University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2023
Assignment 6

Due 11:00am Monday, November 13, 2023

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1. **Brain-warmer: the identity does nothing twice.** Check our relativistic state normalization by squaring the expression for the identity in the 1-particle sector:
$$\mathbb{1}_1^2 \overset{!}{=} \mathbb{1}_1 = \int \frac{đ^d p}{2\omega_{\vec{p}}} |\vec{p}\rangle \langle \vec{p}| .$$

2. **Non-Abelian currents.** [bonus problem] On a previous homework, we studied a complex scalar field. Now, we make a big leap to *two* complex scalar fields, $\Phi_{\alpha=1,2}$, with
$$S[\Phi_\alpha] = \int d^d x dt \left(\frac{1}{2} \partial_\mu \Phi_\alpha^\star \partial^\mu \Phi_\alpha - V\left(\Phi_\alpha^\star \Phi_\alpha\right) \right)$$
Consider the objects
$$Q^i \equiv \frac{1}{2} \int d^d x \mathbf{i} \left(\Pi_\alpha^\dagger \sigma_{\alpha\beta}^i \Phi_\beta^\dagger \right) + h.c.$$
where $\sigma^{i=1,2,3}$ are the three Pauli matrices.
   (a) What symmetries do these charges generate (*i.e.* how do the fields transform)? Show that they are symmetries of S.
   (b) If you want to, show that $[Q^i, H] = 0$, where H is the Hamiltonian.
   (c) Evaluate $[Q^i, Q^j]$. Hence, non-Abelian. Where have you seen this algebra before?
   (d) To complete the circle, find the Noether currents J_μ^i associated to the symmetry transformations you found in part 2a. Show that the resulting charge $Q^i = \int d^d x J_0^i$ agrees with our starting point.
   (e) Generalize to the case of N scalar fields.

3. **Recovering non-relativistic quantum mechanics.**

   Consider a complex scalar field, in the non-relativistic limit,
$$\Phi = \sqrt{2m} e^{-\mathbf{i}mt} \Psi, \quad |\dot{\Psi}| \ll m\Psi.$$
   Recall that in this limit, the antiparticles disappear and the mode expansion is
$$\Psi(x) = \int đ^d p \; e^{+\mathbf{i}\vec{p}\cdot\vec{x}} \mathbf{a}_p, \quad \Psi^\dagger(x) = \int đ^d p \; e^{-\mathbf{i}\vec{p}\cdot\vec{x}} \mathbf{a}_p^\dagger .$$

(a) Show that

$$\hat{P}_i \equiv \int \text{đ}^d p p_i \mathbf{a}_p^\dagger \mathbf{a}_p$$

is the generator of translations and commutes with the Hamiltonian.

(b) Let

$$\hat{X}^i \equiv \int d^d x \Psi^\dagger(x) x^i \Psi(x).$$

A state of one particle at location $\vec{x}$ is

$$|x\rangle = \Psi^\dagger(x) |0\rangle .$$

Show that

$$\hat{X}^i |x\rangle = x^i |x\rangle .$$

(c) Consider the general one-particle state

$$|\psi\rangle = \int d^d x \ \psi(x) \Psi^\dagger(x) |0\rangle = \int d^x x \ \psi(x) |x\rangle .$$

Show that

$$\hat{X}^i |\psi\rangle = \int d^d x x^i \psi(x) |x\rangle$$

and (a little more involved)

$$\hat{P}^i |\psi\rangle = \int d^d x \left(-\mathbf{i} \frac{\partial}{\partial x^i} \psi(x) \right) |x\rangle ,$$

which is the usual action of these operators on single-particle wavefunctions $\psi(x)$.

4. **Combinatorics from 0-dimensional QFT.** [This is a more sophisticated bonus problem. I will not post the solutions of this problem until later. If you have a hard time with it now, please try again in a week.]

Catalan numbers $C_n = \frac{(2n)!}{n!(n+1)!}$ arise as the answer to many combinatorics problems (beware: there is some disagreement in the literature about whether this is C_n or C_{n+1}).

One such problem is: count random walks on a 1d chain with $2n$ steps that start at 0 and end at 0 without crossing 0 in between.

Another such problem is: in how many ways can $2n$ (distinguishable) points on a circle be connected by chords that do not intersect within the circle.

Consider a zero-dimensional QFT with the following Feynman rules:

- There are two fields h and l.
- There is an $\sqrt{t}h^2 l$ vertex in terms of a coupling t.
- The bare l propagator is 1.
- The bare h propagator is 1.
- All diagrams can be drawn on a piece of paper without crossing.
- An annoying extra rule: All the l propagators must be on one side of the h propagators[1].
- There are no loops of h.

The last two rules can be realized from a lagrangian by introducing a large N (below).

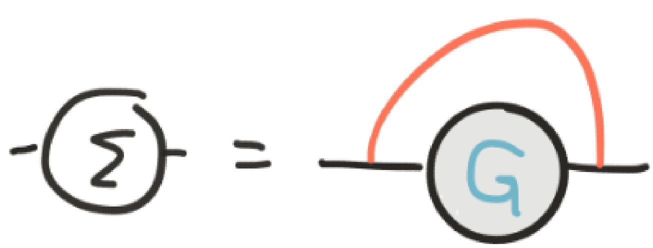
(a) Show that the full two-point green's function for h is

$$G(t) = \sum_n t^n C_n$$

the generating function of Catalan numbers.

(b) Let $\Sigma(t)$ be the sum of diagrams with two h lines sticking out that may not be divided into two parts by cutting a single intermediate line. (This property is called 1PI (one-particle irreducible), and Σ is called the "1PI self-energy of h". We'll use this manipulation all the time later on.) Show that $G(t) = \frac{1}{1-\Sigma(t)}$.

(c) Argue by diagrams for the equation (sometimes this is also called a Schwinger-Dyson equation)



where Σ is the 1PI self-energy of h.

(d) Solve this equation for the generating function $G(t)$.

[1] You'll see in part 4f how to justify this.

(e) If you are feeling ambitious, add another coupling N^{-1} which counts the crossings of the l propagators. The resulting numbers can be called Touchard-Riordan numbers.

(f) How to realize the no-crossings rule? Consider

$$L = \frac{\sqrt{t}}{\sqrt{N}} l_{\alpha\beta} h_\alpha h_\beta + \sum_{\alpha,\beta} l_{\alpha\beta}^2 + \sum_\alpha h_\alpha^2$$

where $\alpha, \beta = 1 \cdots N$. By counting index loops, show that the dominant diagrams at large N are the ones we kept above. Hint: to keep track of the factors of N, introduce ('t Hooft's) double-line notation: since l is a matrix, its propagator looks like: $\begin{matrix}\alpha - - - - - - - - \alpha \\ \beta - - - - - - - - \beta\end{matrix}$, while the h propagator is just one index line α______α, and the vertex is ___!!___. If you don't like my ascii diagrams, here are the respective pictures: $\langle l_{\alpha\beta} l_{\alpha\beta} \rangle =$  ,

$\langle h_\alpha h_\alpha \rangle =$  and the hhl vertex is:  . A closed index loop gives a factor of N.

(g) Use properties of Catalan numbers to estimate the size of non-perturbative effects in this field theory.

(h) There are many other examples like this. Another similar one is the relationship between symmetric functions and homogeneous products. A more different one is the enumeration of planar graphs. For that, see BIPZ.

5. **Scalar Yukawa amplitudes.**

Consider again the scalar Yukawa theory of a complex scalar Φ and a real scalar ϕ. In the following, assume all particles are in momentum eigenstates. Use artisanal methods.

(a) Compute the amplitude for the annihilation of a Φ particle and a $\Phi^\star$ particle into a ϕ particle, at leading order in the coupling g.

(b) Compute the amplitude for $\Phi + \phi \to \Phi + \phi$ scattering to the leading nontrivial order in the coupling. Focus on the generic case where none of the initial momenta is the same as any of the final momenta.

Can you write the answer in a manifestly Lorentz-invariant way?

6. **Fields and forces.** Consider a real free relativistic scalar field of mass m $S[\phi] = \int d^{d+1}x \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \right)$.

(a) Calculate the vacuum expectation value

$$\langle 0 | \mathcal{T} \left(e^{\mathbf{i} \int d^{d+1}x \ \phi(x) J(x)} \right) |0\rangle \equiv e^{\mathbf{i} W[J]}$$

where J is a fixed, external (c-number) source. Use Wick's theorem. Make a series expansion in powers of J and draw some diagrams. To understand the structure of the series, recall the formula on a previous homework for $\left\langle e^{K \cdot q} \right\rangle$ in any gaussian theory.

(b) Now specialize to the case where the source is static and is present for a time $2T$:

$$J(x) = J_{\text{static}} \equiv \theta(T - t)\theta(t + T) \left(\delta^d(x) - \delta^d(x - R) \right)$$

with $T \gg R \gg 1/m$. You should find an answer of the form

$$W \left[J_{\text{static}}(x) \right] = -TV(R)$$

where $V(R)$ is the Yukawa potential.

(c) Chant the following incantation:

Static sources experience a force due to exchange of virtual particles.

Feel happy at having reproduced by canonical methods the answer we found earlier using path integral methods.