orbital}}^{\mu\nu}$  has the form it would have for a scalar field, and  $J_{\text{Dirac}}^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$  are the matrices satisfying the Lorentz algebra.

Convince yourself that the latter matrix specifies how the current acts in the one-particle sector.

#### 4. Meson scattering.

Consider the Yukawa theory with fermions, with  $\mathcal{L}_{\text{int}} = -g\bar{\Psi}\Psi\phi$ , where  $\Psi$  is a Dirac fermion field and  $\phi$  is a real scalar field.

- Draw a Feynman diagram that gives a leading contribution to the scattering amplitude for the process  $\phi\phi \rightarrow \phi\phi$ .
- [Bonus problem] Derive the correct sign of the amplitude by considering the relevant matrix elements of powers of the interaction hamiltonian.
- Evaluate the diagram in terms of a spinor trace and a momentum integral. Do not do the momentum integral. Suppose that the integral is cut off at large  $k$  by some cutoff  $\Lambda$ . Estimate the dependence on  $\Lambda$ .

#### 5. The magnetic moment of a Dirac fermion.

In this problem we consider the hamiltonian density

$$\mathfrak{h}_I = q\bar{\Psi}\gamma^\mu\Psi A_\mu .$$

This describes a local, Lorentz invariant, and gauge invariant interaction between a Dirac fermion field  $\Psi$  and a vector potential  $A_\mu$ . In this problem, we will treat the vector potential, representing the electromagnetic field, as a fixed, classical background field.

Define single-particle states of the Dirac field by  $\langle 0 | \Psi(x) | \vec{p}, s \rangle = e^{-ipx} u^s(p)$ . We wish to show that these particles have a magnetic dipole moment, in the sense that in their rest frame, their (single-particle) hamiltonian has a term  $h_{NR} \ni \mu_B \vec{S} \cdot \vec{B}$  where  $\vec{S} = \frac{1}{2}\vec{\sigma}$  is the particle's spin operator.

- $q$  is a real number. What is required of  $A_\mu$  for  $H_I = \int d^3x \mathfrak{h}_I$  to be hermitian?
- [Bonus problem] How must  $A_\mu$  transform under parity  $P$  and charge conjugation  $C$  in order for  $H_I$  to be invariant? (To answer this, you'll have to

find out how the spinor bilinear transforms, *e.g.* from Peskin.) How do the electric and magnetic fields transform? Show that this allows for a magnetic dipole moment but not an electric dipole moment.

- (c) Show that in the non-relativistic limit

$$\bar{u}(p')\gamma^{\mu\nu}u(p)F_{\mu\nu} = a\xi^\dagger\sigma\cdot\vec{B}\xi'$$

for some constant  $a$  (find  $a$ ). Recall that  $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ . Here  $u, u'$  are positive-energy solutions of the Dirac equation with mass  $m$  and

$$u \xrightarrow{NR} \sqrt{m}(\xi, \xi), u' \xrightarrow{NR} \sqrt{m}(\xi', \xi')$$

in the non-relativistic limit.

- (d) Suppose that  $A_\mu$  describes a magnetic field  $\vec{B}$  that is uniform in space and time.

Show that in the non-relativistic limit

$$\langle \vec{p}', s' | H_I | \vec{p}, s \rangle = \hbar^3 (\vec{p} - \vec{p}') h(\xi, \xi', \vec{B}) + \dots$$

where ... is terms independent of the spin. Find the function  $h(\xi, \xi', \vec{B})$ . You may wish to use the Gordon identity. Rewrite the result in terms of single-particle states with non-relativistic normalization (*i.e.*  $\langle \vec{p} | \vec{p}' \rangle_{NR} = \delta^3(p - p')$ ). Interpret  $h$  as a non-relativistic hamiltonian term saying that the gyromagnetic ratio of the electron is  $-g\frac{|q|}{2m}$  with  $g = 2$ .

- (e) [optional] How does the result change if we add the term

$$\Delta H = \frac{c}{M} \bar{\Psi} F_{\mu\nu} [\gamma^\mu, \gamma^\nu] \Psi \quad ?$$

## 6. Non-relativistic interactions from QFT.

- (a) **Coulomb potential.**

Derive from QED that the force between non-relativistic electrons is a repulsive  $1/r^2$  force law!

- (b) **Pseudoscalar Yukawa theory.**

Consider the theory of a massive Dirac fermion  $\Psi$  and a massive pseudoscalar  $\varphi$  interacting via the term

$$V_5 \equiv g_5 \bar{\Psi} \gamma^5 \Psi \varphi.$$

Convince yourself that this theory is parity invariant (for some assignment of the action of parity on the fields).

List the Feynman rules.

Draw and evaluate the diagrams contributing to  $\Psi\Psi \rightarrow \Psi\Psi$  scattering at leading order in  $g_5$ .

Consider the non-relativistic limit,  $m \gg |\vec{p}|$  and find the effective interaction hamiltonian. If you happen to find zero for the leading term, then it's not the leading term.

**7. Electron-positron scattering.** [Bonus problem]

Draw and evaluate the two diagrams that contribute to  $e^+e^- \rightarrow e^+e^-$  (Bhabha) scattering at tree level in QED. Be careful about the relative sign of their contributions.

Compare with the case of  $e^+e^- \rightarrow \mu^+\mu^-$  and with  $e^-e^- \rightarrow e^-e^-$ .

**8. Supersymmetry.** [Bonus problem] A continuous symmetry that mixes bosons and fermions is called *supersymmetry*.

- (a) The simplest example of a supersymmetric field theory is the theory of a free complex boson and a free Weyl fermion, with Lagrangian is

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger \mathbf{i} \bar{\sigma}^\mu \partial_\mu \chi + F^* F.$$

Here  $F$  is an auxiliary field whose purpose is to make the supersymmetry transformations look nice. Show that the action is invariant under

$$\delta\phi = -\mathbf{i}\epsilon^T \sigma^2 \chi, \delta\chi = \epsilon F + \sigma \cdot \partial\phi \sigma^2 \epsilon^*, \delta F = -\mathbf{i}\epsilon^\dagger \bar{\sigma} \cdot \partial\chi \quad (1)$$

where the symmetry parameter  $\epsilon$  is a 2-component spinor of Grassmann numbers.

- (b) Show that the term

$$\Delta\mathcal{L} = \left( m\phi F + \frac{1}{2} \mathbf{i} m \chi^T \sigma^2 \chi \right) + h.c.$$

is also invariant under the transformation (9). Eliminate  $F$  from the full Lagrangian  $\mathcal{L} + \Delta\mathcal{L}$  by solving its equations of motion, and show that the fermion and boson fields are given the same mass.

- (c) We can include supersymmetric interactions as well. Show that the following field theory is supersymmetric:

$$\mathcal{L} = \partial_\mu \phi_i^* \partial^\mu \phi^i + \chi_i^\dagger \mathbf{i} \bar{\sigma} \cdot \partial \chi_i + F_i^* F_i + \left( F_i \partial_{\phi_i} W + \frac{\mathbf{i}}{2} \partial_{\phi_i} \partial_{\phi_j} W \chi_i^T \sigma^2 \chi_j + h.c. \right)$$

where  $i = 1..n$  and  $W = W(\phi)$  is an arbitrary function of the  $\phi_i$ , called the *superpotential*. For the simple case  $n = 1$  and  $W = g\phi^3/3$  write out the field equations for  $\phi$  and  $\chi$  after eliminating  $F$ .