University of California at San Diego – Department of Physics – Prof. John McGreevy

# Physics 215A QFT Fall 2023
# Assignment 3

**Due 11:00am Monday, October 23, 2023**

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1. **Classical Maxwell theory.** [Peskin problem 2.1, lightly edited] Classical electromagnetism follows from the action

$$S[A] = \int d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\mu A_\mu \right), \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

(a) Derive two of the Maxwell equations (Gauss' Law and Faraday's Law) as the Euler-Lagrange equations of this action, treating the components $A_\mu(x)$ as the dynamical variables:

$$0 = \frac{\delta S[A]}{\delta A_\mu(x)}.$$

Write the equations in the standard form by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

(b) Construct the energy-momentum tensor for this theory, when $j^\mu = 0$. Note that the usual procedure

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu$$

does not result in a symmetric tensor. (It is also not gauge invariant.) To remedy that, we can add to $T^{\mu\nu}$ a term of the form $\partial_\lambda K^{\lambda\mu\nu}$, where $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\widehat{T}^{\mu\nu} \equiv T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu,$$

leads to an energy-momentum tensor $\widehat{T}$ that is symmetric and yields the standard formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2}\left(E^2 + B^2\right), \quad \vec{S} = \vec{E} \times \vec{B}.$$

(c) [Bonus problem] A better way to think about the energy-momentum tensor is to regard it as the response to a change in the background metric. (This is why it appears as a source in Einstein's equations.) To couple the Maxwell theory to a general background metric $g_{\mu\nu}$, we replace all the $\eta_{\mu\nu}$s with $g_{\mu\nu}$s:

$$S[A,g] = \int d^4x \sqrt{g} \left( -\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} + j^\mu A_\mu \right)$$

where the factor of $\sqrt{g} \equiv \sqrt{|\det g|}$ is required to make the integration measure coordinate-invariant, and $g^{\mu\nu}$ is the inverse metric: $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$. Compare the resulting energy-momentum tensor

$$T^{\mu\nu}_g = \frac{2}{\sqrt{g}} \frac{\delta S[A,g]}{\delta g_{\mu\nu}} |_{g_{\mu\nu} = \eta_{\mu\nu}}.$$

with that of the previous part.

Notice that $T^{\mu\nu}_g$ is automatically symmetric and gauge invariant.

[Some useful identities are:

$$\frac{\delta g^{\mu\nu}(x)}{\delta g_{\rho\sigma}(y)} = -g^{\mu\rho} g^{\nu\sigma} \delta^D(x-y) \text{ and } \frac{\delta \det g(x)}{\delta g_{\mu\nu}(y)} = \delta^D(x-y) \det g g^{\mu\nu}.$$

For proofs of these statements see page 93 of this document.]

2. **Maxwell's equations, quantumly.**

(a) Check that the oscillator algebra for the photon creation and annihilation operators

$$[\mathbf{a}_{ks}, \mathbf{a}^\dagger_{k's}] = \delta^3(k-k')\delta_{ss'}. \tag{1}$$

implies (using the mode expansion for $\mathbf{A}$) that

$$[\mathbf{A}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = -\mathbf{i}\hbar \int đ^3k \; e^{\mathbf{i}\vec{k}\cdot(\vec{r}-\vec{r}')} \left( \delta_{ij} - \hat{k}_i \hat{k}_j \right)$$

(and also $[\mathbf{A}_i(\vec{r}), \mathbf{A}_j(\vec{r}')] = 0$ and $[\mathbf{E}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = 0$).

Conclude that it's not possible to simultaneously measure $E_x(\vec{r})$ and $B_y(\vec{r})$.

(b) Using the result of the previous part, check that the wave equation for $\mathbf{A}_i(x)$ follows from the Heisenberg equations of motion

$$\partial_t \vec{\mathbf{E}} = \frac{\mathbf{i}}{\hbar} [\mathbf{H}, \vec{\mathbf{E}}].$$

3. **Goldstone boson.** Here is a simple example of the Goldstone phenomenon, which I mentioned briefly in lecture. Consider again the complex scalar field from a previous assignment.

   Suppose the potential is

$$V(\Phi^\star\Phi) = g\left(\Phi^\star\Phi - v^2\right)^2$$

   where $g, v$ are constants. The important features of $V$ are that (1) it is only a function of $|\Phi|^2 = \Phi\Phi^\star$, so that it preserves the particle-number symmetry generated by $\mathbf{q}$ which was the hero a previous homework problem, and (2) the minimum of $V(x)$ away from $x = 0$.

   Treat the system classically. Write the action $S[\Phi, \Phi^\star]$ in polar coordinates in field space:

$$\Phi(x,t) = \rho e^{\mathbf{i}\theta}$$

   where both $\rho, \theta$ are functions of space and time.

   (a) Consider constant field configurations, and show that minimizing the potential fixes $\rho$ but not the phase $\theta$.

   (b) Compute the mass$^2$ of the $\rho$ field about its minimum, $m_\rho^2 = \frac{1}{2}\partial_\rho^2 V|_{\rho=v}$.

   (c) Now ignore the deviations of $\rho$ from its minimum (it's heavy and slow and hard to excite), but continue to treat $\theta$ as a field. Plug the resulting expression

$$\Phi = ve^{\mathbf{i}\theta(x,t)}$$

   into the action. Show that $\theta$ is a massless scalar field.

   (d) How does the $U(1)$ symmetry generated by $\mathbf{q}$ act on $\theta$?

4. **Angular momentum as Noether charge.**

   Consider an arbitrary local scalar field theory with Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ such that the action is invariant under rotations of space. A rotation acts on a scalar field by

$$\phi(x^i) \mapsto \phi(R^i_j x^j) \tag{2}$$

   where $R$ is a rotation matrix $R^T R = 1$. Consider an infinitesimal rotation $R^i_j = \delta^i_j + \omega^i_j$, for $\omega$ an arbitrary (small) real antisymmetric matrix specifying the plane and angle of rotation. Find the associated Noether currents in terms of $\phi$.

   Relate your answer to the form of the stress tensor. [Hint for interpreting the answer: a rotation is a space-dependent translation.]

If you prefer, you may consider instead the full Lorentz group, which acts by

$$\phi(x^\mu) \mapsto \phi(\Lambda^\mu_\nu x^\nu) \tag{3}$$

where $\Lambda$ is a $D \times D$ matrix preserving the Minkowski metric $\eta_{\mu\nu}$.

Bonus problem: find the associated Noether charges in the case of a field theory of a vector such as electromagnetism, which transforms like $A_\mu(x) \mapsto \left(R^{-1}\right)_\mu^{\ \nu} A_\nu(Rx)$.

The following two problems can be handed either with HW03 or with HW04.

5. **Casimir force is regulator-independent.** [Bonus problem] Suppose we use a different regulator for the sum in the vacuum energy $\sum_j \hbar\omega_j$. The regulator we'll use here is an analog of Pauli-Villars. In the notation introduced in the lecture notes, we replace

$$f(d) \rightsquigarrow \frac{1}{2}\sum_{j=1}^{\infty} \omega_j K(\omega_j)$$

where the function $K$ is

$$K(\omega) = \sum_\alpha c_\alpha \frac{\Lambda_\alpha}{\omega + \Lambda_\alpha}.$$

We impose two conditions on the parameters $c_\alpha, \Lambda_\alpha$:

- We want the low-frequency answer to be unmodified:

$$K(\omega) \overset{\omega \to 0}{\to} 1$$

– this requires $\sum_\alpha c_\alpha = 1$.

- We want the sum over $j$ to converge; this requires that $K(\omega)$ falls off faster than $\omega^{-2}$. Taylor expanding in the limit $\omega \gg \Lambda_\alpha$, we have

$$K(\omega) \overset{\omega \to \infty}{\to} \frac{1}{\omega}\sum_\alpha c_\alpha \Lambda_\alpha - \frac{1}{\omega^2}\sum_\alpha c_\alpha \Lambda_\alpha^2 + \cdots .$$

So we also require $\sum_\alpha c_\alpha \Lambda_\alpha = 0$ and $\sum_\alpha c_\alpha \Lambda_\alpha^2 = 0$.

First, verify the previous claims about $K(\omega)$.

Then compute $f(d)$ and show that with these assumptions, the Casimir force is independent of the parameters $c_\alpha, \Lambda_\alpha$.

[A hint for doing the sum: use the identity

$$\frac{1}{X} = \int_0^\infty ds e^{-sX}$$

inside the sum to make it a geometric series. To do the remaining integral over $s$, Taylor expand the integrand in the regime of interest.]

6. **Casimir energy from balls and springs.** [More difficult bonus problem] Regularize the Casimir energy of a 1d scalar field by discretizing space. If you suppose there are $N \equiv d/a \in \mathbb{Z}$ lattice points in the left cavity

$$| \leftarrow d \rightarrow | \longleftarrow \qquad L - d \qquad \longrightarrow |$$

what answer do you find for the force on the middle plate?

[Hint: you will find the wrong answer! The problem is that with these assumptions $d$ cannot vary continuously. One way to allow $d$ to vary continuously (and get the right answer) is to impose $\phi(0) = 0 = \phi(d)$, but do not assume $d$ corresponds to a lattice site.]