University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2023
Assignment 8

Due 11:00am Monday, November 27, 2023

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1. **Brain warmer.** Mandelstam variables are Lorentz-invariant combinations of the kinematic variables in $2 \leftarrow 2$ scattering

$$s = (p_1 + p_2)^2, t = (p_1 - p_3)^2, u = (p_1 - p_4)^2$$

where p_i^μ are on-shell four-vectors $p_i^2 = m_i^2$ satisfying overall momentum conservation $p_1 + p_2 = p_3 + p_4$. Show that they are not independent but rather satisfy the relation

$$s + t + u = \sum_i m_i^2$$

where i runs over the four external particles.

2. **Particle creation by an external source, continued.**

Consider again the Hamiltonian

$$H = H_0 + \int d^3x \left(-j(t, \vec{x})\phi(x)\right)$$

where H_0 is the free Klein-Gordon Hamiltonian, ϕ is the Klein-Gordon field, and j is a c-number scalar function.

(a) Compute the probability amplitude for the source to create one particle of momentum k. Perform this computation first to $\mathcal{O}(j)$, and then to all orders, using the trick from HW07 to sum the series.

Compute the analogous amplitude for n particles of definite momenta.

(b) Show that the probability of producing n particles (of any momenta) is given by the Poisson distribution,

$$P(n) = \frac{1}{n!}\lambda^n e^{-\lambda}.$$

[Note that for $n > 1$, this requires the measure for the final state phase space, and that's why this part of the problem had to wait for this week.]

(c) Prove the following facts about the Poisson distribution:

$$\sum_{n=0}^{\infty} P(n) = 1, \quad \langle N \rangle \equiv \sum_{n=0}^{\infty} nP(n) = \lambda,$$

that is, $P(n)$ is a probability distribution, and $\langle N \rangle = \lambda$ as predicted. Compute the fluctuations in the number of particles produced $\langle (N - \langle N \rangle)^2 \rangle$.

3. Can there ever be a resonance in a t-channel diagram? Let me break the question down a bit:

   (a) Consider a $2 \leftarrow 2$ scattering process where all the particles have the same mass. Let p_1, p_2 be the momenta of the particles in the initial state. Prove that the Mandelstam variables $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$ cannot be positive when the particles are on-shell $p_i^2 = m^2$.

   (b) Bonus problem: What happens if the particles have different masses? It may be worth distinguishing two cases:
   (a) when the collision is *elastic*, so that the particles retain their identity and therefore $m_1 = m_3$ and $m_2 = m_4$.
   (b) the fully general case where m_i are all different.

4. **Decay of a scalar particle.**

   Consider the following Lagrangian, involving two *real* scalar fields Φ and ϕ:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi - M^2 \Phi^2 + \partial_\mu \phi \partial^\mu \phi - m^2 \phi^2\right) - \mu \Phi \phi^2.$$

   The last term is an interaction that allows a Φ particle to decay into two ϕs, if the kinematics allow it. Calculate the lifetime of the Φ particle to lowest order in μ. In this problem you can set $d = 3$. What is the condition on the masses for a finite lifetime?

5. **Scalar particle scattering cross-sections.**

   What is the leading-order differential cross-section $\frac{d\sigma}{d\Omega}$ for $2 \to 2$ snucleon-snucleon scattering in $d = 3$ space dimensions in the center-of-mass frame?

   What is the total cross section in the limit that the snucleons (the particles being scattered) are massless?

6. If you didn't finish it earlier, try again problem 4 on HW06 (the one about Catalan numbers). I'll post my solution with this problem set.