University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2023
Assignment 4

Due 11:00am Monday, October 30, 2023

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1. **Scale transformations.**

   (a) Check that the following action

   $$S[\phi] = \int d^D x \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - g\phi^p \right) \tag{1}$$

   is invariant under the scale transformation $\phi(x^\mu) \mapsto \lambda^{d_\phi}\phi(\lambda x^\mu)$ for some choice of p and some choice of d_ϕ that depend on D. Check that your answer is the same as the mass dimension of ϕ.

   (b) Find the infinitesimal form of the transformation of ϕ under a scale transformation.

   (c) Find the associated conserved current j_s^μ and conserved charge $\mathbf{s} \equiv \int d^d x j_s^0$ (the dilatation generator).

   (d) Check using the canonical commutation that the dilatation generator $\mathbf{s}$ indeed generates dilatations.

   (e) For a general field theory with Lagrangian density that depends on the fields and their first derivatives, relate the dilatation current j_s^μ to the stress-energy tensor.

2. **Gaussian integrals are your friend.**

   (a) Show that

   $$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2 + jx} = \sqrt{\frac{2\pi}{a}} e^{\frac{j^2}{2a}}.$$

   [Hint: square the integral and use polar coordinates.]

   (b) Consider a collection of variables $x_i, i = 1..N$ and a real, symmetric matrix a_{ij}. Show that

   $$\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i} = \frac{(2\pi)^{N/2}}{\sqrt{\det a}} e^{\frac{1}{2}J^i a_{ij}^{-1} J^j}.$$

   (Summation convention in effect, as always.)

   [Hint: change integration variables to diagonalize a. $\det a = \prod a_i$, where a_i are the eigenvalues of a.]

(c) I include this problem partly because it might be helpful for a future problem. In that regard, for any function of the N variables, $f(x)$, let

$$\langle f(x)\rangle \equiv \frac{\int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j} f(x)}{Z[J=0]}, \quad Z[J] = \int \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}x_i a_{ij} x_j + J^i x_i}.$$

Show that

$$\langle x_i x_j\rangle = \partial_{J_i}\partial_{J_j} \log Z[J]|_{J=0} = a_{ij}^{-1}$$

Also, convince yourself that

$$\left\langle e^{J_i x_i}\right\rangle = \frac{Z[J]}{Z[J=0]}.$$

(d) Note that the number N in the previous parts may be infinite. This is really the only path integral we know how to do.

3. **Gaussian identity.** Show that for a gaussian quantum system

$$\left\langle e^{\mathbf{i}K\mathbf{q}}\right\rangle = e^{-A(K)\left\langle \mathbf{q}^2\right\rangle}$$

and determine $A(K)$. Here $\langle ...\rangle \equiv \langle 0| ... |0\rangle$, vacuum expectation value. Here by 'gaussian' I mean that $\mathbf{H}$ contains only quadratic and linear terms in both $\mathbf{q}$ and its conjugate variable $\mathbf{p}$ (but for the formula to be exactly correct as stated you must assume $\mathbf{H}$ contains only terms quadratic in $\mathbf{q}$ and $\mathbf{p}$; for further entertainment fix the formula for the case with linear terms in $\mathbf{H}$).

I recommend using the path integral representation (with hints from the previous problem). Alternatively, you can use the harmonic oscillator operator algebra. Or, even better, do it both ways.

4. **Zero-phonon processes.**

We wish to understand the probability for a photon to hit (our crude model of) a crystalline solid without exciting any vibrational excitations. It is a nice simultaneous application of *many* of the things we've learned so far (quantization of the radiation field, solution of the harmonic chain, gaussian integrals, the relationship between path integrals and transition amplitudes).

When a photon hits a lattice of atoms, Fermi's golden rule says that the (leading approximation to the) probability for a transition from one state of the lattice $|L_i\rangle$ to another $|L_f\rangle$ is proportional to

$$W(L_i \to L_f) = |\langle L_f| \mathbf{H}_{\mathrm{L}} |L_i\rangle|^2.$$

Here $\mathbf{H}_{\mathrm{L}}$ is the hamiltonian describing the interaction between the photon and an atom in the lattice. For the first parts of the problem, use the following form (to be justified in the last part of the problem):

$$\mathbf{H}_{\mathrm{L}} = Ae^{\mathbf{i}K\mathbf{x}} + h.c. \tag{2}$$

where $\mathbf{x}$ is the (center of mass) position operator of the atom being struck; K is a constant (the photon wavenumber), and (for the purposes of the first parts of the problem) A is a constant. $+h.c.$ means 'plus the hermitian conjugate of the preceding stuff'.

(a) Recalling that $\mathbf{x}$ (up to an additive constant) is part of a collection of coupled harmonic oscillators:

$$\mathbf{x} = nx + \mathbf{q}_n$$

evaluate the "vacuum persistence amplitude" $\langle 0|\, \mathbf{H}_{\mathrm{L}}\, |0\rangle$. You will find the results of the previous problem set useful.

(b) From the previous calculation, you will find an expression that requires you to sum over wavenumbers. Show that in one spatial dimension, the probability for a zero-phonon transition is of the form

$$P_{\text{Mössbauer}} \propto e^{-\Gamma \ln L}$$

where L is the length of the chain and Γ is a function of other variables. Show that this infrared divergence is missing for the analogous model of crystalline solids with more than one spatial dimension. (Cultural remark: these amplitudes are called 'Debye-Waller factors').

(c) Convince yourself that a coupling $\mathbf{H}_{\mathrm{L}}$ of the form (17) arises from the minimal coupling of the electromagnetic field to the constituent charges of the atom, after accounting for the transition made by the radiation field when the photon is absorbed by the atom. 'Minimal coupling' means replacing the momentum operator of the atom $\mathbf{p}$, with the gauge-invariant combination $\mathbf{p} \to \mathbf{p} + \mathbf{A}$. You will also need to recall the form of the quantized electromagnetic field in terms creation and annihilation operators for a photon of definite momentum K.