University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 215A QFT Fall 2023
Assignment 9

Due 11:00am Monday, December 4, 2023

---

1. **Brain warmers on $\mathsf{SO}(3)$.**

   (a) Consider the statement that the rotation generators transform as a vector under rotations:

   $$\left(D_{(j=1)}(\theta)\right)^k_{\ j}\mathbf{J}^j = D_R(\theta)^\dagger \mathbf{J}^k D_R(\theta), \tag{1}$$

   where $D_R(\theta) = e^{-\mathbf{i}\theta^i \mathbf{J}^i}$ and $D_{(j=1)}(\theta) = e^{-\mathbf{i}\theta^i J^i_{(j=1)}}$, with $(J^i_{(j=1)})^j_{\ k} = -\mathbf{i}\epsilon^{ijk}$. Show that to leading nontrivial order in θ (about $\theta = 0$) this is equivalent to the $\mathsf{so}(3)$ Lie algebra,

   $$[\mathbf{J}^i, \mathbf{J}^j] = \mathbf{i}\epsilon^{ijk}\mathbf{J}^k. \tag{2}$$

   (b) Starting from the form of the generators in the vector (spin 1) representation,

   $$(\mathbf{J}^i)^j_{\ k} = -\mathbf{i}\epsilon^{ijk} \tag{3}$$

   (with $\epsilon^{123} = 1$) construct the matrix realizing a rotation by angle θ about the z axis on a vector.

   (c) Here let's show that the equation

   $$[J^i, K^j] = \mathbf{i}\epsilon^{ijk}K^k \tag{4}$$

   can really be interpreted as the statement that "K is a vector". Let's just think about rotations about one axis $\hat{n}$. Suppose J^i and K^k here are operators acting on the representation R. Let

   $$K(s)^k \equiv \hat{D}_R(s)^\dagger K^k \hat{D}_R(s) \tag{5}$$

   (note that k here is an index, not a power) where $\hat{D}_R(s) \equiv e^{-\mathbf{i}s\hat{n}\cdot\vec{J}}$, so s parametrizes the angle of rotation. Show that (6) implies that $K(s)$ satisfies the ODE

   $$\partial_s K(s)^k = \epsilon^{kil}n^i K(s)^l \tag{6}$$

   with initial condition $K(0)^k = K^k$.

   Using uniqueness theorems about solutions of linear ODEs that

   $$K(s)^k = \left(D_{(j=1)}(s\hat{n})\right)^k_{\ j}K^j, \tag{7}$$

   where $\left(D_{(j=1)}(s\hat{n})\right)^k_{\ j}$ are the matrix elements of the spin one representation of this rotation.

2. **Lorentz algebra in $D = 3 + 1$.**

(a) Check the algebra satisfied by rotations and boosts

$$[J^i, J^j] = \mathbf{i}\epsilon^{ijk}J^k, \quad [J^i, K^j] = \mathbf{i}\epsilon^{ijk}K^k, \quad [K^i, K^j] = -\mathbf{i}\epsilon^{ijk}J^k \tag{8}$$

using the explicit matrices in the vector representation given in lecture, namely

$$J^i = \begin{pmatrix} 0 & \\ & \mathbf{J}^i \end{pmatrix} \tag{9}$$

(where the 3×3 matrix $\mathbf{J^i}$ is $(\mathbf{J}^i)^j_k = -\mathbf{i}\epsilon^{ijk}$) and

$$\left(K^i\right)^j{}_0 = \mathbf{i}\delta^i_j = \left(K^i\right)^0{}_j \tag{10}$$

and all other components zero.

For this purpose, I think typing them into Mathematica and writing $K.J - J.K$ etc.... is perfectly acceptable.

(b) Check that in terms of the antisymmetric tensor of operators

$$J^{\mu\nu} = \begin{cases} \epsilon^{ijk}J^k, & \mu\nu = ij \\ K^i, & \mu\nu = 0i \end{cases},$$

(11) can be rewritten as

$$[J^{\mu\nu}, J^{\rho\sigma}] = \mathbf{i}\left(\eta^{\nu\rho}J^{\mu\sigma} + \eta^{\mu\sigma}J^{\nu\rho} - (\mu \leftrightarrow \nu)\right), \tag{11}$$

which is the form of the $\mathsf{so}(d, 1)$ Lie algebra for general d. Make sure you take advantage of symmetries to avoid working too hard.

(c) Show that in $D = 3 + 1$, (11) is equivalent to $\mathsf{su}(2)_L \times \mathsf{su}(2)_R$:

$$[J^i_+, J^j_-] = 0, \qquad [J^i_\pm, J^j_\pm] = \mathbf{i}\epsilon^{ijk}J^k_\pm$$

in terms of

$$J^i_\pm \equiv \frac{1}{2}(J^i \pm \mathbf{i}K^i).$$

(d) **The Shayan-Arani operator.** [Bonus problem] Consider the object $C \equiv \vec{K} \cdot \vec{J}$ that Shasha asked about during lecture. During lecture I said that it's rotation invariant, but I thought that it might not be invariant under the full Lorentz group.

Show that in fact it *is* invariant under the full Lorentz group (i.e. $[\vec{K}, C] = 0, [\vec{J}, C] = 0$). So it is in fact a Casimir for the Lorentz group. That is, in *any* irrep it has to be proportional to the identity operator. It will be some function of the quantum numbers j_L and j_R. Find that function by relating it to other Casimirs and/or computing it in some familiar representations.

3. **A representation of the Clifford algebra gives a representation of Lorentz.** [Bonus problem] Show the following: Given a collection of $k \times k$ matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ (the Clifford algebra, with $\mu, \nu = 0..d$), we can make a k-dimensional representation of $SO(1,d)$ with generators

$$J^{\mu\nu} = \frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu].$$

As an intermediate step, it is helpful to show that

$$[J^{\mu\nu}, \gamma^\rho] \propto \left(\gamma^\mu \eta^{\nu\rho} - \gamma^\nu \eta^{\rho\mu}\right).$$

Convince yourself that this last equation says that γ^ρ transforms as a four-vector, *i.e.*

$$[\gamma^\rho, J^{\mu\nu}_{\text{Dirac}}] = (J^{\mu\nu}_{\text{vector}})^\rho{}_\sigma \gamma^\sigma.$$

4. **Bispinors.**

   (a) Show that any product of gamma matrices between two spinors

   $$V^{\mu_1 \cdots \mu_n} \equiv \bar{\Psi} \gamma^{\mu_1} \cdots \gamma^{\mu_n} \Psi \tag{12}$$

   is a tensor, in the sense that

   $$V^{\mu_1 \cdots \mu_n} \mapsto \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n} V^{\nu_1 \cdots \nu_n}. \tag{13}$$

   (A *tensor* is defined to be an object that transforms this way under Lorentz transformations.)

   (b) Let $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ be just the antisymmetric bit, and similarly for more indices. Show that any bispinor $\Gamma_{ab} \bar{\Psi}_a \Psi_b$ can be decomposed as a sum of these tensors: $\sum_n A_{\mu_1 \cdots \mu_n} \bar{\Psi} \gamma^{\mu_1 \cdots \mu_n} \Psi$, where the As are totally antisymmetric.

5. **Gamma matrices in other dimensions.**

   (a) Find a collection of D matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ in $D = 2$. Are they the smallest possible?

   (b) Find a collection of D matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ in $D = 3$. Are they the smallest possible?

   (c) Find a collection of D matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ in $D = 5$. Are they the smallest possible?

   (d) [bonus problem] Find a collection of D matrices satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ in $D = 6$, or any higher dimension. Are they the smallest possible?

The following problems are about discrete symmetries of scalar field theories. They are a useful warmup to the discussion of discrete symmetries of the Dirac fermion.

6. **Brain warmer: $\mathbb{Z}_2$ symmetry of real scalar field theory.**

   What does the operator

   $$U \equiv e^{\mathbf{i}\pi \sum_k \mathbf{a}_k^\dagger \mathbf{a}_k}$$

   do to the real scalar field

   $$\phi(x) \to \phi'(x) = U\phi(x)U^\dagger$$

   whose ladder operators are $\mathbf{a}, \mathbf{a}^\dagger$?

   For which Lagrangians is this a symmetry?

7. **Charge conjugation in complex scalar field theory.** [Bonus problem]

   Consider again a free complex Klein-Gordon field Φ. Define a discrete symmetry operation (charge conjugation) C, by

   $$\Phi(x) \mapsto C\Phi(x)C^{-1} = \eta_c \Phi^\dagger(x)$$

   where C is a unitary operator, and η_c is an arbitrary phase factor. Assume that the vacuum is invariant under charge conjugation: $C|0\rangle = |0\rangle$.

   (a) Show that the free lagrangian is invariant under C, but the particle number current j^μ changes sign.

   (b) Show that the annihilation operators satisfy

   $$C\mathbf{a}_k C^{-1} = \eta_c \mathbf{b}_k, \quad C\mathbf{b}_k C^{-1} = \eta_c^\star \mathbf{a}_k$$

   and hence show that C interchanges particle and antiparticle states, up to a phase.

8. **Parity symmetry of scalar field theory.** [Bonus problem]

   Under the parity transformation

   $$\vec{x} \mapsto \vec{x}' = -\vec{x}$$

   a real Klein-Gordon transforms as

   $$\phi(t, \vec{x}) \mapsto P\phi(t, x)P^{-1} = \eta_p \phi(t, -\vec{x}) \tag{14}$$

   where P is unitary and $\eta_P = \pm 1$ is the *intrinsic parity* of the field ϕ. Again assume $P|0\rangle = |0\rangle$.

(a) Show that the parity transformation preserves the free Lagrangian (though not the Lagrangian density), for both values of η_P.

(b) Show that an arbitrary n-particle state transforms as

$$P \left| \vec{k}_1, \cdots \vec{k}_n \right\rangle = \eta_P^n \left| -\vec{k}_1, \cdots , -\vec{k}_n \right\rangle .$$

(c) Here we give an explicit realization of the parity operator. Let

$$P_1 \equiv e^{-\mathbf{i}\frac{\pi}{2}\sum_k \mathbf{a}_k^\dagger \mathbf{a}_k}, \quad P_2 \equiv e^{\mathbf{i}\eta_p\frac{\pi}{2}\sum_k \mathbf{a}_k^\dagger \mathbf{a}_{-k}}.$$

Show that

$$P_1 \mathbf{a}_k P_1^{-1} = \mathbf{i}\mathbf{a}_k, \quad P_2 \mathbf{a}_k P_2^{-1} = -\mathbf{i}\eta_p \mathbf{a}_{-k}.$$

Hint: Use the following version of the Campbell-Baker-Hausdorff formula

$$e^{\mathbf{i}\alpha A} B e^{-\mathbf{i}\alpha A} = \sum_{n=0}^{\infty} \frac{(\mathbf{i}\alpha)^n}{n!} B_n$$

where $B_0 \equiv B$ and $B_n = [A, B_{n-1}]$ for $n = 1, 2.....$

Show that $P \equiv P_1 P_2$ is unitary, and satisfies (24).

(d) **Action on the current of a complex scalar field.** Consider now a complex scalar field. Using the results from problems 7 and the preceding parts of 8, find the action of parity on the particle current $j^\mu \mapsto P j^\mu P^{-1}$. (You'll have to extend the action of P from the case of a real field to the complex case.)