

Physics 215A: Quantum Field Theory

Fall 2023

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Last updated: 2024/03/20, 12:10:07

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0.1 Introductory remarks

Quantum field theory (QFT) is the quantum mechanics of *extensive degrees of freedom*. What I mean by this is that at each point of space, there's some stuff that can wiggle. Such a collection of stuff is called a field, and QFT really is just the quantum theory of fields.

It's not surprising that QFT is so useful, since this situation happens all over the place. Some examples of 'stuff' are: the atoms in a solid, or the electrons in those atoms, or the spins of those electrons. A less obvious, but more visible, example is the electromagnetic field, even in vacuum. More examples are provided by other excitations of the vacuum, and it will be our job here to understand those very electrons and atoms that make up a solid in these terms. The vacuum has other less-long-lasting excitations which are described by the Standard Model of particle physics, an important and overwhelmingly successful example of a QFT.

Some examples of QFT are Lorentz invariant ('relativistic'). That's a nice simplification when it happens. Indeed this seems to happen in particle physics. We're going to focus on this case for most of this quarter. Still I would like to emphasize: though some of the most successful applications of QFT are in the domain of high energy particle physics, this is not a class on that subject, and I will look for opportunities to emphasize the universality of QFT.

A consequence of combining quantum mechanics with special relativity is that the number of particles isn't fixed. That is: there are processes where the number of particles changes in time. Later on (in (3.1)), we'll understand in what sense it's a necessary consequence of Lorentz symmetry. The converse is false: particle production can and does happen all the time in non-relativistic situations. This business of particle production is a crucial point of departure and motivation for QFT, worth emphasizing, so let me stop and emphasize it.

Few-particle QM. In classes with the title 'Quantum Mechanics', we generally study quantum systems where the Hilbert space \mathcal{H}_1 holds states of a single particle (or sometimes a fixed small number of them), which are *rays* in \mathcal{H}_1 .

The observables of such a system are represented by hermitian operators acting on \mathcal{H}_1 . For example, the particle has a position \vec{x} and a momentum \vec{p} each of which is a d -vector of operators (for a particle in d space dimensions). The particle could be, for example, an electron (in which case it also has an inherent two-valuedness called spin) or a photon (in which case it also has an inherent two-valuedness called polarization).

Time evolution is generated by a Hamiltonian \mathbf{H} which is made from the position and momentum (and whatever internal degrees of freedom the particle has), in the

sense that $i\hbar\partial_t|\psi\rangle = \mathbf{H}|\psi\rangle$. Finally, the fourth (most ersatz) axiom of QM regards measurement: when measuring an observable \mathbf{A} in a state $|\psi\rangle \in \mathcal{H}$, we should decompose the state in the eigenbasis of \mathbf{A} : $\mathbf{A}|a\rangle = a|a\rangle$, $|\psi\rangle = \sum_a |a\rangle\langle a|\psi\rangle$; the probability to get the answer a is $|\langle a|\psi\rangle|^2$.

By the way: The components of the state vector in the position basis $\langle\vec{x}|\psi\rangle = \psi(\vec{x})$ is a function of space, the wavefunction. This looks like a field. It is not what we mean by a field in QFT. Meaningless phrases like ‘second quantization’ may conspire to try to confuse you about this.

An important point I want to convey (as in the first sentence above) is that a QFT is an ordinary quantum-mechanical system, obeying these same axioms, but with a different Hilbert space, involving a lot more degrees of freedom (dofs).

Now suppose you want to describe quantumly the emission of a photon from an atom with an excited electron. Surely this is something for which we need QM. But it’s not something that can happen within \mathcal{H}_1 , since the number of particles changes during the process. How do you do it?

In the first section of this course we’ll follow an organic route to discovering an answer to this question. This will have the advantage of making it manifest that the four axioms of QM just reviewed are still true in QFT. It will de-emphasize the role of Lorentz symmetry; in fact it will explicitly break it. Actually, Lorentz symmetry will emerge on its own!

‘Divergences’. Another intrinsic and famous feature of QFT discernible from the definition I gave above is its flirtation with infinity. I said that there is ‘stuff at each point of space’: how much stuff is that? Well, there are two senses in which ‘the number of points of space’ is infinite: (1) space can go on forever (the infrared (IR)), and (2) in the continuum, in between any two points of space there are more points (the ultraviolet (UV)). The former may be familiar from statistical mechanics, where it is associated with the thermodynamic limit, which is where the interesting things happen. This limit is not the origin of the famous ‘divergences’ in QFT.

Every QFT is an Effective Field Theory. In fact, the origin of the famous ‘divergences’ is hubris, of trying to use a tool where it is not meant to be used. One lesson I want to emphasize is that every QFT has a regime of usefulness. A QFT which is not meant to describe everything in the world down to arbitrarily short distances is called an Effective Field Theory (EFT). Some QFTs have the decency to tell us when they are wrong (they are called ‘non-renormalizable’).

Weak coupling versus strong coupling. Before getting into it, I want to ad-

vertise the existence of an important dichotomy. This quarter, we're going to study a number of important and physically-relevant quantum field theories quite successfully. This success can be attributed to the fact that these theories are *weakly-coupled*. As we'll see, this means that we can understand them by starting from a large (very large) collection of springs, and doing perturbation theory. To quote from Brian Skinner's great description of his own experience of learning this aspect of quantum field theory: "[So is the universe made of tiny springs...?](#)" Yes, younger Brian, that is exactly what we are saying.

Except: not every quantum field theory is *usefully* described by this machinery. If the tiny springs are coupled to each other too strongly, then this starting point isn't so helpful for describing their groundstate and low-lying excitations. Among the excitations of our vacuum, those that participate in the accurately-named Strong Interactions have this property; so do lots of other quantum field theories that arise in other ways, such as in condensed matter physics. What I mean by "this starting point isn't so helpful" is that the groundstate (the vacuum) may not be adiabatically related to that of the free springs. The multiplicity of phases of matter is reflected in the phases of QFT. Understanding this will be a job for the later quarters of the QFT sequence. Understanding the simple cases well is a necessary first step.

Sources and acknowledgement. The material in these notes is collected from many places, among which I should mention in particular the following:

Peskin and Schroeder, *An introduction to quantum field theory* (Wiley)

Zee, *Quantum Field Theory in a Nutshell* (Princeton, 2d Edition)

Le Bellac, *Quantum Physics* (Cambridge)

Schwartz, *Quantum field theory and the standard model* (Cambridge)

[David Tong's lecture notes](#)

Many other bits of wisdom come from the Berkeley QFT course of Prof. Lawrence Hall.

I also recommend the new book:

Fradkin, *Quantum Field Theory: An Integrated Approach* (Princeton)

which takes a similar perspective to mine.

0.2 Conventions

Following most QFT books, I am going to use the $+ - - -$ signature convention for the Minkowski metric. I am used to the other convention, where time is the weird one, so I'll need your help checking my signs. More explicitly, denoting a small spacetime displacement as $dx^\mu \equiv (dt, d\vec{x})^\mu$, the Lorentz-invariant distance is:

$$ds^2 = +dt^2 - d\vec{x} \cdot d\vec{x} = \eta_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad \eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} .$$

(spacelike is negative). We will also write $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial_t, \vec{\nabla}_x)^\mu$, and $\partial^\mu \equiv \eta^{\mu\nu} \partial_\nu$. I'll use μ, ν, \dots for Lorentz indices, and i, k, \dots for spatial indices.

$D = d + 1$ is the number of dimensions of spacetime. The convention that repeated indices are summed is always in effect unless otherwise indicated.

\equiv means 'equals by definition'. $A \stackrel{!}{=} B$ means we are demanding that $A = B$. $A \stackrel{?}{=} B$ means A probably doesn't equal B .

A consequence of the fact that english and math are written from left to right is that time goes to the left.

A useful generalization of the shorthand $\hbar \equiv \frac{h}{2\pi}$ is

$$d^d k \equiv \frac{d^d k}{(2\pi)^d} .$$

I will also write $\delta^d(q) \equiv (2\pi)^d \delta^{(d)}(q)$. I will try to be consistent about writing Fourier transforms as

$$\int \frac{d^d k}{(2\pi)^d} e^{ikx} \tilde{f}(k) \equiv \int d^d k e^{ikx} \tilde{f}(k) \equiv f(x) .$$

WLOG \equiv without loss of generality. IFF \equiv if and only if.

RHS \equiv right-hand side. LHS \equiv left-hand side. BHS \equiv both-hand side.

IBP \equiv integration by parts. WLOG \equiv without loss of generality.

$+\mathcal{O}(x^n) \equiv$ plus terms which go like x^n (and higher powers) when x is small.

$+h.c. \equiv$ plus hermitian conjugate.

We work in units where \hbar and the speed of light, c , and Boltzmann's constant, k_B , are equal to one unless otherwise noted. When I say 'Peskin' I usually mean 'Peskin & Schroeder'.

Please email me if you find typos or errors or violations of the rules above.

1 From particles to fields to particles again

Here is a way to discover QFT starting with some prosaic ingredients. Besides the advantages mentioned above, it will allow us to check that we are on the right track with well-understood experiments.

In modern physics, when I hear the word 'particle' I reach for my Fourier analyzer...
– J. Ziman,
Electrons and Phonons.

1.1 Quantum sound: Phonons

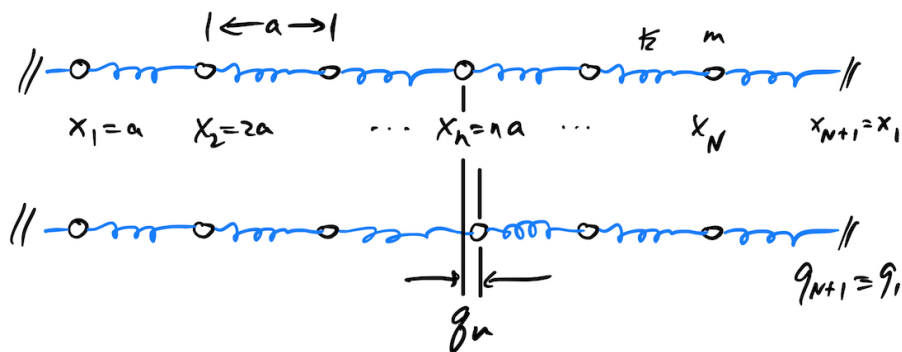
Let's think about a crystalline solid. The specific heat of solids (how much do you have to heat it up to change its internal energy by a given amount) was a mystery before QM. The first decent (QM) model was due to Einstein, where he supposed that the position of each atom is a (independent) quantum harmonic oscillator with frequency ω . This correctly predicts that the specific heat decreases as the temperature is lowered, but is very crude. Obviously the atoms interact: that's why they arrange themselves in a nice crystal pattern, and that's why there are sound waves, as we will see. By treating the elasticity of the solid quantum mechanically, we are going to discover quantum field theory. One immediate benefit of this will be a framework for quantum mechanics where particles can be created and annihilated.

What holds the atoms in a solid in place? Their interactions with their neighbors. Here we use two pieces of physics input: these interactions are translation invariant, and so only depend on the differences of positions $V(q_i - q_j)$, and they decay with distance, so we won't go too far wrong only considering interactions between neighbors.

As a more accurate toy model of a one-dimensional crystalline solid, let's consider a linear chain of particles of mass m , each connected to its neighbors by springs with spring constant κ . When in equilibrium, the masses form a regular one-dimensional crystal lattice (equally spaced mass points). Now let q_n denote the displacement of the n th mass from its equilibrium position x_n and let p_n be the corresponding momentum. Assume there are N masses and (for simplicity) impose periodic boundary conditions: $q_{n+N} = q_n$. The equilibrium positions themselves are

$$x_n = na, n = 1, 2 \dots N$$

where a is the lattice spacing.



The Hamiltonian for the collection of particles is:

$$\mathbf{H} = \sum_{n=1}^N \left(\frac{\mathbf{p}_n^2}{2m} + \frac{1}{2} \kappa (\mathbf{q}_n - \mathbf{q}_{n-1})^2 \right) + \lambda \mathbf{q}^4. \quad (1.1)$$

Notice that this system is an ordinary QM system, made of *particles*. In particular, the whole story below will take place within the fixed Hilbert space of the positions of the N particles.

I've included a heuristic token anharmonic term $\lambda \mathbf{q}^4$ to remind us that we are leaving stuff out; for example we might worry whether we could use this model to describe *melting*. A more precise form would be something like $\sum_{j=2} \lambda_j \sum_n \left(\frac{\mathbf{q}_n - \mathbf{q}_{n-1}}{a} \right)^j$, which describes the next terms in the Taylor expansion of some potential $V(q_n - q_{n-1})$ about the equilibrium configuration. If the springs are stiff, then the positions will not want to depart from their equilibrium values ($q_n - q_{n-1} \ll a$) and, even if the λ_j are all comparable, it might be a good approximation to ignore such terms. This partly explains the ubiquity of harmonic oscillators in discussions of physics¹. By the way, the fact that a solid will suddenly melt if you heat it up or reduce the pressure sufficiently is an indication of what I was saying earlier about the multiplicity of phases of matter reflecting the multiplicity of phenomena in QFT – in a regime where the solid melts, our starting point here in terms of harmonic oscillators will emphatically not be a good description!

Now we'll just set $\lambda = 0$. (It will be a little while before we turn back on the *interactions* resulting from nonzero λ ; bear with me.) This hamiltonian above describes a collection of coupled harmonic oscillators, with a matrix of spring constants $V = \kappa_{ab} \mathbf{q}_a \mathbf{q}_b$. If we diagonalize the matrix of spring constants (which is a real symmetric matrix and so has real eigenvalues), we will have a description in terms of decoupled oscillators, called *normal modes*.

¹The rest of the explanation is that it's the only system we can solve.

Since our system has (discrete) translation invariance, these modes are labelled by a wavenumber k^2 , i.e. the matrix of spring constants κ_{ab} is diagonalized by Fourier transformation:

$$q_k = \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{-ikx_n} q_n, \quad p_k = \frac{1}{\sqrt{N}} \sum_{n=1}^N e^{-ikx_n} p_n, \quad (1.3)$$

(Notice that in the previous expressions I didn't use boldface; that's because this step is really just classical physics. Note the awkward (but in field theory, inevitable) fact that we'll have (field) momentum operators \mathbf{p}_k labelled by a wavenumber aka momentum.) The reason the Fourier kernel e^{ikx} appears here is that it diagonalizes the translation operator \mathbf{T} defined by $\mathbf{T}f(x) = f(x+a)$:

$$\mathbf{T}e^{ikx} \equiv e^{ik(x+a)} = e^{ika} e^{ikx}.$$

In case this expression seems at all mysterious, note that for our discrete, finite chain, \mathbf{T} is represented by the matrix

$$\mathbf{T}_{nm} = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & 0 & 0 & 1 & \\ & & & \ddots & \\ 1 & 0 & & & 0 \end{pmatrix}_{nm}.$$

It might also help to introduce an auxiliary quantum system to think about this linear algebra problem: label the (position) basis states by $|n\rangle, n = 0..N-1$ (with the argument of the ket understood mod N). Then the translation operator is $\mathbf{T} = \sum_n |n-1\rangle\langle n|$. Its eigenvectors are $|k\rangle \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{ikan} |n\rangle$.

Regulators: Here we have our first meaningful encounter with the two ways in which divergences arise in field theory. Here, because N is finite, k takes discrete values ($1 = e^{ikNa}$); this is a long-wavelength “IR” regulator. Because of the lattice structure, k is periodic (only $e^{ikan}, n \in \mathbb{Z}$ appears): $k \equiv k + 2\pi/a$; this is a short-distance “UV” regulator. The range of k can be taken to be

$$0 \leq k \leq \frac{2\pi(N-1)}{Na}.$$

²The inverse transformation is:

$$q_n = \frac{1}{\sqrt{N}} \sum_{k>0}^{2\pi/a} e^{ikx_n} q_k, \quad p_n = \frac{1}{\sqrt{N}} \sum_{k>0}^{2\pi/a} e^{ikx_n} p_k. \quad (1.2)$$

Because of the periodicity in k , we can equivalently label the set of wavenumbers by³:

$$0 < k \leq \frac{2\pi}{a} \quad \text{or} \quad -\frac{\pi}{a} < k \leq \frac{\pi}{a}. \quad (1.4)$$

That is, because of the lattice spacing, there is a largest possible k , and a smallest possible wavelength (which is just the lattice spacing a itself).

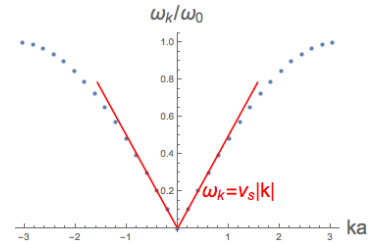
Summary: Because the system is in a box (periodic), k -space is discrete. Because the system is on a lattice, k -space is periodic. There are N oscillator modes altogether. The transformation from real space (modes labelled by n) to Fourier space (modes labelled by k) is an $N \times N$ unitary matrix.

The whole hamiltonian is a bunch of decoupled oscillators, labelled by these funny wave numbers⁴:

$$\mathbf{H} = \sum_k \left(\frac{\mathbf{p}_k \mathbf{p}_{-k}}{2m} + \frac{1}{2} m \omega_k^2 \mathbf{q}_k \mathbf{q}_{-k} \right) \quad (1.5)$$

where the frequency of the mode labelled k is

$$\omega_k \equiv 2\sqrt{\frac{\kappa}{m}} \sin \frac{|k|a}{2}. \quad (1.6)$$



Why might we care about this frequency in (1.6)? For one thing, consider the Heisenberg (or Hamilton) equation of motion for the deviation of one spring:

$$\mathbf{i} \partial_t \mathbf{q}_n = [\mathbf{q}_n, \mathbf{H}] = \mathbf{i} \frac{\mathbf{p}_n}{m}, \quad \mathbf{i} \partial_t \mathbf{p}_n = [\mathbf{p}_n, \mathbf{H}].$$

Combining these gives:

$$m \ddot{q}_n = -\kappa ((q_n - q_{n-1}) + (q_n - q_{n+1})) = -\kappa (2q_n - q_{n-1} - q_{n+1}).$$

In terms of the Fourier-mode operators:

$$m \ddot{\mathbf{q}}_k = -\kappa (2 - 2 \cos ka) \mathbf{q}_k .$$

Plugging in a Fourier ansatz in time $q_k(t) = \sum_{\omega} e^{-i\omega t} q_{k,\omega}$ turns this into a completely algebraic equation (*i.e.* no derivatives) which says $\omega^2 = \omega_k^2 = \left(\frac{4\kappa}{m}\right) \sin^2 \frac{|k|a}{2}$ for

³This range of independent values of the wavenumber in a lattice model is called the *Brillouin zone*. There is some convention for choosing a fundamental domain which prefers the last one but I haven't found a reason to care about this.

⁴Note that $q_n = q_n^*$ implies that $q_k = q_{-k}^*$ (and similarly for p s), so the expression (1.5) is nice and hermitian.

the allowed modes. We see that (the classical version of) this system describes *waves*:

$$0 = (\omega^2 - \omega_k^2) q_{k,\omega} \stackrel{k \ll 2\pi/a}{\simeq} (\omega^2 - v_s^2 k^2) q_{k,\omega}. \quad (1.7)$$

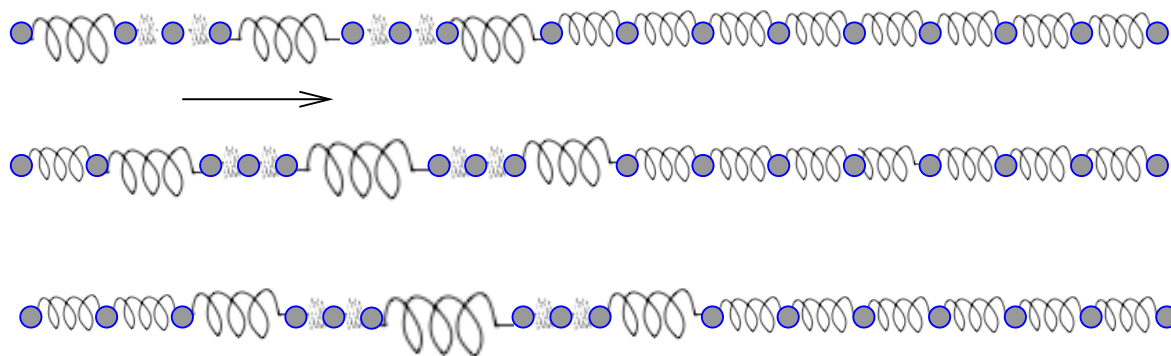
The result for small k is the Fourier transform of the wave equation:

$$(\partial_t^2 - v_s^2 \partial_x^2) q(x, t) = 0. \quad (1.8)$$

v_s is the speed of propagation of the waves, in this case the speed of sound. Comparing to the dispersion relation (1.6), we have found

$$v_s = \left. \frac{\partial \omega_k}{\partial k} \right|_{k \rightarrow 0} = a \sqrt{\frac{\kappa}{m}}.$$

The wave looks something like this:



These are sound waves. So the story I am telling is a quantization of sound waves. Soon we will understand quantization of electromagnetic (EM) waves, too. (Note however that it is bad practice to use ‘quantize’ as a verb meaning some operation we do on a classical system. Really the arrow goes the other way: classical mechanics is a crude approximation to quantum mechanics. There can even be different quantum systems with the same classical limit.)

[End of Lecture 1]

Another way to derive the equations of motion is by varying the action:

$$S[q] = \int dt L, \quad L = \sum_n p_n \dot{q}_n - H = \frac{1}{2} \sum_n m \dot{q}_n^2 - \frac{1}{2} \kappa_{n\ell} q_n q_\ell. \quad (1.9)$$

$$0 = \frac{\delta S}{\delta q_n(t)} = -m \ddot{q}_n - \kappa_{n\ell} q_\ell.$$

By the way, you may have learned some other ways to think about the Lagrange equations of motion. Forget them and use functional derivatives. The only thing you need to know is

$$\frac{\delta q_\ell(s)}{\delta q_n(t)} = \delta_{n\ell} \delta(t-s) \quad (1.10)$$

which says different degrees of freedom are independent. This is just the continuum limit in time of the statement

$$\frac{\partial q_{\ell\alpha}}{\partial q_{n\beta}} = \delta_{\ell n} \delta_{\alpha\beta}$$

that the values of the function at different times and places are independent. Here $q_{\ell\alpha} \equiv q_\ell(t_\alpha)$ and these are just ordinary partial derivatives.

We can put back the terms in the long-wavelength expansion (1.7) that we left out in the wave equation in (1.7). If the next correction is of the form $\omega_k^2 = v_s^2 k^2 + gk^4 + \dots$, then the corresponding wave equation is instead

$$(\partial_t^2 - v_s^2 \partial_x^2 + g \partial_x^4 + \dots) q(x, t) = 0. \quad (1.11)$$

Notice that by dimensional analysis, since the $\partial_x^4 q$ has two more derivatives than the $\partial_x^2 q$ term, g/v_s^2 must have units of length². What length scale is it that suppresses the higher-derivative terms? The (tiny!) lattice spacing. More about the long-wavelength expansion below.

What action produces this wave equation as its equation of motion?

$$S[q] = \int dt \int dx \left(\frac{1}{2} (\partial_t q)^2 - \frac{1}{2} v_s^2 (\partial_x q)^2 + \frac{g}{2} (\partial_x^2 q)^2 + \dots \right). \quad (1.12)$$

Here the independent variable that we must vary to get the equations of motion⁵ ($0 = \frac{\delta S[q]}{\delta q(x,t)}$, which is (1.11)) is $q(x, t)$, a *field*. Thus, we have seen field theory emerge from this lattice model of particles connected by springs.

Here is a slight generalization, by way of recap. Consider the slightly more general class of (still quadratic) Hamiltonians

$$\mathbf{H} = \sum_n \frac{\mathbf{p}_n^2}{2m_n} + \frac{1}{2} \sum_{nm} \kappa_{nm} q_n q_m.$$

⁵To derive (1.11) from this action, you'll need to use the basic fact $\frac{\delta q(x,t)}{\delta q(y,t)} = \delta(x-y)\delta(t-s)$ (plus ordinary calculus); this is the continuum limit in space of the relation (1.10).

We allowed the masses to vary, and we made a whole matrix of spring constants. Notice that only the symmetric part $\frac{1}{2}(\kappa_{nm} + \kappa_{mn})$ of this matrix appears in the Hamiltonian. Also κ must be real so that \mathbf{H} is hermitian. To simplify our lives we can redefine variables

$$Q_n \equiv \sqrt{m_n}q_n, \quad P_n = p_n/\sqrt{m_n}, \quad V_{nm} \equiv \frac{\kappa_{nm}}{\sqrt{m_n m_m}}$$

in terms of which

$$\mathbf{H} = \frac{1}{2} \left(\sum_n P_n^2 + \sum_{nm} V_{nm} Q_n Q_m \right).$$

Notice that $[q_n, p_m] = \mathbf{i}\delta_{nm} \Leftrightarrow [Q_n, P_n] = \mathbf{i}\delta_{nm}$.

Now since V_{nm} is a symmetric matrix, and hence hermitian, it can be diagonalized. That is, we can find an $N \times N$ unitary matrix U so that UVU^\dagger is a diagonal matrix:

$$\sum_{nm} U_{\alpha n} V_{nm} (U^\dagger)_{m\beta} = \delta_{\alpha\beta} \omega_\alpha^2 \quad (1.13)$$

where I assumed that all the eigenvalues of V are positive – otherwise the system would be unstable. U is unitary means:

$$\sum_n U_{\alpha n} (U^\dagger)_{n\beta} = \delta_{\alpha\beta}, \quad \sum_\alpha (U^\dagger)_{n\beta} U_{\beta m} = \delta_{nm}. \quad (1.14)$$

To take advantage of this, we make the change of variables to the *normal modes*

$$\tilde{Q}_\alpha = \sum_n U_{\alpha n} Q_n.$$

Multiplying the BHS of this equation by U^\dagger , we have the inverse relation

$$Q_n = \sum_\alpha (U^\dagger)_{n\alpha} \tilde{Q}_\alpha.$$

Notice that $Q_n = Q_n^\dagger$ is hermitian. This means

$$Q_n = Q_n^\dagger = \sum_n (U^\dagger)_{n\alpha}^* \tilde{Q}_\alpha^\dagger = \sum_n (U^T)_{n\alpha} \tilde{Q}_\alpha^\dagger = \sum_n \tilde{Q}_\alpha^\dagger U_{\alpha n}.$$

Similarly, we define $\tilde{P}_\alpha = \sum_n U_{\alpha n} P_n$.

Now let's look at what this does to the terms in \mathbf{H} (this first equation is sometimes called Parsival's Theorem):

$$\sum_n P_n^2 = \sum_n P_n^\dagger P_n = \sum_{\alpha\beta} \underbrace{\sum_n U_{\alpha n} (U^\dagger)_{n\beta}}_{\stackrel{(1.14)}{=} \delta_{\alpha\beta}} \tilde{P}_\alpha^\dagger \tilde{P}_\beta = \sum_\alpha \tilde{P}_\alpha^\dagger \tilde{P}_\alpha.$$

$$\sum_{nm} V_{nm} Q_n Q_m = \sum_{nm} V_{nm} Q_n^\dagger Q_m = \sum_{\alpha\beta} \underbrace{\sum_{nm} U_{\alpha n} V_{nm} (U^\dagger)_{m\beta}}_{\stackrel{(1.13)}{=} \delta_{\alpha\beta} \omega_\alpha^2} \tilde{Q}_\alpha^\dagger \tilde{Q}_\beta = \sum_{\alpha} \omega_\alpha^2 \tilde{Q}_\alpha^\dagger \tilde{Q}_\alpha.$$

The special case we considered earlier, $\kappa_{nm} = \left(\frac{1}{2}(\mathcal{T} + \mathcal{T}^T) - \mathbb{1}\right)_{nm}$, where each mass is only connected to its two neighbors, is special in three ways:

1. First, it is *local* in the sense that only nearby springs couple to each other, so κ_{nm} is only nonzero when n and m are close together.
2. Second, it has *discrete translation invariance*, that is, it is invariant under $(Q_n, P_n) \rightarrow (Q_{n+1}, P_{n+1})$. This is just because κ_{nm} does not depend explicitly on n and m . Because of the latter property, the normal modes are plane waves, $U_{kn} = e^{ikna}/\sqrt{N}$ which has the consequences that $(U^\dagger)_{kn} = U_{-k,n}$ and hence that $Q_k^\dagger = Q_{-k}$.
3. Finally, it has a certain additional symmetry that also has a right to be called translation invariance from the microscopic point of view. This is the symmetry under $q_n \rightarrow q_n + \epsilon$, which shifts each mass by the same, arbitrary amount. This doesn't stretch the springs at all, and so is a symmetry of H . More precisely, this symmetry arises because the potential depends on q_n only through differences, $V = V(q_n - q_m)$.

Notice that the transformation on the field q_n is non-linear, in the sense that the field shifts under the symmetry (as opposed to a linear transformation like $q \rightarrow Rq$). This is related to the fact that the transformation takes one minimum-energy configuration to another; in this sense, the symmetry is said to be *spontaneously broken*. An important general consequence of a non-linearly realized continuous symmetry is the presence of a massless mode, a Goldstone mode. We've already seen that there is such a massless mode in this system. This is an example of Goldstone's theorem. It's a bit mysterious in this example, but will become clearer later.

QM. So far the fact that quantumly $[\mathbf{q}_n, \mathbf{p}_{n'}] = i\hbar\delta_{nn'}\mathbb{1}$ hasn't really mattered in our analysis (go back and check – we could have derived the wave equation classically)⁶. For the Fourier modes, this implies the commutator

$$[\mathbf{q}_k, \mathbf{p}_{k'}] = \sum_{n,n'} \mathbf{U}_{kn} \mathbf{U}_{k'n'} [\mathbf{q}_n, \mathbf{p}_{n'}] = i\hbar\mathbb{1} \sum_n \mathbf{U}_{kn} \mathbf{U}_{k'n} = i\hbar\delta_{k,-k'}\mathbb{1}.$$

⁶In case you are rusty, or forget the numerical factors like I do, here is a concise summary of the

(In the previous expression I called $\mathbf{U}_{kn} = \frac{1}{\sqrt{N}} e^{-ikx_n}$ the unitary matrix realizing the discrete Fourier kernel.)

To make the final step to decouple the modes with k and $-k$, introduce the annihilation and creation operators⁷

$$\text{For } k \neq 0: \quad \mathbf{q}_k = \sqrt{\frac{\hbar}{2m\omega_k}} \left(\mathbf{a}_k + \mathbf{a}_{-k}^\dagger \right), \quad \mathbf{p}_k = \frac{1}{i} \sqrt{\frac{\hbar m \omega_k}{2}} \left(\mathbf{a}_k - \mathbf{a}_{-k}^\dagger \right). \quad (1.15)$$

They satisfy

$$[\mathbf{a}_k, \mathbf{a}_{k'}^\dagger] = \delta_{kk'} \mathbb{1}.$$

In terms of these, the hamiltonian is

$$\mathbf{H} = \sum_k \hbar \omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} \right) + \frac{p_0^2}{2m}$$

operator solution of the quantum harmonic oscillator:

$$\mathbf{H} = \frac{\mathbf{p}^2}{2m} + \frac{1}{2} m \omega^2 \mathbf{q}^2 = \frac{\hbar \omega}{2} (\mathbf{P}^2 + \mathbf{Q}^2) = \hbar \omega \left(\mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} \right)$$

with

$$\mathbf{a} \equiv \frac{1}{\sqrt{2}} (\mathbf{Q} + i\mathbf{P}), \quad \mathbf{a}^\dagger \equiv \frac{1}{\sqrt{2}} (\mathbf{Q} - i\mathbf{P}).$$

Here I've defined these new operators to hide the annoying factors:

$$\mathbf{Q} \equiv \left(\frac{m\omega}{\hbar} \right)^{1/2} \mathbf{q}, \quad \mathbf{P} \equiv \left(\frac{1}{m\hbar\omega} \right)^{1/2} \mathbf{p}.$$

$$[\mathbf{q}, \mathbf{p}] = i\hbar \mathbb{1} \quad \iff \quad [\mathbf{Q}, \mathbf{P}] = i\hbar \mathbb{1} \quad \iff \quad [\mathbf{a}, \mathbf{a}^\dagger] = \mathbb{1}.$$

The number operator $\mathbf{N} \equiv \mathbf{a}^\dagger \mathbf{a}$ satisfies

$$[\mathbf{N}, \mathbf{a}] = -\mathbf{a}, \quad [\mathbf{N}, \mathbf{a}^\dagger] = +\mathbf{a}^\dagger.$$

So \mathbf{a} and \mathbf{a}^\dagger are lowering and raising operators for the number operator. The eigenvalues of the number operator have to be positive, since

$$0 \leq \|\mathbf{a}|n\rangle\|^2 = \langle n|\mathbf{a}^\dagger \mathbf{a}|n\rangle = \langle n|\mathbf{N}|n\rangle = n \langle n|n\rangle$$

which means that for $n = 0$ we have $\mathbf{a}|n=0\rangle = 0$. If it isn't zero (*i.e.* if $n \geq 1$), $\mathbf{a}|n\rangle$ is also an eigenvector of \mathbf{N} with eigenvalue $n - 1$. It has to stop somewhere! So the eigenstates of \mathbf{N} (and hence of $\mathbf{H} = \hbar \omega (\mathbf{N} + \frac{1}{2})$) are

$$|0\rangle, \quad |1\rangle \equiv \mathbf{a}^\dagger |0\rangle, \quad \dots, \quad |n\rangle = c_n (\mathbf{a}^\dagger)^n |0\rangle \dots$$

where we must choose c_n to normalize these states. The answer that gives $\langle n|n\rangle = 1$ is $c_n = \frac{1}{\sqrt{n!}}$.

⁷You might notice that when $k = 0, \omega_k = 0$. This last step applies to the modes with $\omega_k \neq 0$, hence $k \neq 0$. The 'zero-mode' must be treated specially. It is neglected in many treatments of this topic but actually has a lot of physics in it. If you are curious see [this discussion](#), page 11.

– it is a sum of decoupled oscillators, and a free particle describing the center-of-mass.

The discovery of Fock space. The groundstate is

$$|0\rangle \equiv |0\rangle_{\text{osc}} \otimes |p_0 = 0\rangle, \quad \text{where } \mathbf{a}_k |0\rangle_{\text{osc}} = 0, \forall k, \text{ and } \mathbf{p}_0 |p_0\rangle = p_0 |p_0\rangle.$$

We know this is the groundstate because H is a constant plus a bunch of squares, and the squares all annihilate this state.

An excitation above the groundstate by the creation operator with wavenumber k

$$\mathbf{a}_k^\dagger |0\rangle \propto |\text{one phonon with momentum } \hbar k\rangle \quad (1.16)$$

has energy $\hbar\omega_k$ above the groundstate. (This is an eigenstate because $[N_k, \mathbf{a}_k^\dagger] = \mathbf{a}_k^\dagger$, where $N_k = \mathbf{a}_k^\dagger \mathbf{a}_k$.) The excitation is called a *phonon* with momentum $\hbar k$.⁸ This is what in undergrad QM we would have called “ $|k\rangle$ ”; we can make a state with one phonon in a position eigenstate by taking superpositions:

$$|\text{one phonon at position } x\rangle = \sum_k e^{ikx} |\text{one phonon with momentum } \hbar k\rangle \sim \sum_k e^{ikx} \mathbf{a}_k^\dagger |0\rangle.$$

This is the state that in single-particle QM we would have called “ $|x\rangle$ ”.⁹

The number operator (of the SHO with label k) $\mathbf{N}_k \equiv \mathbf{a}_k^\dagger \mathbf{a}_k$ counts the number of phonons with momentum k . The ground state is the state with no phonons – for this reason we could also call it the ‘vacuum’. We can also make a state with two phonons:

$$|k, k'\rangle = \mathbf{a}_k^\dagger \mathbf{a}_{k'}^\dagger |0\rangle$$

whose energy above the groundstate is

$$E - E_0 = \omega_k + \omega_{k'}. \quad (1.17)$$

Note that all these states have non-negative energy.

So this construction allows us to describe situations where the number of particles $\mathbf{N} = \sum_k \mathbf{N}_k$ can vary! That is, we can now describe dynamical processes in which the number of particles changes. Let me emphasize: In QM, we would describe the Hilbert space of two (distinguishable) particles as a tensor product of the Hilbert space of each.

⁸I put ‘proportional to’ rather than ‘equal’ in (1.16) because there can be a k -dependent normalization factor. We’ll see soon that Lorentz symmetry prefers a particular normalization here which we will adopt.

⁹For now, the fact that $|k\rangle$ and $|x\rangle$ are related by a Fourier transform will have to serve for evidence that $\hbar k$ is the momentum of the particle we’ve just discovered. Later we will show that indeed the state $\mathbf{a}_k^\dagger |0\rangle$ has momentum $\hbar k$ above the groundstate.

How can we act with an operator that enlarges the Hilbert space?? We just figured out how to do it.

We can specify basis states for this Hilbert space

$$\left(\mathbf{a}_{k_1}^\dagger\right)^{n_{k_1}} \left(\mathbf{a}_{k_2}^\dagger\right)^{n_{k_2}} \cdots |0\rangle = |\{n_{k_1}, n_{k_2}, \dots\}\rangle$$

by a collection of *occupation numbers* n_k , eigenvalues of the number operator for each normal mode (and the center-of-mass momentum p_0).

Notice that in this description it is manifest that a phonon has no identity. We only keep track of how many of them there are and what is their wavenumber. They cannot be distinguished. Also notice that we can have as many we want in the same mode – n_k can be any non-negative integer. These are an example of *bosons*. (Further evidence for this is that $\mathbf{a}_k^\dagger \mathbf{a}_{k'}^\dagger = \mathbf{a}_{k'}^\dagger \mathbf{a}_k^\dagger$.)

Lessons from the hard work of the first two lectures:

- Starting from a collection of particles, we chained them together, and made a field; treating this system quantumly, we found a new set of particles. The new particles (the normal modes) are *collective* excitations: their properties can be very different from those of the constituent particles. For example, the constituent particles have dispersion relation $E(p) = \frac{p^2}{2m}$, but the phonons have dispersion relation $E(p) = \hbar\omega_p \stackrel{p \ll \hbar/a}{\sim} v_s |p|$.

And the constituent particles are distinguishable by their locations, but phonons are indistinguishable from each other. The state of such indistinguishable particles is determined merely by specifying a collection of positions (or momenta) and of spin states – we don't need to say which is which (and in fact, we cannot).

- Notice that there are some energies where there aren't any phonon states. In particular, the function (1.6) has a maximum. More generally, in a system with discrete translation invariance, there are *bands* of allowed energies. In the continuum limit (roughly $a \rightarrow 0$ with the correct quantities held fixed), to which we devolve soon, this maximum (which occurs at $k = \frac{\pi}{a}$) goes off to the sky.
- **Lorentz invariance can emerge.** The dispersion relation for the long-wavelength ($ka \ll 1$) sound mode was $\omega^2 = v^2 \vec{k}^2$. This is the Fourier transform of

$$(\partial_t^2 - v^2 \vec{\nabla}^2)\phi(x) = \partial_\mu \partial^\mu \phi(x) = 0, \tag{1.18}$$

a wave equation with Lorentz symmetry (if v is the speed appearing in the Minkowski metric). To get (1.18), we had to ignore the $\mathcal{O}(a^4 k^4)$ terms in the long-wavelength expansion of the dispersion relation, $2(1 - \cos(ka))$. The lattice

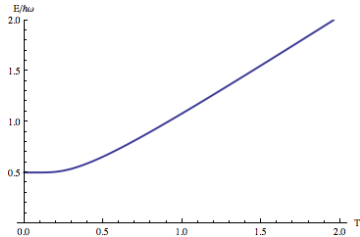
breaks Lorentz symmetry, but its effects go away for $ka \ll 1$. This point might make you think that the Lorentz symmetry which is so precious to us in particle physics could emerge in a similar way, but with a much smaller a than the lattice spacing in solids. There are strong constraints on how small this can be (e.g. [this well-appreciated paper](#)) so it is very useful to treat it as a fundamental principle.

- When $N < \infty$, everything is completely finite. And yet we see that at low energies the system is well-described by a quantum field theory. Even in the thermodynamic limit, $N \rightarrow \infty$ with fixed a , there is a maximum possible ω_k (and a maximum possible k) for each phonon. This is a well-regulated QFT. Observe the tension between the non-symmetric, safe lattice and the symmetric, dangerous continuum.

[End of Lecture 2]

Phonons are real: heat capacity of (insulating) solids. The simplest demonstration that phonons are real is the dramatic decrease at low temperatures of the heat capacity of insulating solids. At high temperatures, the equipartition theorem of classical thermodynamics correctly predicts that the energy of the solid from the lattice vibrations should be T times the number of atoms, so the heat capacity, $C_V = \partial_T E$ should be independent of T .

At low temperatures $T < \Theta_D$, this is wrong. Θ_D is the temperature scale associated with the frequencies of the lattice vibrations (say the maximum of the curve ω_k above). The resolution lies in the thermal energy of a quantum harmonic oscillator for $T < \omega$, the energy goes to a constant $\frac{1}{2}\hbar\omega$, independent of T :



So the heat capacity (the slope of this curve) goes to zero as $T \rightarrow 0$ ¹⁰.

Phonons are real: the Mössbauer effect. Here is another dramatic consequence of the quantization of the lattice vibrations of solids, known as the *Mössbauer effect*, first described in words. The nuclei of the atoms in a solid have various energy levels; when hit with a γ -ray photon, these nuclei can experience transitions from the groundstate to some excited energy level. If an excited nucleus somewhere in the lattice gets hit by a very energetic photon (a γ -ray) of some very specific energy

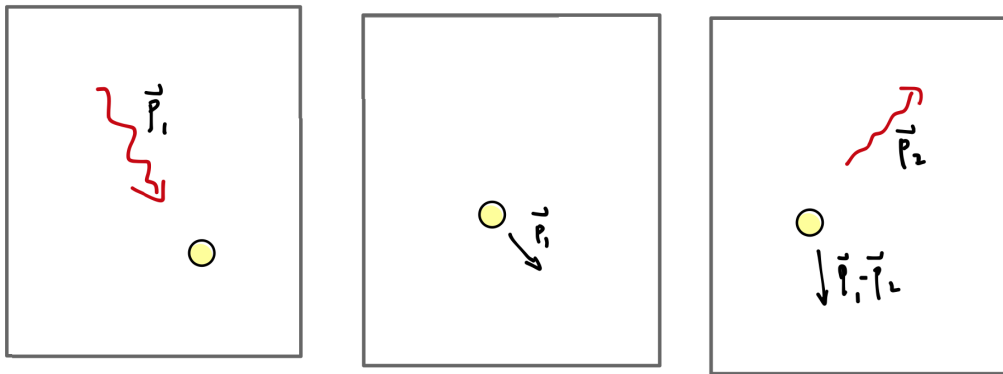
¹⁰More precisely, the energy in thermal equilibrium of our collection of springs is

$$\sum_k \frac{\omega_k}{e^{\omega_k/T} - 1} \stackrel{L \rightarrow \infty, T \ll \Theta_D}{=} \int d^d k \frac{v_s k}{e^{v_s k/T} - 1} = \frac{T^{d+1}}{v_s^{d-1}} \int d^d q \frac{q}{e^q - 1}, \quad q \equiv v_s k/T.$$

So the heat capacity is $C_V \sim T^d \rightarrow 0$ as $T \rightarrow 0$.

$E_\gamma = \Delta E \equiv E_{\text{excited}} - E_0$, the nucleus can absorb and re-emit that photon. The resulting sharp resonant absorption lines at $E_\gamma = \Delta E$ are indeed observed.

This sounds simple, but here is a mystery about this, depicted in the comic strip below: Consider a nucleus alone in space in the excited state, after it gets hit by a photon. The photon carried a momentum $p_\gamma = E_\gamma/c$. Momentum is conserved, and it must be made up by some *recoil* of the absorbing nucleus. This means that not all of the energy of the photon can go into exciting the nucleus. When it emits a photon again, it needn't do so in the same direction. This means that the nucleus remains in motion with momentum $\Delta\vec{p} = \vec{p}_1 - \vec{p}_2$. But if some of its energy $\Delta E = E_{\text{excited}} - E_0$ goes to kinetic energy of recoil, not all of that energy can go to the final photon, and the emitted photon energy will be *less* than E_γ by $E_{\text{recoil}} = \frac{\Delta p^2}{2M}$. This can be as big as $E_{\text{recoil}}^{\text{max}} = \frac{(2\vec{p})^2}{2M} = \frac{(2E_\gamma/c)^2}{2M}$ (in the case of scattering by angle π). So instead of a sharp absorption line, it seems that we should see a broad bump of width $\frac{(E_\gamma/c)^2}{M}$. But we *do* see a sharp line when scattering off a solid!

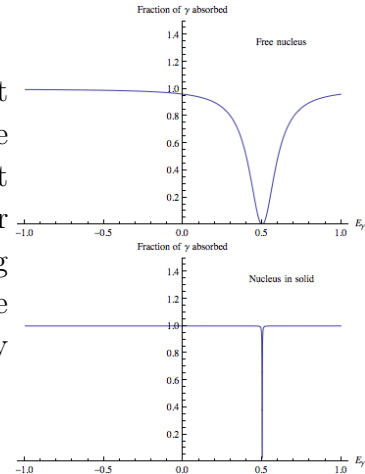


The solution of the puzzle is phonons: for a nucleus in a lattice to recoil requires that the springs are stretched – it must excite a lattice vibration, it must create some phonons. But there is a nonzero probability for it to create *zero* phonons. In this case, the momentum conservation is made up by an acceleration of *the whole solid*, which is very massive (its mass is NM where N is the number of nuclei!), and therefore does not recoil very much at all (it loses only energy $\frac{p_\gamma^2}{2NM}$)¹¹.

¹¹Let's see this in terms of our solution in terms of Fourier modes. Periodic boundary conditions and demanding that the particles are identical says that $|\{q_n\}\rangle \equiv |\{q_n + a\}\rangle$ are the same state. In terms of the $k = 0$ mode, this is

$$q_0 = \frac{1}{\sqrt{N}} \sum_{n=1}^N q_n e^{-i0x_n} \equiv \frac{1}{\sqrt{N}} \left(\sum_{n=1}^N q_n + Na \right), \quad \text{i.e. } q_0 \equiv q_0 + \sqrt{N}a.$$

This allows for very sharp resonance lines. In turn, this effect has allowed for some very high-precision measurements. The different widths in these cartoon absorption spectra don't do justice to the relative factor of N . An essentially similar effect makes it possible to get precise peaks from scattering of X-rays off of a solid (Bragg scattering) – there is a finite amplitude for the scattering to occur without exciting any phonons.



This is actually a remarkable thing: although solids seem ordinary to us because we encounter them frequently, the rigidity of solids is a quantum mechanical emergent phenomenon. You can *elastically* scatter photons off of a solid only because the atoms making up the solid participate in this collective behavior wherein the whole solid acts like a single quantum object!

Interactions. We now pause the celebration to point out some limitations of this analysis. Many aspects of the above discussion are special to the fact that our Hamiltonian was quadratic in the canonical operators q, p . Certainly our ability to completely solve the system is. In particular, the additivity of the energy of the excitations (1.17) is an artifact of this limit where the Hamiltonian is quadratic.

Notice that the number of phonons of *each momentum* $\mathbf{N}_k \equiv \mathbf{a}_k^\dagger \mathbf{a}_k$ is conserved for each k (*i.e.* commutes with \mathbf{H}). This is a lot of symmetry! But if we add generic cubic and quartic terms in \mathbf{q} (or if we couple our atoms to the photon field) even the total number of phonons $\sum_k \mathbf{N}_k$ will no longer be a conserved quantity¹². So a description

This means that the wavefunction for the zeromode must satisfy

$$e^{ip_0 q_0} = e^{ip_0(q_0 + \sqrt{N}a)} \implies p_0 \in \frac{2\pi\mathbb{Z}}{\sqrt{N}a}$$

and the first excited state has energy

$$\frac{p_0^2}{2m} \Big|_{p_0 = \frac{2\pi}{\sqrt{N}a}} = \frac{1}{2} \frac{1}{Nm} \left(\frac{2\pi}{a} \right)^2.$$

The factor of $\frac{1}{Nm}$ is the suppression by the mass of the whole solid. (Recall that $N \sim 10^{24}$.)

¹²Note that it *is* possible to make a non-quadratic action for conserved particles, but this requires adding more degrees of freedom – the required U(1) symmetry must act something like

$$(q^1, q^2) \rightarrow (\cos \theta q^1, \sin \theta q^2).$$

We can reorganize this as a complex field $\Phi = q^1 + \mathbf{i}q^2$ on which the symmetry acts by $\Phi \rightarrow e^{i\theta}\Phi$.

of such particles that forced us to fix their number wouldn't be so great.

For example, think about expanding a general cubic term $\sum_{nml} \lambda_{nml} q_n q_m q_l$ in oscillators. It has terms like

$$a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3}^\dagger, \quad a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3}, \quad a_{k_1}^\dagger a_{k_2} a_{k_3}, \quad a_{k_1} a_{k_2} a_{k_3}$$

all of which change the number of phonons. A quartic term will also contain terms like $a^\dagger a^\dagger a a$ which preserve the number of phonons, but describe an interaction between them, where they exchange momentum.

More generally, I would like to remind you again that not every QFT can be usefully considered as nearly a bunch of harmonic oscillators. Finding ways to think about the ones that can't is an important job for later.

Towards scalar field theory. It is worthwhile to put together the final relation between the 'position operator' and the phonon annihilation and creation operators¹³:

$$\mathbf{q}_n = \sqrt{\frac{\hbar}{2\mu}} \sum_k \frac{1}{\sqrt{\omega_k}} \left(e^{i\mathbf{k}x_n} \mathbf{a}_k + e^{-i\mathbf{k}x_n} \mathbf{a}_k^\dagger \right) + \frac{1}{\sqrt{N}} \mathbf{q}_0 \quad (1.19)$$

and the corresponding relation for its canonical conjugate momentum

$$\mathbf{p}_n = \frac{\mu}{\mathbf{i}} \sqrt{\frac{\hbar}{2\mu}} \sum_k \sqrt{\omega_k} \left(e^{i\mathbf{k}x_n} \mathbf{a}_k - e^{-i\mathbf{k}x_n} \mathbf{a}_k^\dagger \right) + \frac{1}{\sqrt{N}} \mathbf{p}_0.$$

The items in red are the ways in which p and q differ; they can all be understood from the relation $p = \mu \dot{q}$ as you will see on the homework. These expressions are formally identical to the formulae we'll find below expressing a scalar field in terms of creation and annihilation operators (such as Peskin eqns. (2.25) and (2.26)). The stray factors of μ arise because we didn't 'canonically normalize' our fields and absorb the m s into the field, *e.g.* defining $\phi \equiv \sqrt{\mu} q$ would get rid of them. The other difference is because we still have IR and UV regulators in place, so we have sums instead of integrals.

Path integral reminder in a box. [A useful reference is Zee §I.2] If we use the path integral description, some of these things (in particular the continuum, sound-wave limit) are more obvious-seeming.

Let's remind ourselves how the path integral formulation of QM works for a particle in one dimension with $\mathbf{H} = \frac{\mathbf{p}^2}{2m} + V(\mathbf{q})$. The basic statement is the following formula for the propagator – the amplitude to propagate from position eigenstate $|q_0\rangle$ to position eigenstate $|q\rangle$ during a time interval t is

$$\langle q | e^{-i\mathbf{H}t} | q_0 \rangle = \int_{q(0)=q_0}^{q(t)=q} [dq] e^{i \int_0^t ds \left(\frac{1}{2} \dot{q}^2 - V(q) \right)} .$$

¹³Note that relative to (1.15), we relabeled the summation variable $k \rightarrow -k$ in the second term.

Here $[dq] \equiv \mathcal{N} \prod_{l=1}^{M_t} dq(t_l)$ – the path integral measure is defined by a limiting procedure ($M_t \equiv \frac{t}{\Delta t} \rightarrow \infty, \Delta t \rightarrow 0, t$ fixed), and \mathcal{N} is a normalization factor that always drops out of physical quantities so I don't need to tell you what it is.

Recall that the key step in the derivation of this statement is the evaluation of the propagator for an infinitesimal time step:

$$\langle q_2 | e^{-i\mathbf{H}\Delta t} | q_1 \rangle = \langle q_2 | e^{-i\Delta t \frac{p^2}{2m}} e^{-i\Delta t V(\mathbf{q})} | q_1 \rangle + \mathcal{O}(\Delta t^2) .$$

An integral expression for this involving only numbers (no operators, no states) can be obtained by inserting resolutions of the identity

$$\mathbb{1} = \mathbb{1}^2 = \left(\int \overline{d}p |p\rangle\langle p| \right) \left(\int dq |q\rangle\langle q| \right) = \int \overline{d}p \int dq |p\rangle\langle q| e^{ipq}$$

in between the two exponentials. For a more extensive reminder, please see §2.4 of [this document](#).

Two quick but invaluable applications of the path integral:

1. The path integral *explains* the origin of the variational principle and the special role of configurations that solve the equations of motion. They are points of stationary phase in the path integral, where

$$0 = \frac{\delta S}{\delta q(t)} . \tag{1.20}$$

In case you are not familiar with functional derivatives: From the definition of the path integral as a limit it should be clear what is meant by this expression. If we didn't take the limit, the stationary phase condition is that S is extremized in every direction $q_i = q(t_i)$:

$$0 = \frac{\partial S}{\partial q_i} .$$

The functional derivative in (1.20) means the same thing but looks nicer because we can pretend everything is smooth rather than discrete. Since q_i and q_j are independent variables,

$$\frac{\partial q_i}{\partial q_j} = \delta_{ij} \quad \text{and therefore} \quad \frac{\delta q(t)}{\delta q(s)} = \delta(t - s) .$$

With the help of the chain rule, this is the only functional derivative you need to know how to take¹⁴.

Part of this statement is a discovery of the true role of \hbar : its job is to make up the units so that the exponent in $e^{iS[q]/\hbar}$ is dimensionless.

¹⁴If you are not yet comfortable with the machinery of functional derivatives, please work through pages 2-28 through 2-30 of [this document](#) right now.

2. Above I wrote a formula for the real-time propagator. Euclidean path integrals are also very useful, because they compute ground-state expectation values. Here's why:

The vacuum can be prepared by starting in an arbitrary state and acting with e^{-TH} for some large T , and then normalizing (as usual when discussing path integrals, it's best to not worry about the normalization and only ask questions that don't depend on it),

$$|0\rangle = \mathcal{N} e^{-\mathbf{H}T} |\text{any}\rangle.$$

To see this, just expand in the energy eigenbasis. This 'imaginary time evolution operator' $e^{-\mathbf{H}T}$ has a path integral representation just like the real time operator, by nearly the same calculation

$$e^{-\mathbf{H}T} = \int [Dq] e^{-\int_{-T}^0 d\tau L(q(\tau), \dot{q}(\tau))} |q(0)\rangle \langle q(-T)|.$$

Doing the same thing to prepare $\langle 0|$, making a sandwich of $f(\mathbf{q})$ and taking $T \rightarrow \infty$ we can forget about the arbitrary states at the end, and we arrive at

$$\langle f(\mathbf{q}) \rangle \equiv \frac{\langle 0| f(\mathbf{q}) |0\rangle}{\langle 0|0\rangle} = \frac{1}{Z} \int [Dq] e^{-S_E} f(q(0)) = \frac{1}{Z} \int \prod_i^{M_t} dq_i e^{-\sum_j L(q_j)} f(q_0) \quad (1.21)$$

where $Z \equiv \int \prod_i dq_i e^{-\sum_j L(q_j)}$ (here i, j are discrete time labels) and $S_E[q]$ is the euclidean action. Relative to the real time action, there is an important sign¹⁵:

$$S_E[q] = \int d\tau ((\partial_\tau q)^2 + V(q)).$$

This sign makes good sense – it means that configurations with large V are *suppressed* in the euclidean path integral.

This trick of using imaginary time evolution to prepare the groundstate is a crucial one that we will use all the time below¹⁶.

In the special case where L is quadratic in q and \dot{q} , this can be written as

$$Z = \int \prod_i dq_i e^{-q_i D_{ij} q_j}$$

¹⁵By the way, I use square brackets for $S[q]$ to remind myself that S is a functional of q , a machine that eats functions and spits out numbers – for each function $q(\tau)$, $S[q]$ is a number. But in fact my main point here is that for our purposes there is not such a big difference between a functional and a function of many variables (the values of the function at a discrete collection of points).

¹⁶In lecture, I mentioned that we don't know how to do imaginary time evolution in the laboratory. [Here](#) is one recent attempt to do it using measurements. A reason to believe that it shouldn't be possible to do this *efficiently* is that it can be used to solve NP complete problems (by encoding their solution into the groundstate of a Hamiltonian).

where D_{ij} is the (real, symmetric) matrix which discretizes the action. Repeated indices are summed. This is a gaussian integral. This is the path integral point of view on why quadratic terms are special.

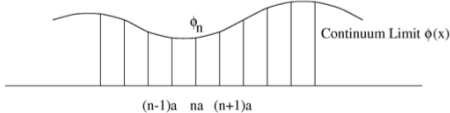
This discussion of path integrals has been for a single degree of freedom q . But as we've seen, we can make a field theory by coupling together a collection of such degrees of freedom q_n . The path integral measure for a field theory is just the product of the measures for each of these q_n – it's just a matter of sticking more labels on the integration variables.

1.2 Scalar field theory

Scalar field theory in one dimension. [Zee §1.3] The (real-time) path integral for our collection of oscillators is

$$Z = \int [dq_1 \cdots dq_N] e^{iS[q]}$$

with $S[q] = \int dt (\sum_n \frac{1}{2} m_n \dot{q}_n^2 - V(\{q\})) \equiv \int dt L(q, \dot{q})$. The potential is $V(\{q\}) = \sum_n \frac{1}{2} \kappa (q_{n+1} - q_n)^2$. Now suppose we have poor eyesight and can't resolve the individual atoms in the chain; rather we're only interested in the *long-wavelength* (small-wavenumber) physics. Sometimes this is described as taking a continuum limit, $a \rightarrow 0, N \rightarrow \infty$. Basically the only thing we need is to think of $q_n = q(x = na)$ as defining a

smooth function:  [Note that the continuum field

is often called $\phi(x)$ instead of $q(x)$ for some reason. At least the letters $q(x)$ and $\phi(x)$ look similar.]

By Taylor expanding $q(x_{n-1})$ about $q(x_n)$, we have

$$(q_n - q_{n-1})^2 \simeq a^2 (\partial_x q)^2 |_{x=na}, \quad a \sum_n f(q_n) \simeq \int dx f(q(x)).$$

The path integral becomes:

$$Z = \int [dq] e^{iS[q]}$$

with $[dq]$ now representing an integral over all configurations $q(t, x)$ (*defined* by this limit) and

$$S[q] = \int dt \int dx \frac{1}{2} (\mu (\partial_t q)^2 - \mu v_s^2 (\partial_x q)^2 - r q^2 - u q^4 - \dots) \equiv \int dt \int dx \mathcal{L} \quad (1.22)$$

where I've introduced some parameters μ, v_s, r, u determined from $m, \kappa, a \dots$ in some ways that we needn't worry about, except to say that they are finite in the continuum limit. The \dots includes terms like $a^4 (\partial_x q)^4$ that are small when $k \ll \frac{1}{a}$, so we ignore them. \mathcal{L} is the Lagrangian *density* whose integral over space is the Lagrangian $L = \int dx \mathcal{L}$.

The equation of motion is the stationary phase condition,

$$0 = \frac{\delta S}{\delta q(x, t)} = -\mu \ddot{q} + \mu v_s^2 \partial_x^2 q - rq - 2uq^3 - \dots$$

In this expression I have written a functional derivative; with our lattice regulator, it is simply a(n extremely useful) shorthand notation for the collection of partial derivatives $\frac{\partial}{\partial q_n}$.

From the phonon problem, we automatically found $r = u = 0$, and the equation of motion is just the wave equation (1.8). This happened because of the symmetry $q_n \rightarrow q_n + \epsilon$. This is the operation that *translates* the whole crystal. It guarantees low-energy phonons near $k = 0$ because it means $q(x)$ can only appear in S via its derivatives. (This is a general property of *Goldstone modes*; more on this later.)

[End of Lecture 3]

The following will be quite useful for our subsequent discussion of quantum light. We can construct a hamiltonian from this action by defining a canonical field-momentum $\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \mu \partial_t q$ and doing the Legendre transformation:

$$H = \sum_n (p_n \dot{q}_n - L_n) = \int dx (\pi(x) \dot{q}(x) - \mathcal{L}) = \int dx \left(\frac{\pi(x)^2}{2\mu} + \mu v_s^2 (\partial_x q(x))^2 + rq^2 + uq^4 + \dots \right). \quad (1.23)$$

(Here in the first step I wrote $L \equiv \sum_n L_n$ to emphasize locality.) I emphasize that the position along the chain x here is just a *label* on the fields, not a degree of freedom or a quantum operator. This dramatic shift of perspective is why I am spending time on this step. (Note that I suppress the dependence of all the fields on t just so it doesn't get ugly, not because it isn't there.)

The field q is called a *scalar field* because it doesn't have any indices decorating it. This is to be distinguished from the Maxwell field, which is a vector field, and which we'll discuss next. (Note that vibrations of a crystal in *three* dimensions actually do involve vector indices! We omit this complication.)

The lattice spacing a and the size of the box Na in the discussion above are playing very specific roles in *regularizing* our 1-dimensional scalar field theory. The lattice spacing a implies a maximum wavenumber or shortest wavelength and so is called an "ultraviolet (UV) cutoff", because the UV is the short-wavelength end of the visible

light spectrum. The size of the box Na implies a maximum wavelength mode which fits in the box and so is called an “infrared (IR) cutoff”.

If we also take the infinite volume limit, then the sums over k become integrals. In this limit we can make the replacement

$$\frac{1}{L^d} \sum_k \rightsquigarrow \int \mathrm{d}^d k, \quad L^d \delta_{kk'} \rightsquigarrow (2\pi)^d \delta^{(d)}(k - k').$$

A check of the normalization factors comes from combining these two rules

$$\forall k', \quad 1 = \sum_k \delta_{k,k'} = \int \mathrm{d}^d k (2\pi)^d \delta^{(d)}(k - k').$$

Continuum (free) scalar field theory in $d + 1$ dimensions. These continuum expressions are easy to generalize to scalar field theory in any number of dimensions. Let’s do this directly in infinite volume and set $\mu = 1$ by rescaling fields (*i.e.* $\phi \equiv \sqrt{\mu}q$). The action is

$$S[\phi] = \int d^d x dt \left(\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} v_s^2 \vec{\nabla} \phi \cdot \vec{\nabla} \phi - V(\phi) \right). \quad (1.24)$$

This is almost what we would have found for the long-wavelength ($ka \ll 1$) description of a d -dimensional lattice of masses on springs, like a mattress (except that there would have been one ϕ for each direction in which the atoms can wiggle). The equation of motion is

$$0 = \frac{\delta S[\phi]}{\delta \phi(x)} = -\partial_t^2 \phi + v_s^2 \nabla^2 \phi - V'(\phi). \quad (1.25)$$

For the harmonic case $V(\phi) = \frac{1}{2} m^2 \phi^2$ we know what we’re doing, and (1.25) is called the Klein-Gordon equation,

$$0 = (\partial_\mu \partial^\mu + m^2) \phi. \quad (1.26)$$

(Notice that I’ve set $v_s = c = 1$ here, and this is where we have committed to a choice of signature convention; take a look at the conventions page §0.2.). In relativistic notation, the Lagrangian density is just $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2)$. This describes free continuum real massive relativistic scalar quantum field theory. (Match the adjectives to the associated features of the lagrangian; collect them all!)

Let’s solve it by canonical methods. The canonical momentum¹⁷ is $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$ and the Hamiltonian (which we can instantly promote to a quantum operator by using boldface symbols) is then

$$\mathbf{H} = \int d^d x \left(\frac{\boldsymbol{\pi}(x)^2}{2} + \frac{1}{2} v_s^2 (\vec{\nabla} \phi \cdot \vec{\nabla} \phi) + \frac{1}{2} m^2 \phi^2 \right).$$

¹⁷More correctly, π is the canonical momentum *density*, since \mathcal{L} is the Lagrangian density not the Lagrangian. But everyone always forgets the word ‘density’ in both cases.

Note that all these terms are positive.

A translation invariant linear problem is solved by Fourier transforms:

$$\phi(x) = \int \mathrm{d}^d k e^{i\vec{k}\cdot\vec{x}} \phi_k, \text{ and } \pi(x) = \int \mathrm{d}^d k e^{i\vec{k}\cdot\vec{x}} \pi_k. \quad (1.27)$$

The hamiltonian is then

$$\mathbf{H} = \int \mathrm{d}^d k \left(\frac{1}{2} \pi_k \pi_{-k} + \frac{1}{2} \left(v_s^2 \vec{k}^2 + m^2 \right) \phi_k \phi_{-k} \right)$$

where $\vec{k}^2 \equiv (-i\vec{k}) \cdot (i\vec{k}) = \vec{k} \cdot \vec{k}$. Just as in (1.5), this is merely a sum of decoupled oscillators, except for the coupling between wavenumbers k and $-k$. Comparing with (1.5), we can read off the normal mode frequencies, aka the dispersion relation:

$$\omega_k^2 = v_s^2 \vec{k}^2 + m^2.$$

$\omega = \pm\omega_k$ is precisely the condition for the Fourier mode $e^{i\vec{k}\cdot\vec{x} - i\omega t}$ to solve the Klein-Gordon equation (1.26).

We can decouple the modes with wavenumber k and $-k$ as above by introducing the ladder operators¹⁸

$$\phi_k \equiv \sqrt{\frac{\hbar}{2\omega_k}} \left(\mathbf{a}_k + \mathbf{a}_{-k}^\dagger \right), \quad \pi_k \equiv \frac{1}{i} \sqrt{\frac{\hbar\omega_k}{2}} \left(\mathbf{a}_k - \mathbf{a}_{-k}^\dagger \right), \quad [\mathbf{a}_k, \mathbf{a}_{k'}^\dagger] = (2\pi)^d \delta^{(d)}(k - k'). \quad (1.28)$$

Their commutator follows from $[\phi(x), \pi(y)] = i\delta^{(d)}(x - y)$ ¹⁹. In terms of the ladder operators,

$$\mathbf{H} = \int \mathrm{d}^d k \hbar\omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} L^d \right).$$

Combining the previous two steps (1.27) and (1.28), the field operators are

$$\begin{aligned} \phi(\vec{x}) &= \int \mathrm{d}^d k \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{i\vec{k}\cdot\vec{x}} \mathbf{a}_k + e^{-i\vec{k}\cdot\vec{x}} \mathbf{a}_k^\dagger \right), \\ \pi(\vec{x}) &= \frac{1}{i} \int \mathrm{d}^d k \sqrt{\frac{\hbar\omega_k}{2}} \left(e^{i\vec{k}\cdot\vec{x}} \mathbf{a}_k - e^{-i\vec{k}\cdot\vec{x}} \mathbf{a}_k^\dagger \right). \end{aligned} \quad (1.29)$$

The mode expansions (1.29) contain a great deal of information. First notice that ϕ is manifestly hermitian. Next, notice that from $\phi(\vec{x}) \equiv \phi(\vec{x}, 0)$ by itself we cannot disentangle \mathbf{a}_k and \mathbf{a}_k^\dagger , since only the combination $\mathbf{a}_k + \mathbf{a}_{-k}^\dagger$ multiplies $e^{i\vec{k}\cdot\vec{x}}$. The

¹⁸Beware that the mode operators \mathbf{a}_k defined here differ by powers of $2\pi/L$ from the finite-volume objects in the previous discussion. These agree with Peskin's conventions.

¹⁹I emphasize that this is really the same equation as our starting point for each ball on springs: $[\mathbf{q}_n, \mathbf{p}_{n'}] = i\hbar \mathbb{1} \delta_{nn'}$.

momentum $\boldsymbol{\pi}$ contains the other linear combination. However, if we evolve the field operator in time using the Heisenberg equation (as you did on the HW), we find

$$\phi(\vec{x}, t) \equiv e^{i\mathbf{H}t} \phi(\vec{x}) e^{-i\mathbf{H}t} = \int \bar{d}^d k \sqrt{\frac{\hbar}{2\omega_k}} \left(e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}}t} \mathbf{a}_k + e^{-i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}}t} \mathbf{a}_k^\dagger \right). \quad (1.30)$$

Indeed we can check that the relation $\boldsymbol{\pi} = \dot{\boldsymbol{\phi}}$ holds.

Negative frequencies. Notice that the dependence on spacetime is via a sum of terms of the form:

$$e^{i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}}t} = e^{-i k_\mu x^\mu} \Big|_{k^0 = \omega_{\vec{k}}}$$

and their complex conjugates. These are precisely all the solutions to the wave equation (1.26). For each \vec{k} , there are two solutions to the second order ODE $(\partial_t^2 + k^2 + m^2)\phi_k(t) = 0$, namely $e^{i\omega t}$ with $\omega = \pm\omega_k$, one with positive frequency and one with negative frequency. You might have worried that solutions with both signs of the frequency mean that the world might explode or something (like it would if we tried to replace the Schrödinger equation for the wavefunction with a Klein-Gordon equation). This danger is evaded in a beautiful way: the coefficient of the positive frequency solution with wavenumber \vec{k} is the destruction operator for the mode; the associated negative frequency term comes with the creation operator for the same mode, as a consequence of reality of the field $\phi = \phi^\dagger$. (Some words about antimatter might be appropriate here, but it will be clearer later when we talk about examples of particles that are not their own antiparticles.)

‘Momentum’. Finally, in a relativistic system, anything we can say about time should also be true of space, up to some signs. So the fact that we were able to generate the time dependence by conjugation with the unitary operator $e^{i\mathbf{H}t}$ (as in (1.30)) says that we should be able to generate the space dependence by conjugating by a unitary operator of the form $e^{-i\vec{\mathbf{P}}\cdot\vec{x}}$. Here $\vec{\mathbf{P}}$ is the last in a long list of objects in this section with a claim to the name ‘momentum’. It is the conserved charge associated with spatial translation symmetry, the generator of spatial translations. Its form in terms of the fields will be revealed below when we speak about Noether’s theorem. For now, let me emphasize that it is distinct from the objects $p_n, \pi(x)$ (which were ‘momenta’ in the sense of canonical momenta of various excitations) and also from the wavenumbers \vec{k} of various modes, which (when multiplied by \hbar) are indeed spatial momenta of single particles. (This statement gives us an expectation for what is the total momentum of a state of a collection of particles which we will check below in §1.4.) In terms of the momentum operator, then, we can write

$$\phi(x^\mu) = e^{i\mathbf{P}_\mu x^\mu} \phi(0) e^{-i\mathbf{P}_\mu x^\mu}$$

with $\mathbf{P}_\mu \equiv (\mathbf{H}, \vec{\mathbf{P}})_\mu$.

1.3 Quantum light: Photons

The quantization of the Maxwell field is logically very similar to the preceding discussion. There are just a few complications from its several polarizations, and from the fact that quantum mechanics means that the vector potential is real and necessary (whereas classically it is just a convenience). This is a quick-and-dirty version of the story. I mention it here to emphasize that the machinery we are developing applies to a system you have already thought a lot about!

Maxwell's equations (with $c = 1$) are:

$$\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \qquad \vec{\nabla} \cdot \vec{B} = 0, \qquad \vec{\nabla} \times \vec{E} = -\partial_t \vec{B}, \qquad (1.31)$$

$$\partial^\mu F_{\mu\nu} = 4\pi j_\nu \qquad \vec{\nabla} \cdot \vec{E} = 4\pi\rho, \qquad \nabla \times \vec{B} = \partial_t \vec{E} + 4\pi\vec{j} \qquad (1.32)$$

(where the familiar electric and magnetic fields are $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$). The first two equations (1.31) can be regarded as constraints on \vec{E} and \vec{B} which mean that their components are not independent. This is annoying for trying to treat them quantumly. To get around this we introduce *potentials* $A_\mu = (\Phi, \vec{A})_\mu$ which determine the fields by taking derivatives and which automatically solve the constraints (1.31):

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad \text{aka} \qquad \vec{E} = -\vec{\nabla}\Phi - \partial_t \vec{A}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}.$$

Potentials related by a *gauge transformation*

$$\vec{A} \rightarrow \vec{A}^\lambda \equiv \vec{A} - \vec{\nabla}\lambda, \quad \Phi \rightarrow \Phi^\lambda \equiv \Phi + \partial_t \lambda \qquad (1.33)$$

for any function $\lambda(\vec{r}, t)$, give the same \vec{E}, \vec{B} (or $F_{\mu\nu}$), and must be regarded as different descriptions of the same physical situation. I emphasize for the first of many times that the transformation (1.33) is *not a symmetry* of the Maxwell theory, but rather a redundancy of our description. There is no such thing as 'gauge symmetry'.

Nevertheless, the Bohm-Aharonov effect is proof that (some of the information in)

the potential is real and useful, despite this redundancy²⁰²¹²². [End of Lecture 4]

We can partially remove the gauge redundancy by choosing our potentials to satisfy *Coulomb gauge*

$$\vec{\nabla} \cdot \vec{A} = 0.$$

In the absence of sources $\rho = 0 = \vec{j}$, we can also set $\Phi = 0$. In this gauge, Ampere's law becomes (restoring factors of c for rhetorical drama)

$$c^2 \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = c^2 \vec{\nabla} \cdot (\vec{\nabla} \cdot \vec{A}) - c^2 \nabla^2 \vec{A} = -\partial_t^2 \vec{A}, \quad \text{i.e. } \boxed{\partial_t^2 \vec{A} - c^2 \nabla^2 \vec{A} = 0}.$$

This wave equation is different from our scalar wave equation (1.8) in three ways:

- we're in three spatial dimensions,
- the speed of sound v_s has been replaced by the actual speed of light c (which is good since the waves in question are light-waves),
- the field \vec{A} is a vector field obeying the constraint $\vec{\nabla} \cdot \vec{A} = 0$. In Fourier space $\vec{A}(x) = \int d^d k e^{i\vec{k} \cdot \vec{x}} \vec{A}(k)$ this condition is

$$0 = \vec{k} \cdot \vec{A}(k)$$

– the vector field is *transverse*.

²⁰In case you haven't seen it before: for a closed curve C , the object

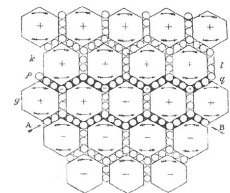
$$\oint_C A_\mu(x) dx^\mu \rightarrow \oint_C A_\mu^\lambda(x) dx^\mu = \oint_C A_\mu(x) dx^\mu + \underbrace{\oint_C \partial_\mu \lambda dx^\mu}_{\stackrel{\text{FTC}}{=} 0} \quad (1.34)$$

is gauge invariant (by the Fundamental Theorem of Calculus), and hence can be physical. Indeed it is physical: The phase acquired by particle of charge e moving along the path C is $e^{ie \oint_C A_\mu dx^\mu}$. This can be observed in an interference experiment.

²¹A brief warning about the previous footnote for future experts, please ignore if you're not interested: actually the allowed gauge transformations are of the form $A \rightarrow A + \mathbf{i}g^{-1}dg$, where g is a single-valued function living in the gauge group, which is $U(1)$ (not \mathbb{R} , if charge is quantized). This includes *large gauge transformations*. For example, if the system lives on a circle with coordinate $\theta \equiv \theta + 2\pi$, then the allowed functions are of the form $g_n(\theta) = e^{in\theta}$ for some $n \in \mathbb{Z}$. The variation of $\oint_{S^1} A$ under the transformation labelled by n is $\oint_{S^1} A \rightarrow \oint_{S^1} A + \mathbf{i} \oint_{S^1} g_n^{-1} dg_n = \oint_{S^1} A + 2\pi n$. The second term is the winding number of the function g_n . But since this is 2π times an integer, the phase acquired by a particle with integer charge q , $e^{iq \oint_{S^1} A}$, is still gauge-invariant.

²²In fact, we also don't know how to formulate quantum mechanics of a charged particle without the vector potential. So this is another strong motivation to introduce it.

In this discussion, we are starting from the continuum field theory description, analogous to (1.22) for the case of sound waves. If you read Maxwell's original paper, you will see that he tried hard to introduce an analog of the balls and springs for the electromagnetic field, a particular UV completion of his equations involving various interlocking cogs and gears (as in the figure at right from his 1861 paper) I actually don't understand it; if you do please explain it to me.



An action that gives rise to Maxwell's equations is

$$S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) = \int d^4x \mathcal{L}_{\text{Maxwell}}. \quad \mathcal{L}_{\text{Maxwell}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2). \quad (1.35)$$

Note that we must regard A as the dynamical variable to obtain (1.32) by $0 = \frac{\delta S}{\delta A_\mu(x)}$. (1.35) is beautifully simple and inevitable: the action should be a Lorentz scalar (meaning that all indices must be contracted using $\eta_{\mu\nu}$) and it must be invariant under (1.33). What other scalar, gauge-invariant quantity can you make out of A_μ ? ²³

The canonical momentum of A is then $\Pi_{A_i} = \frac{\partial \mathcal{L}_{\text{Maxwell}}}{\partial \dot{A}_i} = E^i$. So the Hamiltonian is²⁴:

$$\mathbf{H} = \frac{1}{2} \int d^3x \left(\vec{E}^2 + c^2 \vec{B}^2 \right). \quad (1.37)$$

Here $\vec{E} = -\partial_t \vec{A}$ plays the role of field momentum $\pi(x)$ in (1.23), and $\vec{B} = \vec{\nabla} \times \vec{A}$ plays the role of the spatial derivative $\partial_x q$. We can immediately treat this as a quantum system by analogy with the scalar case, by expanding \vec{A} in terms of a basis of solutions of the equations of motion and promoting their coefficients to creation and annihilation

²³The condition of gauge invariance is easy to solve by writing only functionals of F . A single $F_{\mu\nu}$ has Lorentz indices on it, which we need to contract if we want a Lorentz scalar. We could try taking derivatives of it, but we'd need two derivatives to saturate all the indices, and $\partial^\mu \partial^\nu F_{\mu\nu} = 0$ by antisymmetry of $F_{\mu\nu}$. So we need at least two F s. There are actually other terms we can consider, such as

$$\partial^\rho F_{\mu\nu} \partial_\rho F^{\mu\nu}. \quad (1.36)$$

But notice that this has more derivatives than F^2 , and will therefore be less important in the long-wavelength limit. This doesn't mean that it is absent, but we will ignore all the other terms for now and think about them more later. (The brief version is: (1.36) has two more derivatives than the Maxwell term; dimensional analysis then requires that its coefficient has dimensions of length-squared. If we're interested in wavelengths much longer than this length scale then we can ignore (1.36).) The logic I've just described is a simple example of the Effective Field Theory perspective on the world, one which I hope to proselytize in this course.

²⁴You may also recall that the energy density of a configuration of Maxwell fields is $u = \frac{1}{2} (\vec{E}^2 + \vec{B}^2)$. This result can be obtained either by Legendre transformation of $\mathcal{L}_{\text{Maxwell}}$, or from T_0^0 , the energy momentum tensor, as you'll see on the HW.

operators:

$$\vec{\mathbf{A}}(\vec{r}, t) = \int d^3k \frac{1}{\sqrt{2\omega_k}} \sum_{s=1,2} \left(\mathbf{a}_{\vec{k},s} \vec{e}_s(\hat{k}) e^{i\vec{k}\cdot\vec{r}-i\omega_k t} + \mathbf{a}_{\vec{k},s}^\dagger \vec{e}_s^*(\hat{k}) e^{-i\vec{k}\cdot\vec{r}+i\omega_k t} \right) .$$

where

$$[\mathbf{a}_{\vec{k},s}, \mathbf{a}_{\vec{k}',s'}^\dagger] = \delta^d(k - k') \delta_{ss'} . \quad (1.38)$$

Here we need $\omega_k^2 = c^2 k^2$ to solve the wave equation. There is a bit more to the Maxwell equations: The polarization vectors $\vec{e}(\hat{k})$ conspire to make it so that there are only two independent states for each \vec{k} and they are transverse $\vec{k} \cdot \vec{e}_s(\hat{k}) = 0$, so $s = 1, 2$. The polarization vectors for a given k can be chosen to satisfy a completeness relation:

$$\sum_s e_{si}(\hat{k}) e_{sj}^*(\hat{k}) = \delta_{ij} - \hat{k}_i \hat{k}_j . \quad (1.39)$$

This says that they span the plane perpendicular to \hat{k} . We can also choose them to be normalized: $\hat{e}_s(\hat{k}) \cdot \hat{e}_s^*(\hat{k}) = 1$.

The field momentum is $\vec{\mathbf{E}} = -\partial_t \vec{\mathbf{A}}$:

$$\vec{\mathbf{E}}(\vec{r}, t) = \mathbf{i} \int d^d k \sum_{s=1,2} \sqrt{\frac{\hbar\omega_k}{2}} \left(\mathbf{a}_{\vec{k},s} \vec{e}_s(\hat{k}) e^{i\vec{k}\cdot\vec{r}-i\omega_k t} - \mathbf{a}_{\vec{k},s}^\dagger \vec{e}_s^*(\hat{k}) e^{-i\vec{k}\cdot\vec{r}+i\omega_k t} \right)$$

Also, the magnetic field operator is

$$\vec{\mathbf{B}} = \vec{\nabla} \times \vec{\mathbf{A}} = \int d^d k \sum_s \sqrt{\frac{\hbar}{2\omega_k}} \mathbf{i}\vec{k} \times \left(\mathbf{a}_{\vec{k},s} \vec{e}_s(\hat{k}) e^{i\vec{k}\cdot\vec{r}-i\omega_k t} - \mathbf{a}_{\vec{k},s}^\dagger \vec{e}_s^*(\hat{k}) e^{-i\vec{k}\cdot\vec{r}+i\omega_k t} \right)$$

It seems like the canonical commutator should generalize according to

$$[\phi(x), \pi(x')] = \mathbf{i}\hbar\delta(x - x') \quad \rightsquigarrow \quad [\mathbf{A}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] \stackrel{?}{=} -\mathbf{i}\hbar\delta^3(\vec{r} - \vec{r}')\delta_{ij}$$

where $i, j = 1..3$ are spatial indices. This is not quite true. If we plug in the expressions above and use (1.38) and (1.39) we find instead

$$[\mathbf{A}_i(\vec{r}), \mathbf{E}_j(\vec{r}')] = -\mathbf{i}\hbar \int d^3k e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \left(\delta_{ij} - \hat{k}_i \hat{k}_j \right)$$

Plugging these expressions into the Hamiltonian (1.37), we can write it in terms of these oscillator modes (which create and annihilate *photons*). As for the scalar field, the definitions of these modes were designed to make the Hamiltonian simple:

$$\mathbf{H} = \sum_{\vec{k},s} \hbar\omega_k \left(\mathbf{a}_{\vec{k},s}^\dagger \mathbf{a}_{\vec{k},s} + \frac{1}{2} L^d \right) .$$

As a small check, note that using this Hamiltonian and the canonical commutator, we can reproduce Maxwell's equations:

$$\partial_t^2 \vec{A} = -\partial_t \vec{E} = -\frac{\mathbf{i}}{\hbar} [\mathbf{H}, \vec{E}] = c^2 \vec{\nabla}^2 \vec{A}.$$

The last step is a little bit nontrivial.

So the groundstate is the state $|0\rangle$ annihilated by all the annihilation operators $\mathbf{a}_{\vec{k},s} |0\rangle = 0, \forall \vec{k}, s$. Higher-energy eigenstates of \mathbf{H} can be made by creating photons, *e.g.* $\mathbf{a}_{\vec{k},s}^\dagger |0\rangle$ is a state with one photon. Like phonons, they are identical bosons.

The vacuum energy is

$$E_0 = \frac{1}{2} \sum_{\vec{k},s} \hbar \omega_k = \frac{L^3}{(2\pi)^3} \int d^3k \hbar c k.$$

The fact that \sum_k is no longer a finite sum might be something to worry about. We will see below in §1.6 that this vacuum energy has physical consequences.

Consolidation of understanding.

So far in this chapter, we have studied systems of increasing complexity, all with a quadratic Hamiltonian: the simple harmonic oscillator, a non-interacting scalar field, and the EM field in vacuum (*i.e.* in the absence of charge). All these free field theories have the same structure, in the following sense.

In the following, $\text{Re } \mathbf{A} \equiv \frac{1}{2} (\mathbf{A} + \mathbf{A}^\dagger)$ as usual. The normalization constant is $\mathcal{N} = \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}}$.

$$\mathbf{H}_{\text{SHO}} = \frac{1}{2m} \mathbf{p}^2 + \frac{1}{2} m\omega^2 \mathbf{q}^2 = \hbar\omega \left(\mathbf{a}^\dagger \mathbf{a} + \frac{1}{2} \right)$$

$$[\mathbf{q}, \mathbf{p}] = i\hbar \implies [\mathbf{a}, \mathbf{a}^\dagger] = \mathbb{1}.$$

$$\mathbf{q} = \text{Re } \mathcal{N} \mathbf{a}, \quad \mathbf{p} = m \text{Im } \omega \mathcal{N} \mathbf{a}.$$

$$\mathbf{H}_{\text{scalar}} = \int dx \left(\frac{1}{2\mu} \boldsymbol{\pi}^2 + \frac{1}{2} \mu c^2 (\partial_x \phi)^2 \right) = \sum_k \hbar\omega_k \left(\mathbf{a}_k^\dagger \mathbf{a}_k + \frac{1}{2} \right)$$

$$[\phi(x), \boldsymbol{\pi}(x')] = i\hbar \delta(x - x') \implies [\mathbf{a}_k, \mathbf{a}_{k'}^\dagger] = i\hbar \delta(k - k').$$

$$\phi(x) = \text{Re} \left(\sum_k \mathcal{N}_k e^{ikx} \mathbf{a}_k \right), \quad \boldsymbol{\pi}(x) = \mu \text{Im} \left(\sum_k \omega_k \mathcal{N}_k e^{ikx} \mathbf{a}_k \right).$$

$$\mathbf{H}_{\text{EM}} = \int d^3x \left(\frac{\epsilon_0}{2} \vec{\mathbf{E}}^2 + \frac{\epsilon_0 c^2}{2} \vec{\mathbf{B}}^2 \right) = \sum_{k,s=1,2} \hbar\omega_k \left(\mathbf{a}_{ks}^\dagger \mathbf{a}_{ks} + \frac{1}{2} \right)$$

$$[\mathbf{A}_i(x), \mathbf{E}_j(x')] = i\hbar \left(\delta^3(x - x') \delta_{ij} - \partial_i \partial_j \frac{1}{4\pi|x - x'|} \right) \implies [\mathbf{a}_{ks}, \mathbf{a}_{k's'}^\dagger] = \hbar \delta(k - k') \delta_{ss'}.$$

$$\vec{\mathbf{A}}(x) = \text{Re} \left(\sum_k \mathcal{N}_k e^{i\vec{k}\cdot\vec{x}} \mathbf{a}_{ks} \vec{e}_s(\hat{k}) \right), \quad \vec{\mathbf{E}}(x) = \mu \text{Im} \left(\sum_k \omega_k \mathcal{N}_k e^{i\vec{k}\cdot\vec{x}} \mathbf{a}_{ks} \vec{e}_s(\hat{k}) \right).$$

Note that $\vec{\mathbf{E}}$ is the canonical momentum of $\vec{\mathbf{A}}$.

My point here is to emphasize the parallels between these three sets of equations.

1.4 Lagrangian field theory and symmetries

[Here we fill in the bits of Peskin §2.2 that we missed above.] Let's consider a classical field theory in the Lagrangian description. This means that the degrees of freedom are a set of fields $\phi_r(x)$, where r is a discrete index (for maybe spin or polarization or flavor), and we specify the dynamics by the classical action. If the world is kind to us (in this class we assume this), the action is an integral over space and time of a Lagrangian density

$$S[\phi] \equiv \int d^{d+1}x \mathcal{L}(\phi, \partial^\mu \phi).$$

This important assumption is an implementation of *locality*.

This central object \mathcal{L} encodes the field equations, the canonical structure on the phase space, the Hamiltonian, the symmetries of the theory. We've seen basically two examples so far

$$\mathcal{L}_{\text{KG}} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2$$

and

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2e^2} (E^2 - B^2)$$

with A_μ regarded as the independent degrees of freedom.

A word about units and dimensional analysis: in units with $\hbar = c = 1$, everything has units of mass to some power, called its mass dimension. Energy and momentum $p_\mu = \hbar k_\mu$ have mass dimension $+1$. The space and time coordinates x^μ have mass dimension -1 . The action goes in the exponential of the path integral measure $\int [D\phi] e^{\frac{iS}{\hbar}}$ and so must be dimensionless. So the Lagrangian density has mass dimension $d + 1$. This means that the KG field has mass dimension $\frac{d-1}{2}$ (and the mass m has mass dimension 1 (yay!)). In $d + 1 = 3 + 1$ dimensions, with the Maxwell action, $E \sim \dot{A}$, $B \sim \vec{\nabla} A$ have mass dimension 2 and A has mass dimension one (and e is dimensionless). This is nice because then the covariant derivative $\partial_\mu + A_\mu$ has mass dimension one. Notice that $E^2 + B^2$ has dimension 4 which is good for an energy per unit volume.

object	mass dim.
p_μ	1
x^μ	-1
$\frac{\partial}{\partial x^\mu}$	1
S	0
\mathcal{L}	$d + 1$
ϕ	$\frac{d-1}{2}$
A_μ	1
$\vec{E}, \vec{B}, F_{\mu\nu}$	2

The equation of motion is

$$0 = \frac{\delta S}{\delta \phi_r(x)}.$$

Note the functional derivative. You can check that in the special case when \mathcal{L} depends only on ϕ and $\partial_\mu \phi$ (and no higher derivatives of ϕ) this is the same as the Lagrange EOM

$$0 = \frac{\partial \mathcal{L}}{\partial \phi_r} - \partial^\mu \frac{\partial \mathcal{L}}{\partial (\partial^\mu \phi_r)}$$

(for each r) which I cannot and don't want to remember. Note that since we are interested here in the bulk equations of motion, we ignore boundary terms unless we are interested in field theory on a space with boundary. That is a worthy subject but an unnecessary complication for now.

What's special about ϕ ? From the path integral point of view, the field is just an integration variable. By redefining the field by *e.g.* $\phi \equiv \frac{1}{D}(\chi - B/C)$, we can make the KG theory uglier

$$\mathcal{L} = A + B\chi + \frac{1}{2}C\chi^2 + \frac{1}{2}D\partial^\mu\chi\partial_\mu\chi + \dots$$

One way in which ϕ may be special is if there is a symmetry; for example a \mathbb{Z}_2 symmetry $\phi \rightarrow -\phi$ would forbid terms with odd powers of ϕ in the action. Sometimes, its normalization is meaningful, like in the phonon example where it began its life as the displacement of the atoms from their equilibrium positions. So you see that we are not losing generality except in our neglect of interactions, and in our neglect of terms with more derivatives. The former neglect we will repair little by little in this course, by doing perturbation theory. The latter is justified well by the renormalization group philosophy, which is a subject for next quarter.

Canonical field momentum and Hamiltonian. The Hamiltonian viewpoint in field theory has the great virtue of bringing out the physical degrees of freedom. It has the great shortcoming that it picks out the time coordinate as special and obscures Lorentz symmetry.

The canonical field momentum is defined to be

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi(x))}.$$

Notice that this expression assumes a local Lagrangian density. π is actually a 'field momentum density' in the sense that the literal canonical momentum is $\frac{\partial}{\partial\phi(x)}L = \int d^d x \pi(x)$ (as opposed to \mathcal{L}). I will often forget to say 'density' here.

The hamiltonian is then

$$H = \sum_n p_n q_n - L = \int d^d x \left(\sum_r \pi_r(x) \dot{\phi}_r(x) - \mathcal{L} \right) \equiv \int d^d x \mathfrak{h}.$$

Noether's theorem and the Noether method. Yay, symmetries. Why do physicists love symmetries so much? One reason is that they offer possible resting places along our never-ending chains of 'why?' questions. For example, one answer (certainly the one given in Weinberg's text, but just as certainly not the only one) to the question "Why QFT?" is: quantum mechanics plus Poincaré symmetry.

They are also helpful for solving physical systems. Continuous symmetries are associated with conserved currents. Suppose the action is invariant under a continuous transformation of the fields $\phi, \phi(x) \mapsto \phi'(x)$. The invariance of the action is what makes the transformation a symmetry²⁵. ‘Continuous’ here means we can do the transformation just a little bit, so that the action is preserved under $\phi(x) \mapsto \phi(x) + \epsilon \Delta\phi(x)$, where ϵ is an infinitesimal parameter.

OK, now check out the following slippery maneuver: If the transformation with constant ϵ (independent of space and time) is a symmetry, then the variation of the action with $\epsilon = \epsilon(x, t)$ must be proportional to $\partial_\mu \epsilon$ (at least assuming some smoothness properties of the action), and so that it vanishes $\forall \phi$ when ϵ is constant:

$$S[\phi(x) + \epsilon(x)\Delta\phi(x)] - S[\phi(x)] = \int d^d x dt \partial_\mu \epsilon(x) j^\mu \stackrel{\text{IBP}}{=} - \int d^d x dt \epsilon(x) \partial_\mu j^\mu. \quad (1.40)$$

But if the equations of motion are obeyed, then the action is invariant under *any* variation, including this one, for arbitrary $\epsilon(x)$. But this means that $\partial_\mu j^\mu = 0$, the current j^μ defined by the first equation in (1.40) is conserved. These words are an accurate description of the equation $\partial_\mu j^\mu = 0$ (called the continuity equation) because it means that the charge

$$Q_R \equiv \int_R d^d x j^0$$

in some fixed region of space R can only change by leaving the region (assume the definition of R is independent of time):

$$\partial_t Q_R = \int_R d^d x \partial_t j^0 = - \int_R d^d x \vec{\nabla} \cdot \vec{j} = \int_{\partial R} d^{d-1} x \hat{n} \cdot \vec{j}$$

where in the last step we used Stokes’ theorem.

This trick with pretending the symmetry-transformation parameter ϵ depends on space is called the *Noether method*. More prosaically, the condition that the action is invariant means that the Lagrangian density changes by a total derivative (we assume boundary terms in the action can be ignored):

$$\mathcal{L}(\phi', \partial_\mu \phi') \stackrel{\text{symmetry}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \partial_\mu \mathcal{J}^\mu$$

but on the other hand, by Taylor expansion,

$$\mathcal{L}(\phi', \partial_\mu \phi') \stackrel{\text{calculus}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \left(\frac{\partial \mathcal{L}}{\partial \phi} \Delta\phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu \Delta\phi \right)$$

²⁵Actually, the action only appears in the path integral in the combination $e^{iS/\hbar}$. So it is enough for this object to be invariant. This requires only $S \mapsto S + 2\pi n, n \in \mathbb{Z}$. This loophole is sometimes very important; I hope you will see some examples in later quarters of QFT.

$$\stackrel{\text{IBP}}{=} \mathcal{L}(\phi, \partial_\mu \phi) + \epsilon \left(\underbrace{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}}_{\text{eom}} \right) \Delta \phi + \epsilon \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi \right).$$

By combining the previous two equations for $\mathcal{L}(\phi')$, we see that on configurations that satisfy the EOM, $0 = \partial_\mu j^\mu$ with

$$j^\mu = \sum_r \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_r)} \Delta \phi_r - \mathcal{J}^\mu. \quad (1.41)$$

Notice that I put back the index r on ϕ_r at the last step to emphasize the summation²⁶

There is a converse to the Noether theorem, which is easier to discuss directly in quantum mechanics. Given a conserved charge Q , that is, a hermitian operator with $[H, Q] = 0$, we can make a symmetry transformation of the fields ϕ by

$$\delta \phi \equiv \mathbf{i}\epsilon[Q, \phi]. \quad (1.42)$$

We'll say that Q *generates* the symmetry, for the following reason. (1.42) is the infinitesimal version of the finite transformation

$$\phi \rightarrow \phi' \equiv e^{\mathbf{i}\epsilon Q} \phi e^{-\mathbf{i}\epsilon Q}.$$

The object $\mathbf{U} \equiv e^{\mathbf{i}\epsilon Q}$ is a unitary operator (since Q is hermitian) which represents the action of the symmetry on the Hilbert space of the QFT. It is a symmetry in the sense that it commutes with the time evolution operator $e^{-\mathbf{i}Ht}$.

Some examples will be useful:

- For example, suppose $S[\phi]$ only depends on ϕ through its derivatives, $\mathcal{L}(\phi, \partial\phi) = \mathcal{L}(\partial\phi)$. For definiteness, take $S[\phi] = \int \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$. Then there is a shift symmetry $\phi \rightarrow \phi' \equiv \phi + \epsilon$. Letting ϵ depend on spacetime, the variation of the action is $S[\phi + \epsilon(x)] - S[\phi] = \int \epsilon \partial_\mu \partial^\mu \phi$, so the current is $j_\mu = \partial_\mu \phi$. Let's check the converse: Indeed, the charge $Q = \int_{\text{space}} j_0$ generates the symmetry in the sense that for small ϵ , the variation in the field is

$$\delta \phi \equiv \phi' - \phi = \epsilon = \mathbf{i}\epsilon[Q, \phi]$$

(if we were doing classical mechanics, we should replace $\mathbf{i}[Q, \phi]$ with the Poisson bracket). Using our expression for the current this is

$$\delta \phi = \mathbf{i}\epsilon \left[\int d^d y \underbrace{\dot{\phi}(y)}_{=\pi(y)}, \phi(x) \right] = \epsilon$$

²⁶So far we've been doing classical physics. In QM, there is an extra ingredient, namely the path integral measure. In fact, there are examples of transformations that preserve S , but where $[d\phi]$ is not invariant. This is called an *anomaly*. More on this in a future quarter of QFT.

which is indeed true. In this case the finite transformation is again $\phi \rightarrow \phi + \epsilon$.

- On the homework you've seen an example involving a complex scalar Φ , with $S[\Phi, \Phi^*]$ that is invariant under $\Phi \rightarrow e^{i\epsilon}\Phi = \Phi + i\epsilon\Phi + \mathcal{O}(\epsilon^2)$. This $U(1)$ phase transformation can be rewritten in terms of the real and imaginary parts as an $SO(2)$ rotation. The charge can be written as

$$Q = \int d^d x j^0 = \int \tilde{d}^d p (\mathbf{a}_p^\dagger \mathbf{a}_p - \mathbf{b}_p^\dagger \mathbf{b}_p)$$

where the two sets of creation and annihilation operators are associated with excitations of Φ and Φ^\dagger respectively. (That is, make mode expansions of $\phi_{1,2}$ as we did for a single real scalar field, in terms of mode operators $\mathbf{a}_{1,2}$ respectively. Then let $\mathbf{a} \equiv \mathbf{a}_1 + i\mathbf{a}_2$, $\mathbf{b} \equiv \mathbf{a}_1 - i\mathbf{a}_2$, up to numerical prefactors that I tried to get right in the solutions.) So the particles created by \mathbf{a} and \mathbf{b} have opposite charge (this is essential given the mode expansion $\Phi_k \sim \mathbf{a}_k + \mathbf{b}_{-k}^\dagger$) and can be interpreted as each others' antiparticles: there can be symmetry-respecting processes where an \mathbf{a} particle and \mathbf{b} particle take each other out.

- Consider spacetime translations, $x^\mu \rightarrow x^\mu - a^\mu$. This induces a transformation of the fields by

$$\phi(x) \mapsto \phi(x + a) = \phi(x) + a^\nu \underbrace{\partial_\nu \phi}_{\equiv \Delta_\nu \phi} + \mathcal{O}(a^2).$$

Our transformation parameter is now itself a four-vector, so we'll get a four-vector of currents T_ν^μ . This will be a symmetry as long as the lagrangian doesn't depend explicitly on space and time (so $\partial_\nu \mathcal{L} = 0$) but rather depends on space and time only via the fields (so $0 \neq \frac{d}{dx^\nu} \mathcal{L} \stackrel{\text{chain rule}}{=} \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \partial_\nu \phi \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}$). Let's use the prosaic method for this one: the shift in the Lagrangian density also can be found by Taylor expansion

$$\mathcal{L}(\phi(x), \partial\phi(x)) \mapsto \mathcal{L}(\phi(x + a), \partial\phi(x + a)) \stackrel{\text{Taylor}}{=} \mathcal{L} + a^\mu \frac{d}{dx^\mu} \mathcal{L} = \mathcal{L} + a^\nu \frac{d}{dx^\mu} (\delta_\nu^\mu \mathcal{L}).$$

So the formula (1.41) gives

$$T_\nu^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \underbrace{\partial_\nu \phi}_{\Delta_\nu \phi} - \mathcal{L} \delta_\nu^\mu.$$

For the time translation, the conserved charge T_0^0 gives back the hamiltonian density $\mathfrak{h} = \pi \dot{\phi} - \mathcal{L}$ obtained by Legendre transformation. [\[End of Lecture 5\]](#)

The conserved quantity from spatial translations is the momentum carried by the field, which for the KG field is

$$\mathbf{P}_i = \int d^d x T_i^0 = - \int d^d x \pi \partial_i \phi.$$

For the Maxwell field, this gives the Poynting vector.

There is some ambiguity in the definition of the stress tensor which you'll study on the homework.

Let's check that the expression above for the conserved momentum agrees with our expectations. In particular, in free field theory the total momentum of the state $|\vec{k}_1, \dots, \vec{k}_n\rangle$ should be just the sum of the momenta of the particles, $\vec{P} = \sum_{\ell=1}^n \hbar \vec{k}_\ell$ (with interactions the story can be more complicated). Indeed

$$\mathbf{P}_i = - \int d^d x \pi \partial_i \phi = \int d^d k k_i \mathbf{a}_k^\dagger \mathbf{a}_k$$

agrees with this. (Notice that I used rotation invariance of the vacuum to not worry about a possible constant term.)

- We can do the same thing for spatial rotations, which act on a scalar field by $\phi(x^i) \mapsto \phi(R_j^i x^j)$. An infinitesimal rotation is of the form $R_j^i = \delta_j^i + \omega_j^i$ where ω is an antisymmetric matrix (this guarantees that $RR^T = 1$) specifying the plane and angle of rotation. I'll leave it as an exercise to repeat the analysis and find the conserved current(s) associated with rotation invariance.
- Actually we can do the whole Lorentz group in the same manner. It acts on a scalar field by $\phi(x^\mu) \mapsto \phi(\Lambda_\nu^\mu x^\nu)$. We'll say more about the rest of the Poincaré group, *i.e.* boosts, later on.
- Consider a *scale transformation*, also known as a *dilatation*:

$$\phi(x^\mu) \mapsto \lambda^{d_\phi} \phi(\lambda x^\mu). \tag{1.43}$$

Here d_ϕ is a number that can be different for different fields, called the *scaling dimension* of the field. Classically, the scaling dimension is the same as the mass dimension in the table above ($\frac{D-2}{2}$ for a scalar field ϕ , 1 for a vector potential A_μ). A system is scale invariant if it has no intrinsic length scale – no dimensionful quantities in its definition; an example is the free Maxwell theory. You can study the associated conserved quantity on the homework. I'll say a bit more about scaling symmetry next.

- The study of what is a symmetry of a quantum system of extended dofs is an active subject. For a taste of recent generalizations, take a look at [this recent review](#).

1.5 Naive scale invariance in field theory

The scale transformation plays a special role in our modern understanding of QFT. This modern understanding of QFT is based on the renormalization group. This is the idea that we should understand a system with many degrees of freedom scale-by-scale, allowing for the possibility that our description may change depending on the resolution of the probes with which we study it. The simplest case, then, is when the system is the same at all length scales.

[The following discussion comes from the QFT lectures of Marty Halpern.] Consider a field theory of a scalar field ϕ in D spacetime dimensions, with an action of the form

$$S[\phi] = \int d^D x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - g \phi^p \right)$$

for some constants p, g . Which value of p makes this scale invariant? One way to assess this is to ask: when is g dimensionless?

Naive mass dimensions:

$$[S] = [\hbar] = 0, \quad [x] \equiv -1, \quad [d^D x] = -D, \quad [\partial] = 1$$

The kinetic term tells us the engineering dimensions of ϕ :

$$0 = [S_{\text{kinetic}}] = -D + 2 + 2[\phi] \implies [\phi] = \frac{D-2}{2}.$$

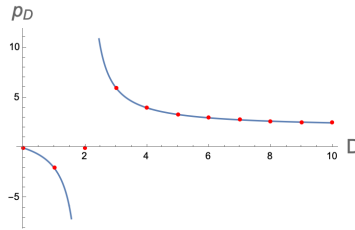
From the path integral representation, we see that quantum field theory in $D = 1$ spacetime dimensions *is* quantum mechanics of a single particle, where the role of the field is played by the position of the particle.

Then the self-interaction term has dimensions

$$0 = [S_{\text{interaction}}] = -D + [g] + p[\phi] \implies [g] = D - p[\phi] = D + p \frac{2-D}{2}$$

We expect scale invariance when $[g] = 0$ which happens when

$$p = p_D \equiv \frac{2D}{D-2},$$



i.e. the scale invariant scalar-field self-interaction in D spacetime dimensions is $\phi^{\frac{2D}{D-2}}$.

D	1	2	3	4	5	6	...	D	∞
$[\phi]$	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$3/2$	2	...	$\frac{D-2}{2}$	∞
scale-inv't $p \equiv p_D$	-2	∞ *	6	4	$10/3$	3	...	$\frac{2D}{D-2}$	2

* What is happening in $D = 2$? The field is dimensionless, and so *any* power of ϕ is naively scale invariant, as are more complicated interactions like $g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j$, where the coupling $g(\phi)$ is a function of ϕ . This allows for scale-invariant *non-linear sigma models*, where the fields are coordinates on a curved manifold with metric $ds^2 = g_{ij}d\phi^i d\phi^j$.

So the case of $D = 1$ says we get a scale invariant theory by putting a particle in a potential that goes like $1/q^2$.

In dimensions where we get fractional powers, this isn't so nice.

Notice that the mass term $\Delta S = \int d^D x \frac{m^2}{2} \phi^2$ gives

$$0 = -D + 2[m] + 2[\phi] \implies [m] = 1 \quad \forall D < \infty$$

– it's a mass, yay.

What are the consequences of this engineering-dimensions calculation in QFT? For $D > 2$, an interaction of the form $g\phi^p$ has

$$[g] = D \cdot \frac{p_D - p}{p_D} \begin{cases} < 0 \text{ when } p > p_D, & \text{non-renormalizable or irrelevant} \\ = 0 \text{ when } p = p_D, & \text{renormalizable or marginal} \\ > 0 \text{ when } p < p_D, & \text{super-renormalizable or relevant.} \end{cases} \quad (1.44)$$

Here I've given names to the different possibilities. To understand these names, consider the 'non-renormalizable' case. Suppose we calculate in QFT some quantity f (say a scattering amplitude) with $[f]$ as its naive dimension, in perturbation theory in g . This is what we'll learn to do using Feynman diagrams soon. Even without knowing anything else, we know the answer will be a series in powers of g :

$$f = \sum_{n=0}^{\infty} g^n c_n$$

with c_n independent of g . So

$$[f] = n[g] + [c_n] \implies [c_n] = [f] - n[g]$$

So if $[g] < 0$, c_n must have more and more powers of some mass (inverse length) as n increases. What dimensionful quantity makes up the difference? Sometimes it is



masses or external momenta. But generically, it gets made up by UV divergences (if everything is infinite, dimensional analysis can fail, nothing is real, I am the walrus). More usefully, in a meaningful theory with a UV cutoff, Λ_{UV} , the dimensions get made up by the UV cutoff, which has $[\Lambda_{UV}] = 1$. Generically (*i.e.* in the absence of cancellations): $c_n = \tilde{c}_n (\Lambda_{UV})^{-n[g]}$, where \tilde{c}_n is dimensionless, and $n[g] < 0$ – it’s higher and higher powers of the cutoff at each order in perturbation theory. This is how non-renormalizable theories got a bad reputation.


Now consider the renormalizable (classically scale invariant) case: Here, $[c_n] = [f]$, for every n , since $[g] = 0$. But in fact, what you’ll get is something like

$$c_n = \tilde{c}_n \log^{\nu(n)} \left(\frac{\Lambda_{UV}}{\Lambda_{IR}} \right), \quad (1.45)$$

where Λ_{IR} is an infrared cutoff or a mass or external momentum, $[\Lambda_{IR}] = 1$. This means that scale invariance is actually broken by quantum effects! More crudely, it is broken by the presence of the UV cutoff.

In this box, I use some things I haven’t explained yet to illustrate the claim (1.45). Think of it as a preview. Some classically scale invariant examples (so that $m = 0$ and the bare propagator is $1/k^2$) where you can see that we get logs from loop amplitudes:

ϕ^4 in $D = 4$:  $\sim \int d^4k \left(\frac{1}{k^2}\right)^2$ ϕ^6 in $D = 3$:  $\sim \int d^3k \int d^3k \left(\frac{1}{k^2}\right)^3$

ϕ^3 in $D = 6$:  $\sim \int d^6k \left(\frac{1}{k^2}\right)^3$ In $D = 2$, even the propagator for a massless scalar field has logs:

$$\langle \phi(x)\phi(0) \rangle = \int d^2k \frac{e^{-ikx}}{k^2} \sim \log \frac{|x|}{\Lambda_{UV}}.$$

The terms involving ‘renormalizable’ in (1.44) are somewhat old-fashioned and come from a high-energy physics point of view where the short-distance physics is unknown, and we want to get as far as we can in that direction with our limited knowledge (in which case the condition ‘renormalizability’ lets us get away with this indefinitely – it lets us imagine we know everything). The latter terms are natural in the opposite situation (like condensed matter physics) where we know some basically correct microscopic description but want to know what happens at low energies. Then an operator like $\frac{1}{M^{2d}}\phi^{2d}$ whose coefficient is suppressed by some large mass scale M is *irrelevant* for physics at energies far below that scale. Inversely, an operator like $m^2\phi^2$ gives a

mass to the ϕ particles, and matters very much (is relevant) at energies $E < m$. In the marginal case, the quantum corrections have a chance to make a big difference.

1.6 Casimir effect: vacuum energy is real

[A. Zee, *Quantum Field Theory in a Nutshell*, §I.9] This subsection has two purposes. One is to show that the $\sum_k \frac{1}{2} \hbar \omega_k$ energy of the vacuum of free fields (such as the radiation field) is real. Sometimes we can get rid of it by choosing the zero of energy (which doesn't matter unless we are studying dynamical gravity). But it is meaningful if we can *vary* ω (or the collection of ω s if we have many oscillators as for the radiation field) and compare the difference.

The other purpose is to give an object lesson in asking the right questions. In physics, the right question is often a question that can be answered by an experiment, at least in principle. Sometimes a computer simulation (even an imagined one) can play the same role. The answers to such questions tend to be well-defined and are less sensitive to our silly theoretical prejudices, *e.g.* about what happens to physics at very short distances.

In the context of the bunch of oscillators making up the radiation field, we can change the spectrum of frequencies of these oscillators $\{\omega_k\}$ by putting it in a box and varying the size of the box. In particular, two parallel conducting plates separated by some distance d experience an attractive force from the change in the vacuum energy of the EM field resulting from their presence. The plates put boundary conditions on the field, and therefore determine which normal modes are present.

To avoid some complications of E&M that are not essential for our point here, we're going to make two simplifications:

- we're going to solve the problem in 1+1 dimensions
- and we're going to solve it for a scalar field.

To avoid the problem of changing the boundary conditions *outside* the plates we use the following device with *three* plates:

$$| \leftarrow d \rightarrow | \leftarrow \quad L - d \quad \rightarrow |$$

(We will consider $L \gg d$, so we don't really care about the far right plate.) The 'perfectly conducting' plates impose the boundary condition that our scalar field $q(x)$ vanishes there. The normal modes of the scalar field $q(x)$ in the left cavity are then

$$q_j = \sin(j\pi x/d), \quad j = 1, 2, \dots$$

with frequencies $\omega_j = \frac{\pi|j|}{d}c$. That is, the mode expansion for the scalar field in the left cavity is

$$q(x, t) = \sum_j q_j(x) e^{-i\omega_j t} a_j + h.c. \quad (1.46)$$

that is, relative to our previous expression, we must set $a_k = -a_{-k}$ to satisfy the boundary conditions. There is a similar expression for the modes in the right cavity which we won't need. We're going to add up all the $\frac{1}{2}\hbar\omega$ s for all the modes in both cavities to get the vacuum energy $E_0(d)$; the force on the middle plate is then $-\partial_d E_0$.

The vacuum energy in the whole region of interest between the outer plates is the sum of the vacuum energies of the two cavities

$$E_0(d) = f(d) + f(L - d)$$

where

$$f(d) = \frac{1}{2}\hbar \sum_{j=1}^{\infty} \omega_j = \hbar c \frac{\pi}{2d} \sum_{j=1}^{\infty} j \stackrel{?!?!}{=} \infty.$$

We have done something wrong. What?

Our crime is hubris: we assumed that we knew what the modes of arbitrarily large mode number k (arbitrarily short wavelength, arbitrarily high frequency) are doing, and in particular we assumed that they cared about our silly plates. In fact, no metal in existence can put boundary conditions on the modes of large enough frequency – those modes don't care about d . The reason a conductor puts boundary conditions on the EM field is that the electrons move around to compensate for an applied field, but there is a limit on how fast the electrons can move (*e.g.* the speed of light). The resulting cutoff frequency is called the *plasma frequency* but we don't actually need to know about all these details. To parametrize our ignorance of what the high-frequency modes do, we must *cut off* (or *regularize*) the contribution of the high-frequency modes. Let's call modes with $\omega_j \gg \pi/a$ high frequency, where a is some short time²⁷. Replace

$$\begin{aligned} f(d) \rightsquigarrow f(a, d) &= \hbar \frac{\pi}{2d} \sum_{j=1}^{\infty} e^{-a\omega_j/\pi} j \\ &= -\frac{\pi\hbar}{2} \partial_a \left(\underbrace{\sum_{j=1}^{\infty} e^{-aj/d}}_{= \frac{1}{1-e^{-a/d}} - 1} \right) \end{aligned}$$

²⁷You can think of a as the time it takes the waves to move by one lattice spacing. If we work in units where the velocity is $c = 1$, this is just the lattice spacing. I will do so for the rest of this discussion.

$$\begin{aligned}
&= + \frac{\pi \hbar}{2d} \frac{e^{a/d}}{(e^{a/d} - 1)^2} \\
&\stackrel{a \ll d}{\simeq} \hbar \left(\underbrace{\frac{\pi d}{2a^2}}_{\rightarrow \infty \text{ as } a \rightarrow 0} - \frac{\pi}{24d} + \frac{\pi a^2}{480d^3} + \dots \right) \quad (1.47)
\end{aligned}$$

Answers that don't depend on a have a chance of being meaningful. The thing we can measure is the force on the middle plate:

$$\begin{aligned}
F &= -\partial_d E_0 = -(f'(d) - f'(L-d)) \\
&= -\hbar \left(\left(\frac{\pi}{2a^2} + \frac{\pi}{24d^2} + \mathcal{O}(a^2) \right) - \left(\frac{\pi}{2a^2} + \frac{\pi}{24(L-d)^2} + \mathcal{O}(a^2) \right) \right) \\
&\stackrel{a \rightarrow 0}{=} -\frac{\pi \hbar}{24} \left(\frac{1}{d^2} - \frac{1}{(L-d)^2} \right) \\
&\stackrel{L \gg d}{=} -\frac{\pi \hbar c}{24d^2} (1 + \mathcal{O}(d/L)) . \quad (1.48)
\end{aligned}$$

This is an attractive force between the plates. (I put the c back in the last line.)

The analogous force between real conducting plates, caused by the change of boundary conditions on the electromagnetic field, has been measured.

An important question is to what extent could we have picked a different cutoff function (instead of $e^{-\pi\omega/a}$) and gotten the same answer for the physics. This interesting question is answered affirmatively in Zee's wonderful book, 2d edition, section I.9 (available electronically [here!](#)).

For example, we could have regularized the sum by imagining that space is a lattice. Then we can use our dispersion relation (1.6) in the sum:

$$\omega_j = 2\sqrt{\frac{\kappa}{m}} \sin \frac{j\pi}{N} = \frac{2c}{a} \sin \frac{j\pi}{N}, \quad N \equiv d/a \in \mathbb{Z}, j = 1..N. \quad (1.49)$$

You'll have the opportunity on the homework to see whether this reproduces the same answer as above²⁸.

²⁸In response to my claim that the answers to physics questions will be independent of the regulator, for reasonable choices of regulator, the following question arises: what happens if we just put a hard cutoff on the frequency, so that

$$f(d) \rightsquigarrow f(N, d) \equiv \frac{\pi \hbar c}{2d} \sum_{j=1}^N j = \frac{\pi \hbar c}{2d} \frac{N(N+1)}{2}.$$

Section 15.2 of Schwarz's textbook explains how to extract the correct answer using this regulator. The key point is that the number of modes N (a) is an integer but (b) it depends on the size of the

The string theorists will tell you that $\sum_{j=1}^{\infty} j = -\frac{1}{12}$, and our calculation above agrees with them in some sense. But what this foolishness means is that if we compute something that is not dependent on the cutoff we have to get the same answer no matter what cutoff we use²⁹. Notice that it is crucial to ask the right questions.

A comment about possible physical applications of the calculation we actually did: you could ask me whether there is such a thing as a Casimir force due to the vacuum fluctuations of *phonons*. Certainly it's true that the boundary of a chunk of solid puts boundary conditions on the phonon modes, which change when we change the size of the solid. The problem with the idea that this might produce a measurable force (which would lead the solid to want to shrink) is that it is hard to distinguish the 'phonon vacuum energy' from the rest of the *energy of formation* of the solid, that is, the energy difference between the crystalline configuration of the atoms and the configuration when they are all infinitely separated. Certainly the latter is not well-described in the harmonic approximation ($\lambda = 0$ in (1.1)).

A few comments about the 3+1 dimensional case of E&M. Assume the size of the plates is much larger than their separation L . Dimensional analysis shows that the force per unit area from vacuum fluctuations must be of the form

$$P = \gamma \frac{\hbar c}{L^4}$$

where γ is a numerical number. γ is not zero!

Use periodic boundary conditions in the xy planes (along the plates). The allowed wave vectors are then

$$\vec{k} = \left(\frac{2\pi n_x}{L_x}, \frac{2\pi n_y}{L_y} \right)$$

with n_x, n_y integers.

We have to do a bit of E&M here. Assume the plates are perfect conductors (this is where the hubris about the high-frequency modes enters). This means that the transverse component of the electric field must vanish at the surface. Instead of plane waves in z , we get standing waves: $\phi(z) \propto \sin(n\pi z/L)$.

cavity. This produces a rapidly oscillating force as d varies, and the wall actually only experiences the *average* force.

These same issues arise in using the lattice regulator. See the homework for some hints.

²⁹A point of view from which this equation is true is the following: The Riemann zeta function is defined (for s in a certain region of the complex plane) to be $\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s}$. This sum has a domain of convergence (which does not include $s = -1$), but it defines an analytic function in that domain, and can therefore be analytically continued to a larger domain (for example by finding an integral representation). The value at $s = -1$ is $\zeta(-1) = -\frac{1}{12}$. This is called zeta-function regularization. If we use this value in E_0 we get the same answer as above.

The frequencies of the associated standing waves are then

$$\omega_n(\vec{k}) = c\sqrt{\frac{\pi^2 n^2}{L^2} + \vec{k}^2}, n = 0, 1, 2, \dots$$

Also, there is only one polarization state for $n = 0$.

So the zero-point energy is

$$E_0(L) = \frac{\hbar}{2} \left(2 \sum'_{n, \vec{k}} \omega_n(\vec{k}) \right)$$

where it's useful to define

$$\sum'_{n, \vec{k}} \equiv \frac{1}{2} \sum_{n=0, \vec{k}} + \sum_{n \geq 1, \vec{k}}$$

Now you can imagine introducing a regulator like the one we used above, and replacing

$$\sum'_{n, \vec{k}} \rightsquigarrow \sum'_{n, \vec{k}} e^{-a\omega_n(\vec{k})/\pi}.$$

and doing the sums and integrals and extracting the small- a (and specifically the a -independent) behavior.

The answer, $\gamma = -\frac{\pi^2}{2 \cdot 5!}$, was predicted by Casimir in 1948 and first unambiguously measured 50 years later.

Comments about the vacuum energy. One conclusion from the above calculation is that the vacuum energy is very UV sensitive – the biggest contributions come from the highest-frequency modes. So even small changes in the short-distance physics makes a big change in E_0 .

We said earlier that the vacuum energy itself is hard to measure, because what we measure is energy differences. But vacuum energy, like all kinds of energy and momentum, gravitates. Positive vacuum energy, when dominant, leads space to *inflate*, *i.e.* to expand in an exponential way in time (see the homework for a demonstration of this statement). Observations of distant supernovae and of the cosmic microwave background show that this is happening right now (for the last few billion years).

Partly because of its UV sensitivity, we can't make a strong prediction for the value of the vacuum energy density E_0 – we simply don't know what happens at wavelengths shorter than an inverse TeV or so. Note that our calculation only works for fields that are well-approximated by balls and springs, *i.e.* a quadratic lagrangian. This is

actually true of the degrees of freedom of the Standard Model at the highest energy scales we know about.

But the observed value of E_0 is very small compared to the scales that appear in particle physics. Our answer is $\int^\Lambda d^3k k \sim \Lambda^4$; if we use $\Lambda \sim \text{TeV}$ we are wrong by 60 orders of magnitude. So there must be some dramatic cancellation. We don't have a good explanation of this. The explanation could be that if the cancellation didn't happen, the universe would have expanded too fast for any structure to form, and so we wouldn't be here to ask the question. This anthropic explanation led the late Steven Weinberg (against his own wishes) to [predict](#) a nonzero value of the cosmological constant on the order of the observed value – a decade before the observation. This kind of reasoning would be disappointing if correct, but cannot be discounted.

[\[End of Lecture 6\]](#)

2 The path integral makes some things easier

2.1 Fields mediate forces

[Zee §1.3] Consider again our chain of balls on springs. Suppose a giant hand reaches in and pushes the atom at position x_n a little bit. This can be described by adding to the hamiltonian a term

$$\delta V(q) = -J_n(t)q_n(t)$$

which applies a force $J_n(t)$ to the n th atom. The giantness of the hand is reflected in the fact that we treat $J_n(t)$ as a fixed c-number function – our little chain of balls do not affect the giant hand.

We can ask, in the presence of such a force, what is the amplitude to go from state I to state F in time T :

$$\langle F | e^{-i \int_0^T dt H(t)} | I \rangle = \int [D\phi] e^{i \int dt d^d x (\frac{1}{2}(\partial\phi)^2 - V(\phi) + J(x)\phi(x))}.$$

As you see, this is a quantity for which we have a path integral representation. Here's a reason to care about this quantity: take the initial and final states to be the ground-state:

$$\langle 0 | e^{-i \int_0^T dt H(t)} | 0 \rangle \simeq e^{-i \int_0^T dt E_{gs}(J)}.$$

If the time-dependence is slow enough, the answer is obtained by the adiabatic approximation: just add up the instantaneous groundstate energy at each time step. This is something we might like to know about, for example to compute the force exerted on the giant hand by our field. Another motivation is that by taking (functional) derivatives with respect to $J(x)$, we can pull down powers of $q(x)$; so this quantity is a generating functional for correlation functions of q .

In order to be able to do this path integral, we retreat to the case where the action is quadratic in ϕ , so that

$$\mathcal{L}(\phi) = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) \stackrel{\text{IBP}}{\equiv} -\frac{1}{2} \phi (\partial^2 + m^2) \phi + \text{total derivative}. \quad (2.1)$$

Going back to the lattice to make the integrals look slightly less scary, we have

$$e^{iW[J]} \equiv \int [D\phi] e^{i \int (\mathcal{L} + J\phi)} = \int_{-\infty}^{\infty} \prod_{n,t}^{M_t, N} dq_{n,t} e^{\frac{i}{2} q_x A_{xy} q_y + i J_x q_x} = \sqrt{\frac{(2\pi i)^{NM_t}}{\det A}} e^{-\frac{i}{2} J_x A_{xy}^{-1} J_y}.$$

Here repeated indices are summed as usual: $q_x A_{xy} q_y = \sum_{x,y} \phi(x) A_{xy} \phi(y)$, etc... So you can see that the matrix A multiplying the quadratic term in this gaussian integral is (a discretization of) $A_{xy} = -\delta^{d+1}(x-y) (\partial_x^2 + m^2)$. It is an $NM_t \times NM_t$ matrix.

Its inverse A^{-1} satisfies by definition $A_{xz}A_{zy}^{-1} = \delta_{xy}$, which is (a discretization of) the differential equation

$$-(\partial^2 + m^2)D(x - y) = \delta(x - y). \quad (2.2)$$

This equation says that D is a *Green's function* for the operator $-(\partial^2 + m^2)$. The fact that there is no special point in spacetime says $A_{xy}^{-1} = D(x - y)$ only depends on the difference of its arguments.

Does this integral actually converge? The integral you'll study on the homework is of the form $\int_{\mathbb{R}} dq e^{-\frac{1}{2}qAq}$, which surely converges if A is a positive matrix. Actually, this is overkill – it is enough to replace $m^2 \rightarrow m^2 - \mathbf{i}\epsilon$ to make all the integrals converge. Here ϵ is an *infinitesimal*, which means $\epsilon^2 = 0$ and $c\epsilon = \epsilon$ for any positive c . Then each $\int dq_{nt}$ will have a factor of $e^{-\epsilon \int q_{nt}^2}$ which suppresses the integrand in the dangerous large-field region³⁰.

The equation (2.2) is translation-invariant and linear so you should not be surprised that it is solved by going to Fourier space (in space and time):

$$D(x) = \int \mathbb{d}^{d+1}k e^{\mathbf{i}k_\mu x^\mu} D_k, \quad \delta^{d+1}(x) = \int \mathbb{d}^{d+1}k e^{\mathbf{i}k_\mu x^\mu}.$$

in terms of which (2.2) becomes the algebraic equation $1 = (k^2 - m^2 + \mathbf{i}\epsilon)D_k$. Hence

$$D_k = \frac{1}{k^2 - m^2 + \mathbf{i}\epsilon} \implies D(x) = \int \mathbb{d}^{d+1}k \frac{e^{\mathbf{i}kx}}{k^2 - m^2 + \mathbf{i}\epsilon}. \quad (2.3)$$

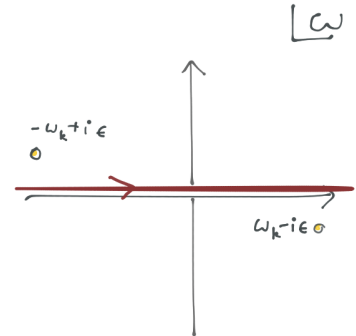
Notice that the shift by ϵ saves the day here: it keeps the integration contour from running right over the pole at $k^2 = m^2$, by moving slightly in the imaginary direction. More explicitly, the denominator

$$k^2 - m^2 + \mathbf{i}\epsilon = \omega^2 - \vec{k}^2 - m^2 + \mathbf{i}\epsilon \quad (2.4)$$

is zero when

$$\omega = \pm \sqrt{\vec{k}^2 + m^2 - \mathbf{i}\epsilon} = \pm (\omega_k - \mathbf{i}\epsilon), \quad \omega_k \equiv \sqrt{\vec{k}^2 + m^2}.$$

In the second step I Taylor expanded $\sqrt{\omega_k^2 - \mathbf{i}\epsilon} = \sqrt{\omega_k^2} - \frac{\mathbf{i}\epsilon}{\omega_k} + \mathcal{O}(\epsilon)^2$ and used the facts that $\omega_k > 0$, and that anything positive times an infinitesimal is an infinitesimal.



We can then do the ω integral by contours³¹: if $t > 0$ ($t < 0$), we can close the

³⁰You could worry that there could be other ways to make the integral well-defined (there are). Another thing you might be bothered by is the boundary conditions on the fields and their relation to the initial and final states. In the next subsection, we'll say more.

³¹We are using the Cauchy residue theorem $\oint_C dz f(z) = 2\pi\mathbf{i} \sum_{z_j} \text{Res}_{z=z_j} f$ where z_j are the poles of f . To remember the sign, consider a small circle C_0 counterclockwise around the origin and $f(z) = 1/z$, so $\oint_{C_0} \frac{dz}{z} = \mathbf{i} \int_0^{2\pi} d\theta = 2\pi\mathbf{i}$.

contour in the UHP (LHP) since the integrand goes like $e^{-\text{Im}\omega t}$, and the integral equals the residue of the pole at $\omega = \pm(\omega_k - \mathbf{i}\epsilon)$ (times $2\pi\mathbf{i}$):

$$D(x) = -\mathbf{i} \int \mathbf{d}^d k \left(\theta(t) \frac{e^{-\mathbf{i}(\omega_k t - \vec{k} \cdot \vec{x})}}{2\omega_k} + \theta(-t) \frac{e^{\mathbf{i}(\omega_k t - \vec{k} \cdot \vec{x})}}{2\omega_k} \right). \quad (2.5)$$

Here $\theta(t) \equiv 1$ if $t > 0$ and 0 otherwise is the Heaviside function. We'll learn to call this *time-ordered* in a moment.

The propagator. Who is $D(x)$, besides some quantity in terms of which we did a Gaussian integral? As you'll see on the homework, the inverse matrix can be extracted via a two-point correlation function:

$$(A^{-1})_{ij} = \int \left(\prod dq \right) q_i q_j e^{\mathbf{i}\frac{1}{2}q A q} / Z = -\mathbf{i} \langle q_i q_j \rangle.$$

Putting back all the labels, the same manipulations show that

$$D(x - y) = \int [D\phi] \phi(x) \phi(y) e^{\mathbf{i}S[\phi]} / Z \stackrel{?}{=} -\mathbf{i} \langle 0 | \hat{\phi}(x) \hat{\phi}(y) | 0 \rangle \quad (2.6)$$

– the amplitude to propagate an excitation created from the vacuum by $\phi(y)$ to be annihilated by $\phi(x)$. The propagator, for short.

(Notice that if the system is Lorentz invariant (which starting from (2.1) it is) then since $D(x)$ is a scalar quantity, it can only depend on x through Lorentz invariants made from x^μ , namely the proper distance $x^2 = t^2 - \vec{x}^2$, and the sign of t .)

Why the “?” in (2.6)? For one thing, in the expression (2.6), $\phi(x)$ and $\phi(y)$ are operators – the order matters. How do I know in which order to write them? (This is in contrast with the path integral expression $\int [D\phi] e^{\mathbf{i}S[\phi]} \phi(x) \phi(y)$, where they are just integration variables, numbers.) To reproduce (2.5) the thing to do is to *time-order* them:

$$\begin{aligned} D(x - y) &= Z^{-1} \int [D\phi] e^{\mathbf{i}S[\phi]} \phi(x) \phi(y) |_{m^2 \rightarrow m^2 - \mathbf{i}\epsilon} \\ &= \langle 0 | \mathcal{T} \phi(x) \phi(y) | 0 \rangle \equiv \theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle. \end{aligned} \quad (2.7)$$

To verify this, plug in the mode expansion (1.29) to see *e.g.*

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{\mathbf{d}^d k \mathbf{d}^d q}{2\sqrt{\omega_k \omega_q}} e^{-\mathbf{i}kx + \mathbf{i}qy} \langle 0 | a_k a_q^\dagger | 0 \rangle = \int \frac{\mathbf{d}^d k}{2\omega_k} e^{-\mathbf{i}k(x-y)}$$

(where $k^0 = \omega_k, q^0 = \omega_q$ to satisfy the KG equation), which reproduces the first term in (2.5)³².

³²The other ways of making the path integral well-defined correspond to other ways of ordering the ϕ s, and other initial and final states.

Now why should we care about the propagator? Look again at $W[J]$. We've learned that (up to the addition of terms independent of J),

$$W[J] = -\frac{1}{2} \int \int d^{d+1}x d^{d+1}y J(x) D(x-y) J(y) = -\frac{1}{2} \int \bar{d}^{d+1}k J_k^* \frac{1}{k^2 - m^2 + i\epsilon} J_k.$$

Here $J(x) = \int \bar{d}^{d+1}k e^{ikx} J_k$, $J_k^* = J_{-k}$. (The last step is another instance of Parseval's theorem, or the statement that Fourier transform maps convolution to multiplication. Just plug in to check.)

We get to pick $J(x)$. Think of our chain of balls and springs as a 1+1d version of a mattress. Let's choose $J = J_1 + J_2$ to describe (in Zee's words) two lumps sitting still on the mattress: $J_a(x) = \delta^3(x - x_a)$, $a = 1, 2$. Then $J_k = \int dx^0 e^{-ik^0 x^0} (e^{i\vec{k}\cdot\vec{x}_1} + e^{i\vec{k}\cdot\vec{x}_2})$. The interaction between the two lumps *mediated* by the mattress field ϕ will then be described by the $J_1 J_2$ cross-terms in $W[J]$:

$$W[J] = 2 \int_x \int_y J_1(x) D(x-y) J_2(y) + \int \int J_1 D J_1 + \int \int J_2 D J_2 \quad (2.8)$$

$$= -\frac{2}{2} \int dx^0 \int dy^0 \int \bar{d}k^0 e^{ik^0(x^0 - y^0)} \int \bar{d}^3k \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}{k^2 - m^2 + i\epsilon} + \dots \quad (2.9)$$

$$= - \int dx^0 \underbrace{\left(\int \bar{d}k^0 2\pi\delta(k^0) \right)}_{=1} \int \bar{d}^3k \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}{k^2 - m^2 + i\epsilon} + \dots \quad (2.10)$$

$$= + \int dx^0 \int \bar{d}^3k \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + m^2 - i\epsilon} + \dots \quad (2.11)$$

where the ... represent the terms involving J_1^2 and J_2^2 .

For this choice of J , the Hamiltonian is time-independent, and

$$e^{iW} = \langle \text{gs} | e^{-iHT} | \text{gs} \rangle = e^{-iE_{\text{gs}}(J)T}, \quad (2.12)$$

where $|\text{gs}\rangle$ is the groundstate of $H(J)$, with eigenvalue $E_{\text{gs}}(J)$. This equality will become clearer in the next subsection. Therefore (2.12) says $W = -E_{\text{gs}}(J)T$. We learn that

$$E_{\text{gs}}(J) = - \int \bar{d}^d k \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + m^2}$$

where $\vec{r} \equiv \vec{x}_1 - \vec{x}_2$. Notice that we can drop the $i\epsilon$ now, because this integrand is nonsingular for real \vec{k} . In $d = 1$, there are poles at $k = \pm im$, and for $x > 0$ ($x < 0$) we

can close the contour in the UHP (LHP) for free to get³³

$$E_{\text{gs}}(J) = -\frac{2\pi\mathbf{i}}{2\pi} \frac{e^{-m|x|}}{2im} = -\frac{e^{-m|x|}}{2m}.$$

Since x is the separation between the lumps, this means that our field has produced an *attractive* force between the lumps

$$F = -\partial_x E_{\text{gs}}(J) = +\frac{1}{2}e^{-m|x|}$$

which falls off exponentially with the separation between the lumps. The attractive nature of this force is familiar to anyone who has sat on a bed next to another person. The range of the potential goes inversely with the mass of the ‘force carrier’ ϕ . The 3d version of this potential $\frac{e^{-mr}}{r}$ (see footnote 33) is called the Yukawa potential.

[End of Lecture 7]

2.2 Euclidean path integral and Wick rotation

[Recommended reading: see Fradkin, chapter 5 for a slightly different perspective on this story.] Here is a route to defining the path integral that makes clearer what is going on with the initial and final states. Along the way, we will see that actually it is the same as the replacement $m^2 \rightarrow m^2 - \mathbf{i}\epsilon$ that we did above in the case of scalar field theory)

For simplicity, let us focus on a single mode of the field – a single harmonic oscillator

$$S[q] = \frac{1}{2} \int dt ((\partial_t q)^2 - \Omega^2 q^2) - \int Jq$$

(where if you like $\Omega^2 = \vec{k}^2 + m^2$ for some fixed k). Consider the replacement $\tau = \mathbf{i}t$ in the action:

$$S[q] = -\frac{1}{2}\mathbf{i} \int d\tau (-(\partial_\tau q)^2 - \Omega^2 q^2) + \mathbf{i} \int d\tau Jq = +\mathbf{i} \int d\tau \left(\frac{1}{2} ((\partial_\tau q)^2 + \Omega^2 q^2) + Jq \right).$$

³³For convenience, here’s the integral in 3d:

$$\begin{aligned} \int d^3k \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2 + M^2} &\stackrel{y \equiv \cos\theta}{=} \frac{1}{(2\pi)^2} \int_0^\infty \frac{k^2 dk}{k^2 + M^2} \underbrace{\int_{-1}^1 dy e^{iky r}}_{=2\frac{\sin kr}{kr}} = \frac{1}{(2\pi)^2 r} \int_{-\infty}^\infty \frac{dk k \sin kr}{k^2 + M^2} \\ &= \frac{1}{(2\pi)^2 r} \left(\frac{1}{2\mathbf{i}} \int_{-\infty}^\infty \frac{dk k e^{ikr}}{k^2 + M^2} + h.c. \right) \quad \text{close contour in UHP for free} \\ &= \frac{1}{(2\pi)^2 r} \frac{1}{2\mathbf{i}} 2\pi\mathbf{i} \frac{\mathbf{i}M e^{i(M)r}}{2\mathbf{i}M} \cdot 2 = \frac{e^{-Mr}}{4\pi r}. \end{aligned}$$

With this replacment, the path integral becomes

$$Z_E[J] = \int [dq] e^{-\int d\tau (\frac{1}{2}((\partial_\tau q)^2 + \Omega^2 q^2) + Jq)} = \int [dq] e^{-S_E[q]}. \quad (2.13)$$

This is a different integral than before if we regard τ as a real variable. This integrand suppresses configurations with large q , and large $\partial_\tau q$, and the integral is therefore totally well-defined. The euclidean action is³⁴

$$S_E[q] = \int d\tau \left(\frac{1}{2} ((\partial_\tau q)^2 + \Omega^2 q^2) + Jq \right) = \int d\tau \left(\frac{1}{2} q (-\partial_\tau^2 + \Omega^2) q + Jq \right)$$

where $(-\partial_\tau^2 + \Omega^2)$ is a positive operator (meaning all of its eigenvalues are positive)³⁵. Call its inverse G_E (or G for short), which then, by definition, satisfies

$$(-\partial_\sigma^2 + \Omega^2) G(\sigma, \tau) = \delta(\sigma - \tau) \quad .$$

The fact that our system is time-translation invariant means $G(\sigma, \tau) = G(\sigma - \tau)$. We can solve this equation in fourier space: $G(\sigma) = \int \tilde{d}\omega e^{i\omega\sigma} G_\omega$ makes it algebraic:

$$G_\omega = \frac{1}{\omega^2 + \Omega^2}$$

and we have

$$G(\sigma) = \int \tilde{d}\omega \frac{e^{i\omega\sigma}}{\omega^2 + \Omega^2} = e^{-\Omega|\sigma|} \frac{1}{2\Omega}. \quad (2.14)$$

(This is the same integral we did above, but it will help to say it again. Do it by residues: the integrand has poles at $\omega = \pm i\Omega$ (see the figure 1 below). The absolute value of $|\sigma|$ is crucial, and comes from the fact that the contour at infinity converges in the upper (lower) half plane for $\sigma > 0$ ($\sigma < 0$)). The integral is thus completely

³⁴It is called euclidean because the $(\partial_\tau q)^2$ has the same sign as the spatial derivatives $(\partial_x q)^2$, so this is the action we get in euclidean spacetime with metric $\delta_{\mu\nu}$, rather than $\eta_{\mu\nu}$. Exercise: put back the spatial derivative terms and check that this is the case.

³⁵ An important perspective on the uniqueness of the euclidean Green's function and the non-uniqueness of the real-time Green's function: in euclidean time, the Green's function is the inverse of an operator of the form $-\partial_\tau^2 + \Omega^2$ which is *positive* (\equiv all its eigenvalues are positive) – recall that $-\partial_\tau^2 = \hat{p}^2$ is the square of a hermitian operator. If all the eigenvalues are positive, the operator has no kernel, so it is completely and unambiguously invertible. If $A = \sum_n \lambda_n |n\rangle\langle n|$, then the inverse is unambiguously $A^{-1} = \sum_n \frac{1}{\lambda_n} |n\rangle\langle n|$ as long as none of the $\lambda_n = 0$. This is why there are no poles on the axis of the (euclidean) ω integral in (2.14). In real time, in contrast, we are inverting something like $+\partial_t^2 + \Omega^2$ which annihilates modes with $\partial_t = i\Omega$ (if we were doing QFT in $d > 0 + 1$ this equation would be the familiar $p^2 - m^2 = 0$). These are called ‘on-shell states’, they are actual states in the spectrum of the Hamiltonian of the system. So the operator we are trying to invert has a kernel – it’s not actually invertible on the whole Hilbert space – and this is the source of the ambiguity. In frequency space, this is reflected in the presence of poles of the integrand *on* the contour of integration; the choice of how to negotiate them encodes the choice of Green's function.

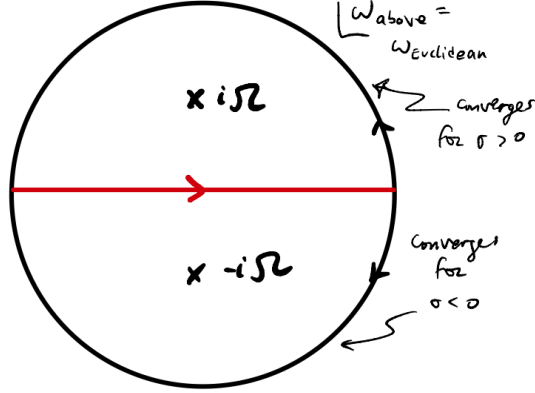


Figure 1: Poles of the integrand of the ω integral in (2.14).

well-defined with no ambiguity and the result is

$$Z_E[J] = \# \sqrt{\det G_E} e^{-\frac{1}{2} \int d\tau \int d\sigma J(\tau) G_E(\tau-\sigma) J(\sigma)}. \quad (2.15)$$

Now we address several questions about this calculation.

1. What does the euclidean path integral compute? It computes ground state expectation values. Why does the euclidean path integral put the oscillator into its groundstate? This I explained already during our path integral review in the previous section: The point in life of the euclidean path integral is to prepare the groundstate from an arbitrary state:

$$\int_{q(\beta)=q_0} [dq] e^{-S[q]} = \langle q_0 | e^{-\beta \mathbf{H}} | \dots \rangle = \psi_{\text{gs}}(q_0) \quad (2.16)$$

– the euclidean-time propagator $e^{-\beta \mathbf{H}}$ beats down the amplitude of any excited state relative to the groundstate, for large enough β .

More slowly: the path integral representation for the real-time propagation amplitude is

$$\langle f | e^{-i\mathbf{H}t} | i \rangle = \int_{f \leftarrow i} [dq] e^{i \int_0^t dt L}.$$

On the RHS here, we sum over all paths between i and f in time t (i.e. $q(0) = q_i, q(t) = q_f$), weighted by a phase $e^{i \int dt L}$. But that means you also know a representation for

$$\sum_f \langle f | e^{-\beta \mathbf{H}} | f \rangle \equiv \text{tr} e^{-\beta \mathbf{H}}$$

– namely, you sum over all *periodic* paths $q_i = q_f$ in *imaginary time* $\beta = i t$. So:

$$Z(\beta) = \text{tr} e^{-\beta \mathbf{H}} = \oint [dq] e^{-\int_0^\beta d\tau L} = \oint [dq] e^{-S_{\text{eucl}}[q]}$$

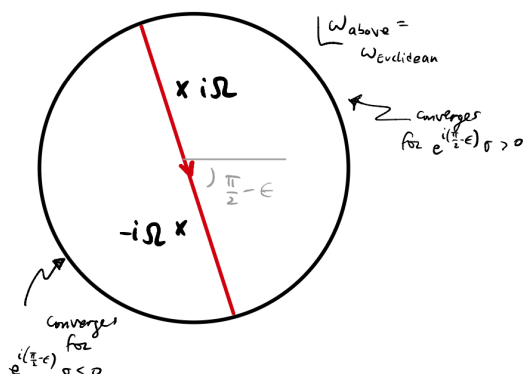
The LHS is the partition function in quantum statistical mechanics. The RHS is the euclidean functional integral we've been using. [For more on this, see Zee §V.2]

The period of imaginary time, $\beta \equiv 1/T$, is the inverse temperature. We've been studying the limit as $\beta \rightarrow \infty$. Taking $\beta \rightarrow \infty$ means $T \rightarrow 0$, and you'll agree that at $T = 0$ we project onto the groundstate (if there's more than one groundstate we have to think more).

2. What is the relationship between the Euclidean path integral we just did and the real-time path integral that we started with? I claim that the real-time calculation that keeps the oscillator in its groundstate is the *analytic continuation* of the one we did above, where we replace

$$\omega_{\text{Mink}} = e^{-i(\pi/2-\epsilon)} \omega_{\text{above}} \quad (2.17)$$

where ϵ is (a familiar) infinitesimal. In the picture of the euclidean frequency plane in Fig. 1, this is a rotation by *nearly* 90 degrees. We don't want to go all the way to 90 degrees, because then we would hit the poles at $\pm i\Omega$. The replacement (2.17) just means that if we integrate over *real* ω_{Mink} , we rotate the contour in the integral over ω as follows:



as a result we pick up the same poles at $\omega_{\text{above}} = \pm i\Omega$ as in the euclidean calculation. Notice that we had better also rotate the argument of the function, σ , at the same time to maintain convergence, that is:

$$\omega_{\text{eucl}} = -i\omega_{\text{Mink}}, \quad \omega_{\text{eucl}} t_{\text{eucl}} \stackrel{!}{=} \omega_{\text{Mink}} t_{\text{Mink}} \implies t_{\text{eucl}} = +it_{\text{Mink}}. \quad (2.18)$$

So the answer to the question at the beginning of this long paragraph is: we are rotating the contour of integration (in the complex t -plane) in the action integral.

So this is giving us a contour prescription for the real-frequency integral. The result is the *Feynman* contour, and it is the same as what we got from $m^2 \rightarrow m^2 - i\epsilon$:

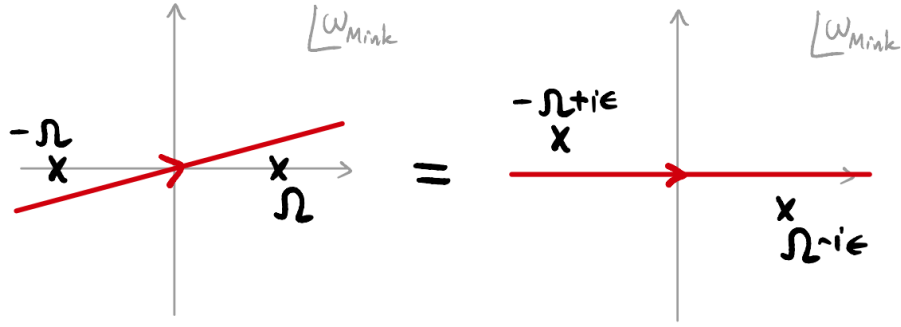


Figure 2: The Feynman contour in the ω_{Mink} complex plane.

depending on the sign of the (real) time separation of the two operators (recall that t is the difference), we close the contour around one pole or the other, giving the *time-ordered* propagator. For the case of a free scalar field, the replacement $m^2 \rightarrow m^2 - \mathbf{i}\epsilon$ had the same effect of rotating the real-frequency contour away from the poles. It is also the same as shifting the frequency by $\Omega \rightarrow \Omega - \mathbf{i}\epsilon$, as indicated in the right part of Fig. 2. This last prescription works even in a case where there is no m^2 term, *i.e.* for a massless scalar field.

Notice for future reference that the euclidean action and real-time action are related by

$$S_{\text{eucl}}[Q] = \int dt_{\text{eucl}} \frac{1}{2} \left(\left(\frac{\partial Q}{\partial t_{\text{eucl}}} \right)^2 + \Omega^2 Q^2 \right) = -\mathbf{i} S_{\text{Mink}}[Q] = -\mathbf{i} \int dt_{\text{Mink}} \frac{1}{2} \left(\left(\frac{\partial Q}{\partial t_{\text{Mink}}} \right)^2 - \Omega^2 Q^2 \right).$$

because of (2.18). This means the path integrand is $e^{-S_{\text{eucl}}} = e^{\mathbf{i} S_{\text{Mink}}}$.

Time-ordering. To summarize the previous discussion: in real time, we must choose a state, and this means that there are many Green's functions, not just one: $\langle \psi | q(t)q(s) | \psi \rangle$ depends on $|\psi\rangle$, unsurprisingly.

But we found a special one, the one that arises by analytic continuation from the euclidean Green's function, which is unique (as we discussed in Footnote 35). It is

$$G(t) = \langle \mathcal{T}(q(t)q(0)) \rangle,$$

the time-ordered, or Feynman, Green's function, and I write the time-ordering symbol \mathcal{T} to emphasize this. I emphasize that from our starting point above, the time ordering arose because we have to close the contour in the UHP (LHP) for $t > 0$ ($t < 0$).

Let's pursue this one more step. We can define the time-ordered product of more than two local operators by induction, or equivalently in the obvious way: put them

in the order of their times. The same argument tells us that the generating functional for time-ordered correlation functions of q is

$$Z[J]_{i\epsilon} \equiv \int [Dq] e^{iS_0|_{m^2 \rightarrow m^2 - i\epsilon}} e^{i\int Jq} = \langle e^{i\int Jq} \rangle = \langle 0 | \mathcal{T} e^{i\int J\hat{q}} | 0 \rangle. \quad (2.19)$$

Here the time-ordered exponential on the RHS is defined by putting each term in its Taylor expansion in time order. On the LHS we emphasize that the path integral is defined by the $i\epsilon$ prescription, by analytic continuation from the Euclidean integral. With this prescription, the dependence on the choice of initial and final states in the path integral goes away, because only the vacuum contributes.

This quantity (2.19) has the picturesque interpretation as the *vacuum persistence amplitude*, in the presence of the source J . One reason to care about $Z[J]$ is that its derivatives give correlation functions:

$$\langle 0 | \mathcal{T} \phi(x_1) \phi(x_2) \cdots | 0 \rangle = \left(-i \frac{\delta}{\delta J(x_1)} \right) \left(-i \frac{\delta}{\delta J(x_1)} \right) \cdots \ln Z[J]_{J=0}$$

– it is a generating functional for such correlators of ϕ s. So we see that in general, the correlation functions that are computed by this “ $i\epsilon$ prescription” of Wick rotating from Euclidean spacetime are time-ordered:

$$\frac{1}{Z} \int [D\phi] e^{iS_{m^2 \rightarrow m^2 - i\epsilon}} f(\phi) = \langle 0 | \mathcal{T} f(\phi) | 0 \rangle.$$

Causality. In other treatments of this subject, you will see the Feynman contour motivated by ideas about causality. We’ll discuss this in §3. This was not the logic of our discussion but it is reassuring that we end up in the same place. Note that even in 0+1 dimensions there is a useful notion of causality: effects should come after their causes.

2.3 Feynman diagrams from the path integral

In this subsection we will be brave in one sense and cowardly in another. We’ll be brave in that we are going to think about actual interactions, terms of higher order than quadratic in the Lagrangian.

The previous subsection was a sophisticated discussion of QFT in 0 + 1 dimensions, since we focussed on a single mode. To help demystify some more of the structure we’ll discover in QFT, let’s regress even further, and consider the case of QFT in 0 + 0 dimensions. By the path-integral representation, this means ordinary integrals. If everything is positive, this is probability theory – a very special case of QFT. My

intention here is to develop some intuition about perturbation theory and Feynman diagrams in a context where everything is under complete control.

Suppose we want to do the integral

$$Z(J) = \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}m^2q^2 - \frac{g}{4!}q^4 + Jq} \equiv \int dq e^{-S(q)}. \quad (2.20)$$

It is the path integral for ϕ^4 theory with fewer labels. For $g = 0$, this is a gaussian integral which we know how to do. For $g \neq 0$ it's not an elementary function of its arguments. (It's a function with someone's name attached to it, which basically just means that we don't know how to do the integral despite our best efforts.)

If you like, we are studying the probability distribution $P(q) \equiv \frac{1}{Z}e^{-S(q)}$. We'll be interested mostly in the *moments* of the distribution $\langle q^n \rangle = \int_{-\infty}^{\infty} dq q^n P(q)$, which are the analogs of correlation functions.

We can develop a (as we'll see, non-convergent!) series expansion in g by writing it as

$$Z(J) = \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}m^2q^2 + Jq} \left(1 - \frac{g}{4!}q^4 + \frac{1}{2!} \left(-\frac{g}{4!}q^4 \right)^2 + \dots \right)$$

and integrating term by term. And the term with q^{4n} (that is, the coefficient of $\frac{1}{n!} \left(-\frac{g}{4!} \right)^n$) is

$$\int_{-\infty}^{\infty} dq e^{-\frac{1}{2}m^2q^2 + Jq} q^{4n} = \left(\frac{\partial}{\partial J} \right)^{4n} \int_{-\infty}^{\infty} dq e^{-\frac{1}{2}m^2q^2 + Jq} = \left(\frac{\partial}{\partial J} \right)^{4n} e^{\frac{1}{2}J\frac{1}{m^2}J} \sqrt{\frac{2\pi}{m^2}}.$$

So, resumming the exponential series,

$$Z(J) = \sqrt{\frac{2\pi}{m^2}} e^{-\frac{g}{4!} \left(\frac{\partial}{\partial J} \right)^4 e^{\frac{1}{2}J\frac{1}{m^2}J}}. \quad (2.21)$$

Note the similarity, for $g = 0$, with our expression for $W[J]$ in the previous subsections.

Wick's Theorem is the key ingredient here. I think this is clearer in a slightly more complicated example. Consider:

$$Z[J] \equiv \int \prod_{n=1}^{\infty} dq_n e^{-\frac{1}{2}q_n A_{nm} q_m + J_n q_n - \frac{g}{4!} \sum_n q_n^4} \quad (2.22)$$

$$\stackrel{?}{=} \int \prod_{n=1}^{\infty} dq_n e^{-\frac{1}{2}q_n A_{nm} q_m + J_n q_n} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{-g}{4!} \right)^j \left(\sum_n q_n^4 \right)^j \quad (2.23)$$

If we are interested in a particular coefficient in this expansion in g and J , we can use Wick's theorem:

$$\frac{1}{Z(0)} \int \prod dq e^{-\frac{1}{2} q_n A_{nm} q_m} q_{n_1} \cdots q_{n_k} = \begin{cases} 0, & k \text{ odd} \\ \sum (\text{contractions}), & k \text{ even} \end{cases}. \quad (2.24)$$

What I mean by a *contraction* is associated with a choice of pairing of the q s in the integrand; a pair $q_n q_m$ produces a factor of $(A^{-1})_{nm}$. So for example, the simplest case is the familiar

$$\frac{1}{Z(0)} \int \prod dq e^{-\frac{1}{2} q_n A_{nm} q_m} q_i q_j = (A^{-1})_{ij} = \langle q_i q_j \rangle \equiv \langle ij \rangle.$$

The next one is

$$\frac{1}{Z(0)} \int \prod dq e^{-\frac{1}{2} q_n A_{nm} q_m} q_{n_1} q_{n_2} q_{n_3} q_{n_4} = \langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 24 \rangle + \langle 14 \rangle \langle 23 \rangle.$$

This is the statement that the moments of a gaussian distribution factorize, and are determined by the covariance A^{-1} (and the mean, which is zero here if we set $J = 0$). For the special case with just one q , this is

$$\frac{1}{Z(0)} \int dq e^{-\frac{1}{2} m^2 q^2} q^4 = \frac{3}{m^2}$$

since the three terms are the same.

Here is a proof of Wick's theorem: by completing the square, the LHS of (2.24) is

$$\frac{1}{Z(0)} \partial_{J_{n_1}} \cdots \partial_{J_{n_k}} \left(\sqrt{\frac{(2\pi)^N}{\det A}} e^{J(A^{-1})J/2} \right) \Big|_{J=0} = \partial_{J_{n_1}} \cdots \partial_{J_{n_k}} e^{J(A^{-1})J/2} \Big|_{J=0}.$$

But in order to get a nonzero answer here when we set $J = 0$, we have to choose a pairing among the n_i , and the result is the sum over such pairings (contractions) of the product of $(A^{-1})_{n_i n_j}$ for each pairing.

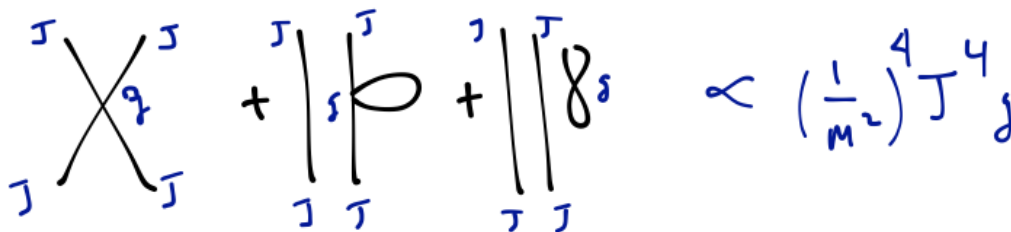
Feynman diagrams. (2.21) is a double expansion in powers of J and powers of g . This expansion can be described usefully in terms of diagrams. The idea of the diagrams is to help us keep track of the combinatorics of the Wick contractions. Each diagram is associated with an amplitude, and I will conflate the two. By the principle of superposition (the amplitude for something to happen is the sum of the amplitudes of all possible ways it could have happened), a physical quantity of interest is given by the sum of the allowed diagrams.

Here are the Feynman rules for this 0+0 dimensional field theory, say for the partition function: the allowed diagrams are all graphs with 4-valent vertices (“interactions”) and 1-valent vertices (“external lines”). There is a factor of $1/m^2$ for each edge (the propagator), and a factor of $(-g)$ for each 4-point vertex (the coupling), and a factor of J for each external line (the source). There is a further purely numerical factor which you can figure out, but let’s not worry about it right now.

So for example if we are interested in the contribution going like gJ^4 in $\tilde{Z}(J) \equiv Z(J)/Z(0)$, it comes from

$$\frac{1}{Z(0)} \int dq e^{-\frac{1}{2}m^2q^2} \left(\dots + \frac{1}{4!} (Jq)^4 + \dots \right) \left(1 - \frac{g}{4!} q^4 + \dots \right)$$

and we have to sum over all ways of contracting these 8 qs in the integrand. These contractions correspond to the following three diagrams:



I emphasize that only the topology of the diagrams matters: you can draw them however you want preserving the connectivity. It’s useful, however, to put the external lines in the same place in each diagram, in order to avoid over-counting.

[End of Lecture 8]

A systematic method to draw all the contractions contributing to the term going like $J^n g^\ell$ is to

- first draw n dots (associated with the $n Jqs$) with one nubbin coming out,
- then draw ℓ vertices (associated with the ℓgq^4 s) with four nubbins coming out.
- These nubbins represent the factors of q in the integrand $\frac{1}{Z(0)} \int dq e^{-\frac{1}{2}m^2q^2} q^{n+4\ell} \cdot J^n g^\ell \cdot \frac{1}{n!} \frac{1}{(4!)^\ell \ell!}$. Now draw all possible ways of connecting these nubbins by ‘propagators’, that is, lines connecting the nubbins that indicate which qs got contracted with which in each term on the RHS of Wick’s theorem.

There is a symmetry factor that comes from expanding the exponential: if the diagram has some symmetry preserving the external labels, the multiplicity of diagrams

does not completely cancel the $1/n!$. Please do not worry about these numerical factors right now.

As another (simpler and more important) example, consider the analog of the two-point function when $J = 0$:

$$G \equiv \langle q^2 \rangle |_{J=0} = \frac{\int dq q^2 e^{-S(q)}}{\int dq e^{-S(q)}} = -2 \frac{\partial}{\partial m^2} \log Z(J=0).$$

At $J = 0$, the only external lines arise from the (two) explicit insertions of q , so this is given by the sum of all diagrams with two external lines. In perturbation theory this is:

$$G \simeq \text{---} + \text{---} \circ + \text{---} \circ \circ + \text{---} \circ \circ \circ + \text{---} \bigcirc + \mathcal{O}(g^3) \\ = m^{-2} \left(1 - \frac{1}{2} g m^{-4} + \frac{2}{3} g^2 m^{-8} + \mathcal{O}(g^3) \right) \quad (2.25)$$

The Feynman diagrams we're going to draw all the time this quarter are the same idea, but with more labels. Here is a too-quick preview: Notice that each of the qs in our integral could come with a label, $q \rightarrow q_a$. Then each line in our diagram would be associated with a matrix $\left((m^2)^{-1} \right)_{ab}$ which is the inverse of the quadratic term $q_a m_{ab}^2 q_b$ in the action. If our diagrams have loops we get free sums over the label. If that label is conserved by the interactions, the vertices will have some delta functions. In the case of translation-invariant field theories we can label lines by the conserved momentum k . Each comes with a factor of the free propagator $\frac{i}{k^2 + m^2 + i\epsilon}$, each vertex conserves momentum, so comes with $i g \delta^D(\sum k) (2\pi)^D$, and we must integrate over momenta on internal lines $\int d^D k$.

2.4 Brief comments about large orders of perturbation theory

- The perturbation series for (2.20) about $g = 0$ does *not* converge. How do I know? One way to see this is to notice that if I made g even infinitesimally negative, the integral itself would not converge (the potential would be unbounded below), and $Z_{g=-|\epsilon|}$ is not defined. Therefore Z_g as a function of g cannot be analytic in a neighborhood of $g = 0$. This argument is due to Dyson, and applies also in most QFTs. This means there is more to QFT than perturbation theory: the perturbation series does not define the field theory amplitudes.

The expansion of the exponential in the integrand is clearly convergent for each q . The place where we went wrong is exchanging the order of integration over q and summation over n .

- The integral actually does have a name – it’s a Bessel function:

$$Z(J=0) = \frac{2}{\sqrt{m^2}} \sqrt{\rho} e^\rho K_{\frac{1}{4}}(\rho), \quad \rho \equiv \frac{3m^4}{4g}$$

(for $\text{Re} \sqrt{\rho} > 0$), as `Mathematica` will tell you. Because we know about Bessel functions, in this case we can actually figure out what happens at strong coupling, when $g \gg m^4$, using the asymptotics of the Bessel function.

- In this case, the perturbation expansion too can be given a closed form expression:

$$Z(J=0) \simeq \sqrt{\frac{1}{m^2}} \sum_n \frac{(-1)^n 2^{2n+\frac{1}{2}}}{n! (4!)^n} \Gamma\left(2n + \frac{1}{2}\right) \left(\frac{g}{m^4}\right)^n. \quad (2.26)$$

- The expansion for G is of the form

$$G \simeq m^{-2} \sum_{n=0}^{\infty} c_n \left(\frac{g}{m^4}\right)^n.$$

When n is large, the coefficients satisfy $c_{n+1} \stackrel{n \gg 1}{\simeq} -\frac{2}{3} n c_n$ (you can see this by looking at the coefficients in (2.26)) so that $|c_n| \sim n!$. Up to order-one factors, c_n here is the number of diagrams with n vertices. This factorial growth of the number of diagrams is general in QFT and is another way to see that the series does not converge. It is called an *asymptotic series*.

- The fact that the coefficients c_n grow means that there is a *best* number of orders to keep. The errors start getting bigger when $c_{n+1} \left(\frac{g}{m^4}\right) \sim c_n$, that is, at order $n \sim \frac{3m^4}{2g}$. So if you want to evaluate G at a certain value of the coupling, you should stop at a certain order of the expansion in n .
- I said above that the fact that the perturbation series doesn’t converge means that it doesn’t define the field theory amplitudes. What does it miss? To answer this, consider trying to do the integral (2.20) by saddle point (at $J=0$ for simplicity):

$$0 = S'(q_\star) = m^2 q_\star + \frac{g}{3!} q_\star^3.$$

(Note the resemblance to the equations of motion.) This has *three* solutions:

$$q_\star = 0, \quad q_\star = \pm i \sqrt{\frac{3!m^2}{g}}.$$

The expansion about the ‘trivial’ saddle at $q_\star = 0$ (where the action is $S(q_\star = 0) = 0$) reproduces the perturbation series described above. At the other saddles,

$$S\left(q_\star = \pm i \sqrt{\frac{3!m^2}{g}}\right) = -\frac{3m^4}{2g}, \quad (2.27)$$

which means their contribution would go like $e^{+\frac{3m^4}{2g}}$, which actually would blow up at weak coupling, $g \rightarrow 0$. These saddles are not on the contour of integration and don't contribute for small positive g , but more generally (as for example when $m^2 < 0$), there will be effects that go like $e^{-\frac{a}{|g|}}$. This is a function whose series expansion in g at $g = 0$ is identically zero. You can never find it by doing perturbation theory in g about $g = 0$. The general name for such contributions is *instantons*.

- A technique called *Borel resummation* can sometimes produce a well-defined function of g from an asymptotic series whose coefficients diverge like $n!$. The idea is to make a new series

$$B(z) \equiv \sum_{m=0}^{\infty} \frac{c_m}{m!} z^m$$

whose coefficients are ensmallened by $n!$. Then to get back $Z(g)$ we use the identity

$$1 = \frac{1}{n!} \int_0^{\infty} dz e^{-z} z^n$$

and do the Laplace transform of $B(z)$:

$$\int_0^{\infty} dz B(z) e^{-z/g} = \sum_{m=0}^{\infty} c_m \frac{\int_0^{\infty} dz e^{-z/g} z^m}{m!} = g \sum_{m=0}^{\infty} c_m g^m = gZ(g).$$

This procedure requires both that the series in $B(z)$ converges and that the Laplace transform can be done. In fact this procedure works in this case.

The fact that the number of diagrams at large order grows like $n!$ is correlated with the existence of saddle-point contributions to $Z(g)$ that go like $e^{-a/g}$. This is because they are both associated with singularities of $B(z)$ at $z = a$; such a singularity means the sum of $\frac{c_n}{n!} z^n$ must diverge at $z = a$. (More generally, non-perturbative effects that go like $e^{-a/g^{1/p}}$ (larger if $p > 1$) are associated with (faster) growth like $(pn)!$. For more on this, see [this classic work](#) by Steve Shenker.)

- The function $G(g)$ can be analytically continued in g away from the real axis, and can in fact be defined on the whole complex g plane. It has a branch cut on the negative real axis, across which its discontinuity is related to its imaginary part. The imaginary part goes like $e^{-\frac{a}{|g|}}$ near the origin and can be computed by a tunneling calculation like (2.27).

How did we know Z has a branch cut? One way is from the asymptotics of the Bessel function. But, better, *why* does Z satisfy the Bessel differential equation

as a function of the couplings? The answer, as you'll check on the homework, is that the Bessel equation is a Schwinger-Dyson equation,

$$0 = \int_{-\infty}^{\infty} \frac{\partial}{\partial q} (\text{something } e^{-S(q)}) \quad (2.28)$$

which results from demanding that we can change integration variables in the path integral. This equation is true for any choice of 'anything,' as long as it doesn't grow too fast at large q .

The generalization of (2.28) in field theory is a way to derive True Facts that do not depend on perturbation theory. For example, for a scalar field:

$$0 = \int [D\phi] \frac{\delta}{\delta\phi(x)} (\text{anything } e^{iS[\phi]})$$

is true as long as 'anything' doesn't grow too fast at large fields. As you'll study on the homework, this is the sense in which the equations of motion are true quantum mechanically: the field $\hat{\phi}(x)$ satisfies its equations of motion *up to contact terms*, meaning except for when $\phi(x)$ collides with some other operator in the expectation value (like $\hat{\phi}(y)$ here).

For a bit more about this, you might look at sections 3 and 4 of [this recent paper](#) from which I got some of the details here. See also the giant book by Zinn-Justin. There is a deep connection between the large-order behavior of the perturbation series about the trivial saddle point and the contributions of non-trivial saddle points. The keywords for this connection are *resurgence* and *trans-series* and a starting reference is [here](#).

3 Lorentz invariance and causality

[Peskin §2.2, 2.3, 2.4] Now we take Lorentz invariance to be an exact symmetry and see what its consequences are for QFT.

Relativistic normalization of 1-particle states. The Fock space of a real scalar field is spanned by the states $|\vec{p}_1, \dots, \vec{p}_n\rangle \propto \mathbf{a}_{p_1}^\dagger \dots \mathbf{a}_{p_n}^\dagger |0\rangle$ where $\mathbf{a}_p |0\rangle = 0$. Now it is time to turn that \propto into an $=$. Fock space is a direct sum of sectors labelled by the number of particles: $\sum_k N_k = 0, 1, 2, \dots$ (Without interactions, the hamiltonian is block diagonal in this decomposition.) In the no-particle sector, it is clear what we should do: $\langle 0|0\rangle = 1$.

A one-particle state is $|p\rangle \equiv c_p \mathbf{a}_p^\dagger |0\rangle$. How best to choose c_p ?

(This discussion is shaded because it contains equations that will not be true in the normalization we'll use below. In this regard, beware the section of Peskin called "how not to quantize the Dirac equation".) Suppose we choose $c_p \stackrel{?}{=} 1$. Then

$$\langle \vec{k} | \vec{p} \rangle \stackrel{?}{=} \langle 0 | \mathbf{a}_k \mathbf{a}_p^\dagger | 0 \rangle \stackrel{\mathbf{a}_k | 0 \rangle = 0}{=} \langle 0 | [\mathbf{a}_k, \mathbf{a}_p^\dagger] | 0 \rangle = (2\pi)^d \delta^{(d)}(\vec{k} - \vec{p}) \equiv \delta(\vec{k} - \vec{p}).$$

Suppose the previous equation is true in my rest frame F . Since $1 = \int \bar{d}^d p \delta(p - k)$, we see that $\bar{d}^d p \delta(p - k)$ is Lorentz invariant.

To figure out more explicitly the Lorentz transformation properties of the delta function, we can use

$$\delta(f(x)) = \sum_{\text{zeros } x_0 \text{ of } f} \frac{\delta(x - x_0)}{|f'(x_0)|}.$$

If another F' is obtained by a boost in the x direction, $p'_\mu = \Lambda_{\mu\nu} p_\nu$,

$$\begin{pmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix} \quad \Rightarrow \quad \begin{aligned} \frac{dp'_x}{dp_x} &= \gamma \left(1 - \beta \frac{dE}{dp_x} \right) \\ &= \gamma \left(1 - \beta \frac{p_x}{E} \right) \\ &= \frac{\gamma}{E} (E - \beta p_x) = \frac{E'}{E} \end{aligned}$$

where we used $E^2 = \vec{p}^2 + m^2 = p_x^2 + p_\perp^2 + m^2$ and $2E \frac{dE}{dp_x} = 2p_x$ and $\frac{dE}{dp_x} = \frac{p_x}{E}$.

So

$$\delta^{(d)}(\vec{p} - \vec{k}) = \frac{\bar{d}^d p'}{\bar{d}^d p} \delta^{(d)}(\vec{p}' - \vec{k}') = \frac{dp'_x}{dp_x} \delta^{(d)}(\vec{p}' - \vec{k}') = \frac{E'}{E} \delta^{(d)}(\vec{p}' - \vec{k}'). \quad (3.1)$$

Which means that in F' we would have

$$\langle \vec{k}' | \vec{p}' \rangle \stackrel{?}{=} \frac{E'}{E} \delta^{(d)}(\vec{p}' - \vec{k}').$$

There is a special frame, it's no good.

There is an easy fix:

$$|\vec{p}\rangle \equiv \sqrt{2\omega_{\vec{p}}} a_{\vec{p}}^\dagger |0\rangle.$$

In that case the calculation in the shaded text is replaced by

$$\langle \vec{k} | \vec{p} \rangle = \sqrt{4\omega_k \omega_p} \delta^{(d)}(k - p) = 2\omega_p \delta^{(d)}(k - p)$$

while

$$\langle \vec{k}' | \vec{p}' \rangle = 2\omega_p \frac{\omega_{p'}}{\omega_p} \delta^{(d)}(k' - p').$$

So the expression is the same in any frame, yay.

Now you can ask why the factor of $\sqrt{2}$. We'd like to use these states to resolve the identity in the 1-particle sector, $\mathbb{1}_1 \equiv \sum_i |i\rangle\langle i|$. I claim that the following expression does this and makes Lorentz symmetry manifest:

$$\begin{aligned} \mathbb{1}_1 &\stackrel{?}{=} \int \underbrace{\mathrm{d}^{d+1}k}_{E>0} \theta(k^0) 2\pi \delta(k^2 - m^2) |\vec{k}\rangle \langle \vec{k}| \\ &= \int \mathrm{d}^d k \int dk^0 \theta(k^0) \left(\frac{\delta(k^0 - \sqrt{k^2 + m^2})}{2k^0} \right) |\vec{k}\rangle \langle \vec{k}| = \int \frac{\mathrm{d}^d k}{2\omega_{\vec{k}}} |\vec{k}\rangle \langle \vec{k}|. \end{aligned}$$

We used the general fact $\delta(f(x)) = \sum_{x_0|f(x_0)=0} \frac{1}{|f'(x_0)|} \delta(x - x_0)$. Because of the $\theta(k^0)$, only the positive-energy solution contributes.

So in retrospect, a quick way to check the normalization is to notice that the following combination is Lorentz invariant:

$$\frac{\mathrm{d}^d k}{2\omega_k} = \int dk_0 \theta(k_0) \mathrm{d}^d k \delta(k^2 - m^2) = \frac{\mathrm{d}^d k'}{2\omega_{k'}}.$$

Actually, this statement has a hidden assumption, that $m^2 \geq 0$. In that case, the 4-vector k^μ satisfying $k^2 = m^2 \geq 0$ is timelike (or lightlike), and no Lorentz transformation connected to the identity can change the sign of k_0 , it can only move it around within the interior of the forward lightcone (or the forward lightcone itself). So the $\theta(k_0)$ is Lorentz invariant.

Notice that our convenient choice of normalization doesn't show that our Hamiltonian description of scalar field theory is actually Lorentz invariant. For example, we have

$$[\phi(\vec{x}), \pi(\vec{y})]_{ETCR} = i\delta^{(d)}(\vec{x} - \vec{y})$$

at equal times, in one frame. What about other frames?

A second reason to study commutators is ...

3.1 Causality and antiparticles

Causality in general: This is the very reasonable condition on our description of physics that *events should not precede their causes*.

It will be worthwhile to think about how to implement this condition in a QFT. (The following discussion is based on appendix D of [this paper](#).) Suppose B wants to send a message to A . How does he do this? He applies a unitary operator³⁶, call it \mathbf{B} (with $\mathbf{B}^\dagger \mathbf{B} = 1$), localized near B , to the shared state of their quantum many body system $|\psi\rangle_{ABE}$. (Here E is for ‘environment’, the rest of the world besides A and B .) Then A waits for a time t and measures some observable \mathbf{A} . To send a different message, he should apply a different operator, say B' .

Under the circumstances just stated, the expectation for A ’s measurement of \mathbf{A} is

$$\langle \mathbf{A} \rangle_B = \langle \psi | \mathbf{B}^\dagger \underbrace{e^{i\mathbf{H}t} \mathbf{A} e^{-i\mathbf{H}t}}_{=\mathbf{A}(t)} \mathbf{B} | \psi \rangle = \langle \mathbf{A}(t) \rangle - \langle \mathbf{B}^\dagger [\mathbf{B}, \mathbf{A}(t)] \rangle.$$

Therefore, if $[\mathbf{B}, \mathbf{A}(t)] = 0$, the expectation doesn’t depend on what B did. In fact, replacing \mathbf{A} with \mathbf{A}^η for any η and using ($[\mathbf{B}, \mathbf{A}(t)] = 0 \implies [\mathbf{B}, \mathbf{A}(t)^\eta] = 0$) shows that all the moments of the distribution for A ’s measurement will also be independent of what B did, so no message gets through³⁷.

So nonvanishing commutators are essential for sending information. Notice that entangled states of well-separated local degrees of freedom (such as a Bell pair of distant spins $|\uparrow_0 \downarrow_r\rangle - |\downarrow_0 \uparrow_r\rangle \in \mathcal{H}_0 \otimes \mathcal{H}_r$) are insufficient for sending signals: whatever we do to the spin at 0 has no effect on the outcomes of measurements of the spin at r .

³⁶‘Applying an operator’ is more complicated than it seems. Actually it means ‘changing the Hamiltonian so that the time evolution operator is \mathbf{B} ’.

³⁷The loophole-seeking reader (ideally, this is everyone) will worry that a distribution is not in general determined by its moments. (For example, there are perfectly normalizable distributions with finite averages but where the higher moments are all infinite, such as $p(x) = \frac{\sqrt{2}a^3/\pi}{x^4+a^4}$ on the real line: $\langle x^2 \rangle = a^2$, but $\langle x^{2n} \rangle = \infty$ for $n > 1$.) What we would really like to show is that the conditional probability $p(a|B)$ is independent of B , in which case for sure A couldn’t learn anything about what B did. Here a runs over the set of eigenvalues of the operator \mathbf{A} , $\mathbf{A}|a\rangle = a|a\rangle$. That is

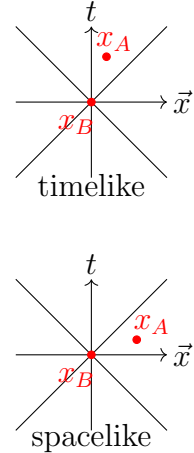
$$p(a|B) = \langle \psi | \mathbf{B}^\dagger \underbrace{e^{i\mathbf{H}t} |a\rangle \langle a| e^{-i\mathbf{H}t}}_{=\mathbf{P}_a(t)} \mathbf{B} | \psi \rangle = \langle \mathbf{P}_a(t) \rangle - \langle \mathbf{B}^\dagger [\mathbf{B}, \mathbf{P}_a(t)] \rangle.$$

Does $[\mathbf{A}, \mathbf{B}] = 0$ imply that the projector onto a particular eigenvalue of a commutes with \mathbf{B} ? In a finite dimensional Hilbert space, it does for sure, since $0 = [\mathbf{A}^\eta, \mathbf{B}] = \sum_a a^\eta [\mathbf{P}_a, \mathbf{B}]$ is true for all η , which gives infinitely many equations for $[\mathbf{P}_a, \mathbf{B}]$. In the case of infinite dimensional \mathcal{H} I think there is some room for functional analysis horror. On the other hand, any measurement has finite resolution. Thanks to Sami Ortoleva and Chuncheong Lam for help with this point.

Here's a proof: the spin operators at 0, \vec{S}_0 and the spin operators at r , \vec{S}_r commute: $[S_0^i, S_r^j] = 0$, since \vec{S}_0 acts the identity on \mathcal{H}_r and vice versa. The groundstate of a quantum field is somewhat similar: there is quite a bit of entanglement between nearby points in space, but the equal-time commutators $[\phi(\vec{x}), \phi(\vec{y})] = 0$ vanish, so it can't be used to send instantaneous signals.

[End of Lecture 9]

Causality in relativistic QFT. [Peskin §2.4] In a Lorentz invariant system, 'precede' is sometimes a frame-dependent notion. If A is in the future lightcone of B , *i.e.* $0 < (x_A - x_B)^2 = (t_A - t_B)^2 - (\vec{x}_A - \vec{x}_B)^2$ and $t_A > t_B$, then everyone agrees that A is after B . This is the easy case. But if A and B are spacelike separated, $0 > (x_A - x_B)^2 = (t_A - t_B)^2 - (\vec{x}_A - \vec{x}_B)^2$, then there is a frame in which they occur at the same time, and frames where they occur in either order. This is the dangerous case.



So: causality will follow if, for all local operators \mathbf{A}, \mathbf{B} , $[\mathbf{A}(x_A), \mathbf{B}(x_B)] = 0$ whenever x_A and x_B are spacelike separated, $0 > (x_A - x_B)^2$. Recall that spacelike separated means that there is a Lorentz frame where A and B are at the same time.

A general operator in a scalar QFT can be made from ϕ_s and $\partial_\mu \phi_s$, so the general statement of causality (in free scalar QFT) will follow from considering commutators of

$$\phi(x) = \int \frac{d^d p}{\sqrt{2\omega_{\vec{p}}}} \left(\mathbf{a}_{\vec{p}} e^{-ip_\mu x^\mu} + \mathbf{a}_{\vec{p}}^\dagger e^{+ip_\mu x^\mu} \right) |_{p^0 = \omega_{\vec{p}}} \equiv \phi^{(+)} + \phi^{(-)}$$

at different points in spacetime. Here we have decomposed the field into positive- and negative-frequency parts. Notice that since $\phi^{(+)}$ ($\phi^{(-)}$) only involves annihilation (creation) operators, $[\phi^{(\pm)}(x), \phi^{(\pm)}(y)] = 0$ for any x, y . Using the ladder algebra $[a, a^\dagger] = 1$,

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^d p}{2\omega_{\vec{p}}} \left(e^{-ip_\mu(x-y)^\mu} - e^{+ip_\mu(x-y)^\mu} \right) \\ &= \underbrace{\int d^{d+1} p \, 2\pi \delta(p^2 - m^2) \theta(p^0)}_{\text{Lorentz inv't}} \left(e^{-ip_\mu(x-y)^\mu} - e^{+ip_\mu(x-y)^\mu} \right) \end{aligned} \quad (3.2)$$

Here comes the slippery stuff: Suppose $x - y$ is spacelike. Let's choose a frame

where they (the points labelled by x, y) are at the same time, and let Λ be the Lorentz matrix that gets us there: $\Lambda_\nu^\mu(x-y)^\nu = (0, \vec{\Delta}x)^\mu \equiv \tilde{x}^\mu$. Then we can change the integration variable to $\tilde{p}^\mu = \Lambda_\nu^\mu p^\nu$, so that

$$p_\mu(x-y)^\mu = p^\mu \eta_{\mu\nu} (x-y)^\mu = \tilde{p}^\sigma \underbrace{\Lambda_\sigma^\mu \eta_{\mu\nu} \Lambda_\rho^\nu}_{=\eta_{\sigma\rho}} \tilde{x}^\rho = \tilde{p}_\mu \tilde{x}^\mu.$$

Then

$$[\phi(x), \phi(y)] \stackrel{(x-y)^2 < 0}{=} \int \mathrm{d}^{d+1}\tilde{p} \, 2\pi\delta(\tilde{p}^2 - m^2)\theta(\tilde{p}^0) \underbrace{\left(e^{-i\tilde{p}\cdot\vec{\Delta}x} - e^{+i\tilde{p}\cdot\vec{\Delta}x} \right)}_{\text{odd under } \tilde{\vec{p}} \rightarrow -\tilde{\vec{p}}} = 0. \quad (3.3)$$

We conclude that $[\phi(x), \phi(y)] = 0$ if $(x-y)^2 < 0$, *i.e.* for spacelike separation. By now you've computed enough of these commutators that you know this oddness is why the equal-time commutator of ϕ s vanishes; Lorentz invariance shows that this implies its vanishing for all spacelike separation.

The same argument works for $[\phi, \pi]$ and $[\pi, \pi]$. For $[\phi, \pi]$, the strict inequality $(x-y)^2 < 0$ is important – they do not commute when light-like separated (this is the Lorentzian version of coincident, $(x-y)^2 = 0$).

So: vanishing equal-time commutators (for nonzero separation) plus Lorentz symmetry implies causality.

Notice that the argument fails if $(x-y)^2 > 0$, since then we can't get rid of the time component of the exponents by a Lorentz transformation, and they don't cancel. It *is* possible to send signals inside the light cone.

Now let's think more about the bit that is nonzero:

$$[\phi(x), \phi(y)] = \underbrace{[\phi^{(+)}(x), \phi^{(-)}(y)]}_{\equiv \hat{\Delta}^+(x-y)} + \underbrace{[\phi^{(-)}(x), \phi^{(+)}(y)]}_{\equiv \hat{\Delta}^-(x-y)}.$$

Because $[\mathbf{a}, \mathbf{a}^\dagger] \propto \mathbb{1}$, $\hat{\Delta}^\pm$ is a c-number, independent of what state it acts on. So, for any normalized state $|\psi\rangle$,

$$\begin{aligned} \Delta^+(x-y) &= \langle \psi | \hat{\Delta}^+(x-y) | \psi \rangle = \langle 0 | \hat{\Delta}^+(x-y) | 0 \rangle \\ &= \langle 0 | [\phi^{(+)}(x), \phi^{(-)}(y)] | 0 \rangle \\ &= \langle 0 | \phi^{(+)}(x)\phi^{(-)}(y) | 0 \rangle - \langle 0 | \phi^{(-)}(y) \underbrace{\phi^{(+)}(x) | 0 \rangle}_{=0: \phi^+ \ni \mathbf{a}, \mathbf{a} | 0} \rangle \\ &= \langle 0 | \phi(x)\phi(y) | 0 \rangle \end{aligned}$$

where in the last step we again used $\phi^+ | 0 \rangle = 0$ (and $\langle 0 | \phi^- = 0$).³⁸ This is a

³⁸Note from the definition that $\Delta^\pm(x-y) = [\phi^{(\pm)}(x), \phi^{(\mp)}(y)] = -\Delta^\mp(y-x)$.

vacuum-to-vacuum amplitude, or *propagator*³⁹, in the sense that

$$\phi(y) |0\rangle = |\text{'particle created at } y'\rangle$$

$$\langle 0| \phi(x) = \langle \text{'particle destroyed at } x'|$$

That is, Δ^+ is the amplitude to propagate the excitation created by the field from y to x .

$$\Delta^+(x-y) = \sum_p (x \cdot \text{---} \leftarrow \text{---} \cdot y) = \int \frac{\mathrm{d}^d p}{2\omega_{\vec{p}}} e^{-ip \cdot (x-y)} \Big|_{p^0 = \omega_{\vec{p}}}. \quad (3.4)$$

This integral can be done in terms of functions with names⁴⁰, but the most useful information is about its asymptotics in the very timelike ($t \equiv |x^0 - y^0| \gg |\vec{x} - \vec{y}|$) and very spacelike ($|x^0 - y^0| \ll |\vec{x} - \vec{y}| \equiv r$) limits.

$$\Delta^+(x-y) \simeq \begin{cases} e^{-imt}, & |x^0 - y^0| \gg |\vec{x} - \vec{y}|, (x-y)^2 \equiv t^2 \\ e^{-mr}, & |x^0 - y^0| \ll |\vec{x} - \vec{y}|, (x-y)^2 \equiv -r^2 \end{cases}$$

You can read more about how to arrive at these expressions in Peskin (page 27); the spacelike case is related by a Lorentz boost (to the rest frame) to the calculation of the Yukawa potential that we did in the last section.

Notice that this quantity $\langle \phi(x)\phi(y) \rangle$ is *not* zero outside the lightcone. There are nonzero spatial correlations (and indeed entanglement) in the scalar field vacuum. What gives? Nothing. Causality only requires the vanishing of *commutators* outside the lightcone, which we already showed in (3.3).

The cancellation in (3.3) can be interpreted as destructive interference between particles and antiparticles. It's clearer for the complex scalar field, where $\Phi \equiv \Phi^{(+)} + \Phi^{(-)}$ with

$$\Phi^{(+)} = \int \frac{\mathrm{d}^d p}{\sqrt{2\omega_p}} e^{-ipx} \mathbf{a}_p, \quad \Phi^{(-)} = \int \frac{\mathrm{d}^d p}{\sqrt{2\omega_p}} e^{+ipx} \mathbf{b}_p^\dagger$$

(with the expressions for the + and - frequency components for Φ^* following by taking hermitian conjugates).

³⁹We'll understand its connection to the time-ordered propagator just below.

⁴⁰Specifically, in four spacetime dimensions and spacelike separation, $(x-y)^2 \equiv -r^2$, $\Delta^+(x-y) = \frac{m}{4\pi^2 r} K_1(mr)$.

So consider the analogous⁴¹

$$D(x-y) \equiv \langle 0 | \Phi(x) \Phi^*(y) | 0 \rangle = \langle 0 | [\Phi^+(x), \Phi^{*-}(y)] | 0 \rangle = \underbrace{\Delta_{\mathbf{a}}^+(x-y)}_{\text{from } [\mathbf{a}, \mathbf{a}^\dagger]}$$

$$D^*(y-x) \equiv \langle 0 | \Phi^*(y) \Phi(x) | 0 \rangle = \langle 0 | [\Phi^{*+}(y), \Phi^-(x)] | 0 \rangle = -\underbrace{\Delta_{\mathbf{b}}^-(y-x)}_{\text{from } [\mathbf{b}, \mathbf{b}^\dagger]}$$

So if we consider the commutator,

$$\langle 0 | [\Phi(x), \Phi^*(y)] | 0 \rangle = D(x-y) - D^*(y-x)$$

$$= \sum_p \left(\underbrace{(x \cdot \text{---} \leftarrow p \text{---} \cdot y)}_{\text{particle}} - \underbrace{(x \cdot \text{---} -p \rightarrow \text{---} \cdot y)}_{\text{antiparticle!}} \right)$$

then in the spacelike case, the antiparticle bit from the first term of the commutator cancels the particle bit of the second, as above in (3.3). Antimatter makes QFT causal.

3.2 Propagators, Green's functions, contour integrals

[This subsection is mostly in a different color because I am mostly skipping it in lecture: we already learned all these things from the point of view of the path integral in the last section. I'm leaving it in because it has nice pictures, and because it has a possibly-useful summary of the discussion of choosing the contour for the frequency integrals in the propagator.]

Any Green's function for the Klein-Gordon operator (propagator for a real free scalar field) can be represented as :

$$\Delta_C(x) = \underbrace{\int_C \mathrm{d}^{d+1}p}_{\equiv \int_C \mathrm{d}p_0 \int \mathrm{d}^d p} e^{-i p_\mu x^\mu} \frac{\mathbf{i}}{p^2 - m^2}$$

for some choice of contour C . To see that this is related to the object we discussed above, first note that the denominator is

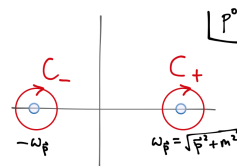
$$p^2 - m^2 = (p_0 - \omega_{\vec{p}}) (p_0 + \omega_{\vec{p}}) , \quad \omega_{\vec{p}} \equiv \sqrt{\vec{p} \cdot \vec{p} + m^2},$$

⁴¹Notice the annoying fact that the positive frequency component of Φ^* , $\Phi^{*(+)}$, is actually the dagger of the *negative* frequency component of Φ :

$$(\Phi^*)^{(+)} = \left(\Phi^{(-)} \right)^* .$$

so there are two poles, which seem to be on the real axis; this means that our integral is ambiguous and we need more information, indeed some physical input.

We can specify the contour C by taking linear combinations of C_{\pm} which are small circles going *clockwise* around the poles at $\pm\omega_{\vec{p}}$.



These basic integrals are (for a reminder about Cauchy's theorem see footnote 31):

$$\int_{C_+} d^{d+1}p e^{-ip_{\mu}x^{\mu}} \frac{\mathbf{i}}{p^2 - m^2} = \int d^d p \frac{1}{2\omega_{\vec{p}}} e^{-ipx} \Big|_{p_0=\omega_{\vec{p}}} = \Delta^+(x). \quad (3.5)$$

$$\int_{C_-} d^{d+1}p e^{-ip_{\mu}x^{\mu}} \frac{\mathbf{i}}{p^2 - m^2} = \int d^d p \frac{1}{-2\omega_{\vec{p}}} e^{-ipx} \Big|_{p_0=-\omega_{\vec{p}}} = -\Delta^+(-x) \stackrel{\text{let } \vec{q} \equiv -\vec{p}}{=} \Delta^-(x).$$

These are the same Δ^{\pm} we encountered in (3.4) as commutators of ϕ^{\pm} .

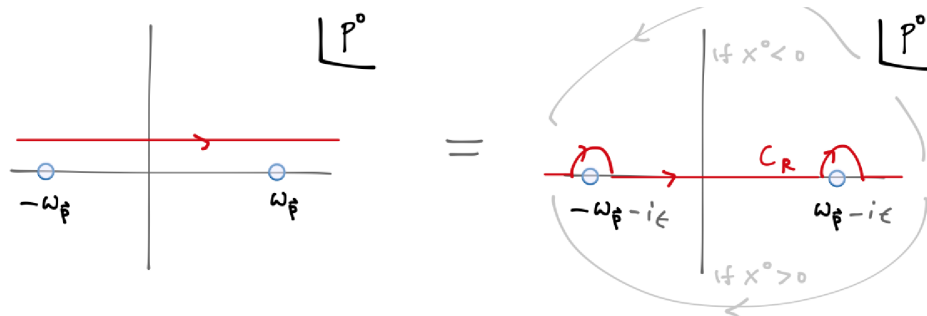
These are two particular choices of contour, and others are also interesting. Consider the *retarded* propagator,

$$\Delta_R(x - y) \equiv \theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle.$$

This is the Green's function that governs linear response:

$$\Delta_R(x - y) = \frac{\delta \langle \phi(x) \rangle_J}{\delta J(y)}.$$

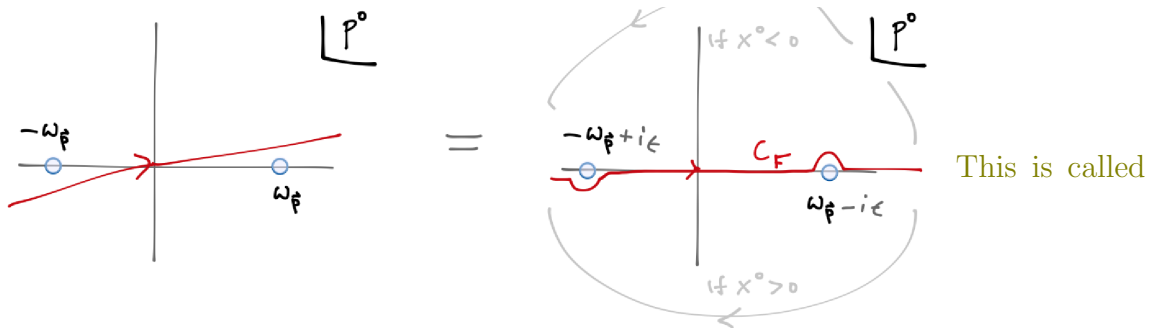
(I explain how to see this in §4.9; for now, notice that the $\theta(x^0 - y^0)$ guarantees that the response happens after the perturbation that causes it.) It is also a Green's function for the KG operator. We can reproduce Δ_R by routing our contour to go above the poles in the complex p^0 plane: if $x^0 - y^0 > 0$, then the factor $e^{-ip^0(x^0 - y^0)}$ decays when $\text{Im} p^0 < 0$, so we can close the contour for free in the LHP, and we pick up both poles; by contrast, if $x^0 - y^0 < 0$, we must close in the UHP and we pick up no poles and get zero. Notice that we could get the same result by replacing $p^0 \rightarrow p^0 + i\epsilon$ in the denominator, where ϵ is an *infinitesimal* (this means that $\epsilon^2 = 0$ and $\epsilon c = \epsilon$ for any positive quantity c).



So $D_R(x, y) = \theta(x^0 - y^0) (\Delta^+(x, y) + \Delta^-(x, y)) = \theta(x^0 - y^0) (\Delta^+(x, y) - \Delta^+(y, x))$. If we route the contour *below* the poles, we get the *advanced* propagator, where the response only occurs before the source. Surprisingly, the advanced propagator is sometimes useful in physics.

Another interesting way to navigate the poles is by replacing $p^2 - m^2$ with $p^2 - m^2 + i\epsilon$. This shifts the poles to

$$p^0 = \pm \omega_p \sqrt{1 - i\epsilon/\omega_p} = \pm \omega_p (1 - i\epsilon).$$



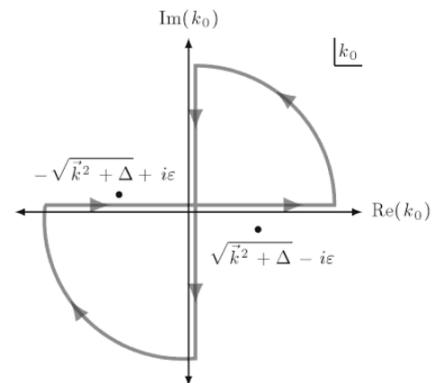
the *Feynman contour*, C_F , and we saw in the last section that it arises by continuation from Euclidean spacetime. Consider again the *euclidean* propagator, where we get rid of the relative sign in the metric :

$$I_E(\vec{x}) \equiv \int d^D p \frac{-i}{\sum_{i=1}^D p_i^2 + m^2} e^{-i \sum_{i=1}^D p_i x_i}.$$

Its poles are at $p^D = \pm i \sqrt{\vec{p} \cdot \vec{p} + m^2}$, far from the integration contour, so there is no problem defining it. Now consider smoothly rotating the contour by varying θ from 0 to $\pi/2 - \epsilon$ in $p^0 \equiv e^{i\theta} p^D$. The Feynman contour is the analytic continuation of the euclidean contour, and the ϵ is the memory of this.

To be more precise about this ‘smooth rotation’ of the contour, we can use Cauchy’s theorem again, with the contour at right (the figure is from Matthew Schwarz’s book). The semicircular arcs are at $|k^0| = \Lambda$, and give a contribution that decays like an inverse power of Λ , and we must take $\Lambda \rightarrow \infty$. The integrand is analytic inside the contour, no poles, so the integral $\int \frac{e^{-ikx}}{k^2 - m^2} = 0$.

Therefore the real-frequency integral (the real axis) and euclidean integral (minus the imaginary axis contribution, with $x^D = it$) are equal.



⁴²In position space, the Feynman propagator is

$$\Delta_F(x) \equiv \int_{C_F} d^{d+1}p e^{-ip_\mu x^\mu} \frac{\mathbf{i}}{p^2 - m^2} = \theta(x^0)\Delta^+(x) + \theta(-x^0)\Delta^+(-x) = \theta(x^0)\Delta^+(x) - \theta(-x^0)\Delta^-(x).$$

If $x^0 > 0$ (< 0), we must close the contour in the LHP (UHP) and get Δ^+ (Δ^-). Recalling that $\Delta^+(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$,

$$\Delta_F(x-y) = \langle 0|(\theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x))|0\rangle \equiv \langle 0|\mathcal{T}(\phi(x)\phi(y))|0\rangle.$$

The \mathcal{T} is the *time-ordering symbol*: the operators at the earliest times go on the right, so we can regard time as increasing to the left.

The propagator is a Green's function. So we've learned that

$$\frac{\mathbf{i}}{p^2 - m^2} \equiv \tilde{\Delta}(p)$$

is the Fourier transform of $\Delta(x)$, the momentum-space propagator (either retarded or Feynman). From this we can see that $\Delta(x)$ is a Green's function for the differential operator $\partial_\mu\partial^\mu + m^2$ in the sense that

$$(\partial_\mu\partial^\mu + m^2)\Delta(x) = -\mathbf{i}\delta(x)$$

(by plugging in the Fourier expansion of Δ and of the delta function, $\delta^{(d+1)}(x) = \int d^{d+1}p e^{-ipx}$, and differentiating under the integral). Notice that this did not depend on the choice of contour, so this equation in fact has several solutions, differing by the routes around the poles (hence by Δ^\pm , which are solutions to the homogeneous equation, without the delta function). On the homework, you will show this directly from the position space definitions.

⁴²You might worry about the fact that the exponential in the integrand grows or decays with Λ depending on the sign of t . It's not the exponential that kills the integral, but it also doesn't cause a problem:

The contribution from the first quadrant is of the form (for large Λ):

$$\frac{1}{\Lambda} \int_0^{\pi/2} d\theta e^{-\Lambda t \sin \theta} = \frac{\pi}{2\Lambda} (I_0(\Lambda t) - L_0(\Lambda t)) \stackrel{\Lambda \gg \text{everyone}}{\sim} \frac{\pi}{2\Lambda^2 t}.$$

(Here I just used Mathematica, where $L_0(x)$ is called `StruveL[0, x]` in Mathematica and $I_0(x)$ is `BesselI[0,x]`.) The contribution from the third quadrant is

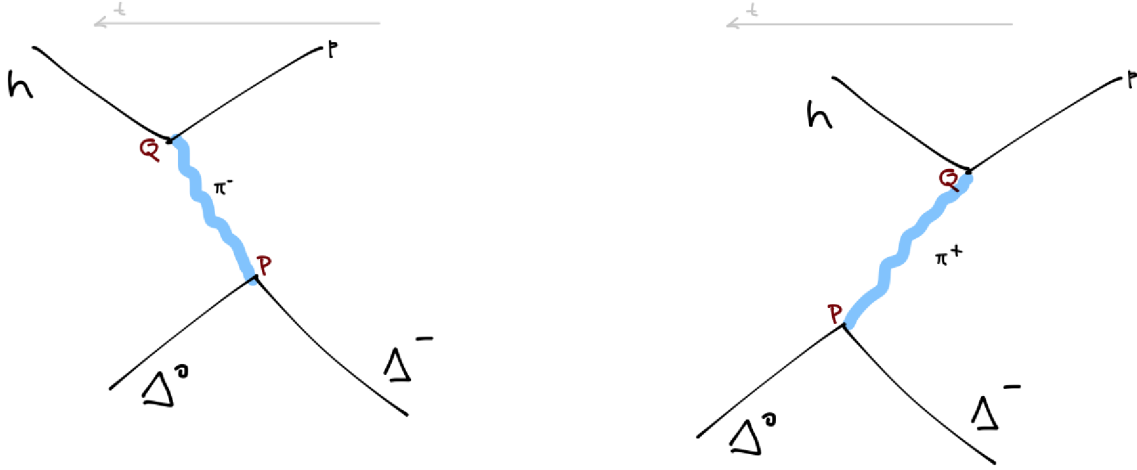
$$\frac{1}{\Lambda} \int_\pi^{3\pi/2} d\theta e^{-\Lambda t \sin \theta} = \frac{\pi}{2\Lambda} (I_0(\Lambda t) + L_0(\Lambda t)) \stackrel{\Lambda \gg \text{everyone}}{\sim} \frac{\pi}{2\Lambda^2 t}.$$

Thanks to Dominic Holland for alerting me to this potential problem.

Physics preview. Here is a preview of the physics of the Feynman propagator. Imagine we introduce some interactions, such as a cubic term in the Lagrangian, *e.g.*

$$\mathcal{L} \ni \phi_p(x)\phi_n(x)\phi_\pi(x) + h.c. \quad (3.6)$$

where the fields appearing here destroy or create particles with the names in the subscripts. Here are two stories we might tell about a collection of such particles.



In both pictures, time goes to the left. In the first picture, a Δ^- emits a π^- , becoming a Δ^0 at spacetime point P . This π^- propagates to Q where it is absorbed by a p , which turns into an n . In the second picture, a p emits a π^+ at Q , and becomes n ; that π^+ is absorbed by a Δ^- which becomes a Δ^0 .

The Feynman propagator includes both of these stories automatically.

Antiparticles going backwards in time. This story is clearer if we discuss the complex scalar, where particles (created by \mathbf{a}^\dagger) and antiparticles (created by \mathbf{b}^\dagger) are distinct:

$$\Phi(x) = \underbrace{\int \frac{d^d p}{\sqrt{2\omega_p}} \mathbf{a}_p e^{-ipx}}_{\equiv \Phi^+(x)} + \underbrace{\int \frac{d^d p}{\sqrt{2\omega_p}} \mathbf{b}_p^\dagger e^{+ipx}}_{\equiv \Phi^-(x)}.$$

The commutator is

$$\begin{aligned} [\Phi(x), \Phi^*(y)] &= \underbrace{[\Phi^+(x), \Phi^{*-}(y)]}_{\text{from } [\mathbf{a}, \mathbf{a}^\dagger], \text{ particles}} + \underbrace{[\Phi^-(x), \Phi^{*+}(y)]}_{\text{from } [\mathbf{b}, \mathbf{b}^\dagger], \text{ antiparticles}} \\ &= \Delta_{\mathbf{a}}^+(x-y) + \underbrace{\Delta_{\mathbf{b}}^-(x-y)}_{=-\Delta_{\mathbf{b}}^+(y-x)} \\ &= \int_{C_+} d^{d+1} e^{-ip(x-y)} \frac{\mathbf{i}}{p^2 - m^2} + \int_{C_-} d^{d+1} e^{-ip(x-y)} \frac{\mathbf{i}}{p^2 - m^2}. \end{aligned} \quad (3.7)$$

The propagator that we'll really need to compute S -matrix elements is

$$\begin{aligned}
\Delta_F(x-y) &\equiv \langle 0 | \mathcal{T}(\Phi(x)\Phi^*(y)) | 0 \rangle \\
&= \theta(x^0 - y^0) \underbrace{\langle 0 | \Phi(x)\Phi^*(y) | 0 \rangle}_{=\langle 0 | \Phi^+(x)\Phi^{*-}(y) | 0 \rangle = \langle 0 | [\Phi^+(x), \Phi^{*-}(y)] | 0 \rangle = \Delta_{\mathbf{a}}^+(x-y), \text{ particles}} \\
&\quad + \theta(y^0 - x^0) \underbrace{\langle 0 | \Phi^*(y)\Phi(x) | 0 \rangle}_{=-\langle 0 | [\Phi^{*+}(y), \Phi^-(x)] | 0 \rangle = -\Delta_{\mathbf{b}}^+(x-y), \text{ antiparticles}}
\end{aligned}$$

Signals propagate inside the lightcone. Returning to our discussion of causality, we showed that in general $[B, A(t)] \neq 0$ is required for B to successfully send a signal to A at time t . In a relativistic theory we can now say a little more. If $A(0)$ is a local operator at the point $\vec{x} = 0$, then the *support* of $A(t)$ is only in the region $|\vec{x}| \leq ct$, the forward lightcone of the event $(\vec{x}, t) = (0, 0)$. By the *support* of the operator, I mean the set of points in space where it acts non-trivially, *i.e.* as something other than the identity operator. We know this because we showed that in a relativistic QFT, all local operators commute for any spacelike separation. An operator that commutes with all other operators is proportional to the identity.

3.3 Interlude: where is 1-particle quantum mechanics?

Earlier, we started with non-relativistic quantum mechanics (NRQM) of a bunch of particles and arrived at a QFT describing a different set of particles (phonons). But I also said that *everything* is quanta of a QFT, including NRQM. Let's take a break from the development of QFT to see how that happens.

[Tong, §2.8] Consider a relativistic complex free massive scalar field Φ , with mass m . The minimum energy (above the vacuum) of a single-particle state is $\omega_{p=0} = m$; in its rest frame, the wavefunction is e^{-imt} . Consider the change of variables:

$$\Phi(\vec{x}, t) =: e^{-imt} \Psi(\vec{x}, t) \frac{1}{\sqrt{2m}}. \quad (3.8)$$

The Klein-Gordon equation is

$$0 = \partial_t^2 \Phi - \nabla^2 \Phi + m^2 \Phi = \frac{1}{\sqrt{2m}} e^{-imt} \left(\ddot{\Psi} - 2im\dot{\Psi} - \nabla^2 \Psi \right),$$

where the terms with m^2 cancel; so far we've just changed variables. The non-relativistic limit is $|\vec{p}| \ll m$ which says $\omega_p = \sqrt{\vec{p}^2 + m^2} \simeq m + \frac{p^2}{2m} + \dots$. The point of factoring out the phase e^{-imt} is that the time evolution of Ψ looks like $e^{-i(E-m)t} \sim e^{-i\frac{p^2}{2m}t}$; so in the NR limit, $|\ddot{\Psi}| \ll m|\dot{\Psi}|$ meaning that we can ignore that term in the KG equation. The remaining equation of motion is

$$i\partial_t \Psi = -\frac{1}{2m} \nabla^2 \Psi + \dots. \quad (3.9)$$

The \dots are higher derivative terms coming from the expansion of the dispersion relation, which I will now drop. This looks like the Schrödinger equation for a particle in no potential, in position space, but that is a coincidence: Ψ is not a wavefunction. This equation (3.9) is the eom associated with the lagrange density

$$\mathcal{L} = i\Psi^* \dot{\Psi} - \frac{1}{2m} \vec{\nabla} \Psi^* \cdot \vec{\nabla} \Psi$$

(which you can also get by plugging (3.8) into $\partial_\mu \Phi^* \partial^\mu \Phi$), from which $\pi_\Psi = i\Psi^*$, $\pi_{\Psi^*} = 0$. The ETCRs are then

$$[\Psi(\vec{x}), \Psi^*(\vec{y})] = \delta^d(\vec{x} - \vec{y}), \quad [\Psi, \Psi] = 0 = [\Psi^*, \Psi^*]$$

and the Hamiltonian is

$$H = \int d^d x \frac{1}{2m} \vec{\nabla} \Psi^* \cdot \vec{\nabla} \Psi.$$

The solution in terms of creation operators is then (a first-order linear ODE has a single solution)

$$\Psi(x) = \int d^d p e^{i\vec{p}\cdot\vec{x}} \mathbf{a}_p, \quad \Psi^*(x) = \int d^d p e^{-i\vec{p}\cdot\vec{x}} \mathbf{a}_p^\dagger$$

with $[\mathbf{a}_p, \mathbf{a}_q^\dagger] = (2\pi)^d \delta^d(p - q)$ as before. The hamiltonian is then

$$H = \int d^d p \frac{\vec{p}^2}{2m} \mathbf{a}_p^\dagger \mathbf{a}_p \quad (3.10)$$

(with no normal-ordering constant – the vacuum of this non-relativistic theory is extremely boring and has no dance of birth and death of the fluctuating particle-antiparticle pairs).

The crucial point here is that the antiparticles are gone, despite the fact that the field is complex. They disappeared when we dropped the second-order time derivative (recall that a first-order equation has half as many integration constants as a second-order equation). In the limit we've taken, we don't have enough energy to make any. (More precisely, we're promising to only study states where there is not enough energy to make any antiparticles.) The states are

$$\mathbf{a}_{p_1}^\dagger \cdots \mathbf{a}_{p_n}^\dagger |0\rangle \equiv |\{p\}\rangle, \quad \mathbf{a}_p |0\rangle = 0$$

and these are energy eigenstates with energy $H |\{p\}\rangle = \sum_i \frac{p_i^2}{2m} |\{p\}\rangle$, a nice non-relativistic (NR) dispersion for each particle. The particle-number symmetry is still present $\Psi \rightarrow e^{-i\alpha} \Psi$, but now the Noether current is

$$j^\mu = \left(\Psi^* \Psi, \frac{\mathbf{i}}{2m} \Psi^* \vec{\nabla} \Psi + h.c. \right)^\mu = \left(\rho, \vec{j} \right)^\mu.$$

Now we can find the QM of a single particle which cannot go away (since we got rid of the antiparticles), with some position and momentum *operators*. In fact the momentum operator is just the charge associated with translation invariance, and takes the form (just like on the homework)

$$\vec{\mathbf{P}} = \int d^d p p \vec{a}_p^\dagger \mathbf{a}_p$$

and $\vec{\mathbf{P}} |\{p\}\rangle = \sum_a p_a \vec{a}_a |\{p\}\rangle$. What's the position operator? A state with a particle at position \vec{x} is

$$|\vec{x}\rangle = \Psi^*(x) |0\rangle = \int d^d p e^{-i\vec{p}\cdot\vec{x}} \mathbf{a}_p^\dagger |0\rangle.$$

If we let

$$\vec{\mathbf{X}} \equiv \int d^d x \Psi^*(x) \vec{x} \Psi(x)$$

then indeed $\vec{\mathbf{X}}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$. To see that the Heisenberg algebra $[\mathbf{X}, \mathbf{P}] = \mathbf{i}$ works out, consider the general 1-particle state

$$|\psi\rangle = \int d^d x \psi(x) |x\rangle.$$

The function $\psi(x)$ here is the usual position-basis Schrödinger wavefunction. You can check on the homework that

$$\mathbf{X}^i |\psi\rangle = \int d^d x x^i \psi(x) |x\rangle, \quad \mathbf{P}^i |\psi\rangle = \int d^d x \left(-\mathbf{i} \frac{\partial}{\partial x^i} \psi(x) \right) |x\rangle$$

which implies the Heisenberg commutator. Finally, the hamiltonian (3.10) gives the time evolution equation

$$\mathbf{i} \partial_t \psi = -\frac{\nabla^2}{2m} \psi$$

which really is the Schrödinger equation.

Many particles that one studies in NR QM are actually fermions (e, p, n, \dots) and therefore not described by a scalar field. But in the 1-particle sector, who can tell? No one. Later we'll see the NR limit of the Dirac equation, which is basically the same, but with some extra juicy information about spin.

Next we will speak about 'interactions'. This term is used in two ways. In NR QM, it is sometimes used to describe an external potential $V(x)$ appearing as an extra term in the Schrödinger equation

$$\mathbf{i} \partial_t \psi = -\frac{\nabla^2}{2m} \psi + V(x) \psi(x).$$

Such a term explicitly violates translation symmetry. It can be accomplished by adding to the action the *quadratic* term

$$\Delta S_V = - \int d^d x \Psi^*(x) \Psi(x) V(x) = - \int d^d x \rho(x) V(x).$$

This says that nonzero density of particles at x costs energy $V(x)$. A second sense of 'interaction' which is how it will be used forever below is interaction *between* particles. With only one particle this cannot happen. NR QM theory does accommodate more than one particle, and we can consider an interaction between them like

$$\Delta S = - \int d^d x \int d^d y \Psi^*(x) \Psi(x) V(x-y) \Psi^*(y) \Psi(y).$$

If $V(x-y) = \delta^d(x-y)$, this interaction is local.

[End of Lecture 10]

4 Interactions, not too strong

4.1 Where does the time dependence go?

[Peskin chapter 4.2] Now we must get straight where to put the time dependence. Different ways of doing the book-keeping are called different ‘pictures’. At some reference time, say $t = 0$, they all agree: label states by $|\psi, 0\rangle$ and operators by $\mathcal{O}(0)$. At a later time, in picture P , these evolve to $|\psi, t\rangle_P, \mathcal{O}_P(t)$. Physics, such as any amplitude like

$${}_P \langle \phi, t | \mathcal{O}_P(t) | \psi, t \rangle_P \quad (4.1)$$

is independent of P . Let’s assume the hamiltonian H has no explicit time dependence.

In Heisenberg picture ($P = H$), $|\psi, t\rangle_H \equiv |\psi, 0\rangle$ for all time, and the burden of the time dependence is all on the operators

$$\mathcal{O}_H(t) = U_H^\dagger(t) \mathcal{O}(0) U_H(t). \quad (4.2)$$

The Heisenberg equations of motion are

$$i\dot{\mathcal{O}}_H = [\mathcal{O}_H, H_H] \quad (4.3)$$

so in particular $\dot{H}_H = 0$ so $H_H(t) = H_H(0) = H$. Then (4.3) is solved by (4.2) with $U_H(t) = e^{-itH}$. An example of an operator is the free field:

$$\Phi_H(\vec{x}, t) = \int \frac{d^d p}{\sqrt{2\omega_p}} (\mathbf{a}_p e^{-ip_\mu x^\mu} + \mathbf{b}_p^\dagger e^{ip_\mu x^\mu}) |_{p_0=\omega_p}$$

which you time-evolved this way on the homework. In fact, this equation is basically the whole story of free field theory. The field makes particles that don’t care about each other.

In Schrödinger picture ($P = S$), $\frac{d}{dt} \mathcal{O}_S = \partial_t \mathcal{O}_S$ time dependence of operators comes only from explicit, external dependence in the definition of the operator (which will not happen here), so $\mathcal{O}_S(t) = \mathcal{O}(0)$, and (4.1) then requires

$$|\psi, t\rangle_S = U_H(t) |\psi, 0\rangle.$$

And the unitary evolution operator is

$$U_H(t) = e^{-iH(0)t} = e^{-iH_S t} = U_S = U$$

so does not require a picture label.

Interactions. So, in an interacting field theory, all we need to do is to find U to figure out what it does. For better or worse, this isn’t a realistic goal in general. This

quarter we are going to focus on the special case where the interactions are weak, so that the hamiltonian takes the form

$$H = H_0 + V$$

where H_0 is quadratic in fields (linear terms are allowed but annoying) and we assume that the interaction term V is proportional to a small parameter. This by no means exhausts all interesting questions in field theory; on the other hand, a surprisingly enormous amount of physics can be done using this assumption.

Interaction picture. ($P = I$) Under the circumstances just described, it is very convenient to use a hybrid picture where the time-dependence of the operators is as in the Heisenberg picture for the hamiltonian with $V \rightarrow 0$. This free field evolution is solvable (and is what we've been doing all along so far):

$$\mathcal{O}_I(t) \equiv U_0^\dagger \mathcal{O}(0) U_0, \quad U_0(t) \equiv e^{-iH_0 t}. \quad (4.4)$$

Note that in this picture, $H_0(t) = H_0(0) = H_0$. Equivalently, $i\dot{\mathcal{O}}_I = [\mathcal{O}_I, H_0]$, where in this expression, crucially, $H_0 = H_0(\Phi_I)$ is made from interaction picture fields, whose evolution we know from above; for example, for a complex scalar,

$$\Phi_I(\vec{x}, t) = \int \frac{d^d p}{\sqrt{2\omega_p}} (\mathbf{a}_p e^{-ip_\mu x^\mu} + \mathbf{b}_p^\dagger e^{ip_\mu x^\mu}).$$

The catch is that the interaction-picture states are still time-dependent:

$$\underbrace{{}_H \langle \varphi, t | \mathcal{O}_H(t) | \psi, t \rangle_H}_{= \langle \varphi, 0 | U_H^\dagger(t) \mathcal{O}(0) U_H(t) | \psi, 0 \rangle} \stackrel{(4.1)!}{=} {}_I \langle \varphi, t | \underbrace{\mathcal{O}_I(t)}_{= U_0^\dagger(t) \mathcal{O}(0) U_0(t)} | \psi, t \rangle_I$$

$\forall \varphi, \psi$ which says that

$$| \psi, t \rangle_I = U_0^\dagger(t) U_H(t) | \psi, 0 \rangle \equiv U_I(t) | \psi, 0 \rangle, \quad \text{that is,} \quad U_I(t) = U_0^\dagger(t) U_H(t). \quad (4.5)$$

In the interaction picture, the interaction hamiltonian itself evolves according to

$$i \frac{d}{dt} V_I = [V_I, H_0] \implies V_I(t) = U_0^\dagger V(0) U_0.$$

So for example, if $V(0) = \int d^d x g \phi^3(x, 0)$, then using $1 = U_0 U_0^\dagger$ repeatedly,

$$U_0^\dagger V(0) U_0 = g \int d^d x U_0^\dagger \phi^3(x, 0) U_0 = g \int d^d x U_0^\dagger \phi(x, 0) U_0 U_0^\dagger \phi(x, 0) U_0 U_0^\dagger \phi(x, 0) U_0 = g \int d^d x (\phi_I(x, t))^3.$$

This trick wasn't special to ϕ^3 and works for any local interaction:

$$(V(t))_I = V|_{t=0}(\phi_I(t))$$

– just stick the interaction-picture-evolved fields into the form of the interaction at $t = 0$, easy.

How do the states evolve? Notice that $[U_0^\dagger, U_H] \neq 0$, if the interactions are interesting. So

$$\begin{aligned}
\partial_t |\psi, t\rangle_I &= \partial_t (U_I(t) |\psi, 0\rangle) = \partial_t \left(\underbrace{U_0^\dagger(t)}_{=e^{-iH(0)t}} \underbrace{U_H(t)}_{=e^{-i(H_0+V)t}} |\psi, 0\rangle \right) \\
&= U_0^\dagger \left(\underbrace{iH_0 - iH}_{=-iV(0)} \right) U_H |\psi, 0\rangle \\
&= -iU_0^\dagger V(0) \underbrace{U_H U_H^\dagger}_{=1} U_0 |\psi, t\rangle_I = -i \underbrace{U_0^\dagger V(0) U_0}_{=V(t)} |\psi, t\rangle_I. \tag{4.6}
\end{aligned}$$

Here $V(t) \equiv V_I(t)$ is the interaction-picture evolved interaction Hamiltonian. We've learned that

$$\boxed{i\partial_t |\psi, t\rangle_I = V(t) |\psi, t\rangle_I}.$$

Alternatively, the interaction-picture evolution operator satisfies

$$i\partial_t U_I(t) = V(t) U_I(t).$$

Notice how this differs from the Heisenberg evolution equation (4.3): although the full H is time-independent, $V_I(t)$ actually does depend on t , so $[V(t), V(t')] \neq 0$, and so the solution is not just a simple exponential. We'll find a nice packaging for the solution next in the form of Dyson's expansion.

Peskin's notation for this object is $U_I(t) = U(t, t_0)|_{t_0=0}$. We can figure out how to change the reference time from zero as follows:

$$\begin{aligned}
|\psi, t\rangle_I = U_I(t) |\psi, 0\rangle, \quad |\psi, t'\rangle_I = U_I(t') |\psi, 0\rangle &\implies |\psi, 0\rangle = U_I^\dagger(t') |\psi, t'\rangle \\
|\psi, t\rangle_I = \underbrace{U_I(t) U_I^\dagger(t')}_{=U(t, t')} |\psi, t'\rangle_I
\end{aligned}$$

From which we infer that

$$U(t, t') = U_0^\dagger(t) U_H(t) U_H^\dagger(t') U_0(t') = e^{iH_0 t} e^{-iH(t-t')} e^{-iH_0 t'}.$$

From now on we drop the $P = I$ subscripts: everything is I .

Definition of S -matrix. What are we going to do with the evolution operator? Here is a basic (only slightly naive) three-step framework for doing particle physics (not yet for making predictions):

- At time t_i , specify (*e.g.* measure) all the particle types, spins, momenta in the form of an initial state $|i\rangle$ in the QFT Hilbert space.
- Wait. At time t , the state is

$$U(t, t_i) |i\rangle = |\psi, t\rangle.$$

- At time t_f , measure all the particle data and get some state $|f\rangle$.

Quantum mechanics says that the probability for this outcome is

$$|\langle f | U(t_f, t_i) |i\rangle|^2.$$

(One way in which this is significantly naive is that the space of outcomes is continuous, so we must instead make probability distributions. More soon.) Because particle interactions are a fast-moving business, a useful idealization is to take $t_i \rightarrow -\infty$ and $t_f \rightarrow \infty$, and let

$$S_{fi} \equiv \langle f | \hat{S} |i\rangle, \quad \hat{S} \equiv U(\infty, -\infty)$$

the S -matrix (‘ S ’ is for ‘scattering’).

This has only three ingredients: initial state, time evolution operator, and final state. Let’s focus on the middle of the sandwich some more:

Dyson expansion. We need to solve the equation

$$\partial_t |\psi, t\rangle = -iV(t) |\psi, t\rangle, \quad \text{with initial condition } |\psi, t_i\rangle = |i\rangle.$$

Here’s a “solution”:

$$|\psi, t\rangle = |i\rangle + (-i) \int_{t_i}^t dt_1 V(t_1) |\psi, t_1\rangle$$

as you can check by the fundamental theorem of calculus. The only small problem is that we don’t know $|\psi, t_1\rangle$. But we can use this expression for that too:

$$\begin{aligned} |\psi, t\rangle &= |i\rangle + (-i) \int_{t_i}^t dt_1 V(t_1) \left(|i\rangle + (-i) \int_{t_i}^{t_1} dt_2 V(t_2) |\psi, t_2\rangle \right) \\ &= |i\rangle + (-i) \int_{t_i}^t dt_1 V(t_1) |i\rangle + (-i)^2 \int_{t_i}^t dt_1 \int_{t_i}^{t_1} dt_2 V(t_1) V(t_2) |\psi, t_2\rangle \end{aligned}$$

Why stop there? Two comments: (1) This generates a series in V ; so this is a good idea when $V \propto \lambda \ll 1$. (2) Notice the time-ordering: the range of integration restricts $t_1 \geq t_2$, and the earlier operator $V(t_2)$ is to the right. The result of doing this infinitely many times is

$$|\psi, t\rangle = \sum_{n=0}^{\infty} (-i)^n \int_{t_i}^t dt_1 \int_{t_i}^{t_1} dt_2 \cdots \int_{t_i}^{t_{n-1}} dt_n V(t_1) V(t_2) \cdots V(t_n) |i\rangle = U(t, t_i) |i\rangle. \tag{4.7}$$

Since (4.7) is true for any $|i\rangle$, it tells us a formula for U .

To review, the equation we are trying to solve is:

$$\underbrace{\mathbf{i}\partial_t |\psi, t\rangle}_{=V(t)|\psi, t\rangle} = \mathbf{i}\partial_t U |\psi, t_i\rangle = \mathbf{i}\partial_t U U^\dagger |\psi, t\rangle.$$

This is true for all $|\psi, t\rangle$, so it means $\mathbf{i}\partial_t U U^\dagger = V$. Multiplying the BHS on the right by U gives

$$\implies \partial_t U = -\mathbf{i}VU.$$

We might expect that an equation like this has a solution which is something like $U \stackrel{?}{\sim} e^{-\mathbf{i}Vt}$.

Now we must deal with what Lawrence Hall calls “the wretched $n!$ ”. Starting from our series solution (4.7),

$$\begin{aligned} U(t, t_i) &= \sum_{n=0}^{\infty} (-\mathbf{i})^n \int_{t_i}^t dt_1 \int_{t_i}^{t_1} dt_2 \cdots \int_{t_i}^{t_{n-1}} dt_n V(t_1)V(t_2)\cdots V(t_n) \\ &= \sum_{n=0}^{\infty} (-\mathbf{i})^n \int_{t_i}^t dt_1 \int_{t_i}^{t_1} dt_2 \cdots \int_{t_i}^{t_{n-1}} dt_n \mathcal{T}(V(t_1)V(t_2)\cdots V(t_n)) \\ &= \sum_{n=0}^{\infty} (-\mathbf{i})^n \frac{1}{n!} \int_{t_i}^t dt_1 \int_{t_i}^{t_1} dt_2 \cdots \int_{t_i}^{t_{n-1}} dt_n \mathcal{T}(V(t_1)V(t_2)\cdots V(t_n)) \end{aligned} \quad (4.8)$$

In the first step I used the fact that the operators are already time ordered (this followed from the differential equation we are solving, since the V always acts from the left). In the second step we used the fact that the time-ordered integrand doesn’t change if we permute the labels on the times. So we can just average over the $n!$ possible orderings of n times. If we pull out the time-ordering symbol, this is an exponential series:

$$U(t, t_i) = \mathcal{T} \left(e^{-\mathbf{i} \int_{t_i}^t dt' V(t')} \right).$$

The time-ordered exponential is defined by its Taylor expansion – it’s really just a recipe (or nice packaging) for generating the Dyson expansion.

4.2 S -matrix

Taking the times to $\pm\infty$ in the previous equation gives an expression for the S -matrix:

$$\hat{S} = U(-\infty, \infty) = \mathcal{T} \left(e^{-\mathbf{i} \int_{-\infty}^{\infty} dt V(t)} \right). \quad (4.9)$$

The practical value of these expressions is that they give a (compact) recipe for evaluating the time evolution operator as a series in powers of the small parameter in front of $V(0)$: we know $V(t)$ in terms of things like $\mathbf{a}, \mathbf{a}^\dagger$, can pull them down term-by-term.

I should have called the previous expression the ‘ S -operator’, since the thing we are after is really the S -matrix elements, $\langle f | \hat{S} | i \rangle$, for which we still need $\langle i |$ and $| f \rangle$. Here we encounter some small trouble. Can we just use the states like $\sqrt{2\omega_p} \mathbf{a}_p^\dagger | 0 \rangle$ (the eigenstates of the free hamiltonian) which we’ve grown to love? In fact, even the vacuum $| 0 \rangle$ is not an eigenstate of the actual $H_0 + V$ (since $[H_0, V] \neq 0$), so it will not stay where we put it. The vacuum of the interacting theory $|\Omega\rangle$ is itself an object of mystery (a boiling sea of virtual particles and antiparticles), and the stationary excited states are too (a particle carries with it its disturbance of the vacuum). We’ll learn to deal with this in perturbation theory, but here’s an expedient: pick a function $f(t)$ which is **zero at one end, one in the middle, and then zero again at the far end**. Now replace the interaction hamiltonian V with $f(t)V(t)$. Then, if we take $t_i <$ the time before which the interaction turns on, and $t_f >$ the time after which we turn it off, then we can use the free hamiltonian eigenstates. This is in fact wrong in detail, but it will get us started. We’ll fix it later in our discussion of the LSZ reduction formula.

Example. Let’s think about ‘scalar Yukawa theory’. Simplifying the notation a bit, the whole Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} M^2 \phi^2 + \frac{1}{2} \partial_\mu \Phi^* \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^* \Phi + \mathcal{L}_I \quad (4.10)$$

$$\text{with } \mathcal{L}_I = -g \Phi^* \Phi \phi.$$

In terms of our notation for perturbation theory, $H = H_0 + V$, this means $V = -\int d^d x \mathcal{L}_I$. The mode expansions are

$$\phi = \int \frac{d^d p}{\sqrt{2\omega_p}} (\mathbf{a}_p e^{-ipx} + \mathbf{a}_p^\dagger e^{ipx}) \Big|_{p^0 = \omega_p}$$

$$\Phi = \int \frac{d^d p}{\sqrt{2E_p}} (\mathbf{b}_p e^{-ipx} + \mathbf{c}_p^\dagger e^{ipx}) \Big|_{p^0 = E_p}$$

where I’ve written $\omega_p \equiv \sqrt{M^2 + p^2}$, $E_q \equiv \sqrt{m^2 + q^2}$. Notice that $\Phi \rightarrow e^{-i\alpha} \Phi$ is a symmetry; the conserved charge is

$$q = N_c - N_b.$$

But the ϕ particles are not conserved.⁴³

⁴³You might notice a possible problem with this theory: what happens to the quadratic term for Φ when ϕ is very negative? Let’s not take it too seriously.

Artisanal⁴⁴ meson⁴⁵ decay. [Tong §3.2.1] The interaction \mathcal{L}_I can change the number of ϕ particles. What is the amplitude for a ϕ particle to decay to a Φ -anti- Φ pair? So consider

$$|i\rangle = \sqrt{2\omega_p} a_p^\dagger |0\rangle, \quad |f\rangle = \sqrt{2E_{q_1} 2E_{q_2}} b_{q_1}^\dagger c_{q_2}^\dagger |0\rangle.$$

The S -matrix element between these states is

$$\langle f | \hat{S} | i \rangle = \langle f | \mathcal{T} \left(1 + (-i) \int d^{d+1}x g \phi_x \Phi_x^* \Phi_x + \mathcal{O}(g^2) \right) | i \rangle$$

where all operators are in interaction picture. The first term dies because $\langle f | i \rangle = 0$. In the $\mathcal{O}(g)$ term, the time-ordering doesn't matter because all the operators are at the same time. The $\phi \sim a + a^\dagger$ takes a one-particle state into a superposition of states with zero and two ϕ -particles. We need to end up with zero ϕ -particles. The leading-order nonzero term is

$$\begin{aligned} &= -ig \int d^{d+1}x \langle f | \Phi_x^* \Phi_x \int \frac{d^d k}{\sqrt{2\omega_k}} e^{-ikx} \underbrace{\mathbf{a}_k \mathbf{a}_p^\dagger}_{=\delta^d(k-p)|0\rangle} |0\rangle \sqrt{2\omega_p} \\ &= -ig \int d^{d+1}x e^{-ipx} \langle 0 | \mathbf{b}_{q_1} \mathbf{c}_{q_2} \sqrt{4E_{q_1} E_{q_2}} \int \frac{d^d k_1}{\sqrt{2E_{k_1}}} e^{ik_1 x} \mathbf{b}_{k_1}^\dagger \int \frac{d^d k_2}{\sqrt{2E_{k_2}}} e^{ik_2 x} \mathbf{c}_{k_2}^\dagger |0\rangle \end{aligned}$$

⁴⁴*Artisanal* means doing something by hand when it can be automated. We are doing it this way now to understand the process of mechanization that we will introduce next.

⁴⁵Notice that ϕ is just like the field we discussed earlier (in §2.1), which mediates an attractive force between sources with a range $1/M$. In \mathcal{L}_I we have replaced the force J exerted by the ‘giant hand’ with $g|\Phi|^2$, where Φ is now also a dynamical variable.

The effect (that a scalar field of mass M mediates an attractive force of range $1/M$) is named after Yukawa. The context in which Yukawa thought about this was the atomic nucleus. An atomic nucleus is made up of a bunch of nucleons (protons and neutrons), all of which have non-negative charge and therefore repel each other by Coulomb interaction. But somehow, nuclei stay together. Yukawa’s idea was that there is some short-ranged but very strong force holding them together, and the ϕ particle is the mediator of the force. M , then, is what determines the size of the nucleus, and the quanta of ϕ are called pions.

And the role of the nucleons is played by (the quanta of) Φ in our story here. That’s why I made Φ a complex field, because nucleons carry some conserved charges and are not their own antiparticles.

This model has a number of shortcomings. One is that the nucleons are actually spin-half fermions and not scalars. We’ll learn how to fix that in §5, and it’s not a real problem. I will call the particles created by c^\dagger and b^\dagger snucleons and anti-snucleons, respectively, in recognition of the fact that they are scalars.

A second problem is that we’ve left out the information that distinguishes protons and neutrons, and also there are various species of pions. This ‘isospin’ symmetry information is easy to include and very helpful. A more important shortcoming is the fact that a ‘very strong force’ is required to overcome the Coulomb repulsion. This is accomplished by the fact that the coupling g is not very small in a realistic model, and perturbation theory in g is not always so great.

Meson is an old-fashioned word for a scalar particle, and more specifically is another name for pion.

$$= -ig \int d^{d+1}x e^{i(q_1+q_2-p)x} = -ig(2\pi)^{d+1} \delta^{d+1}(q_1 + q_2 - p)$$

This is a small victory. The delta function imposes conservation of energy and momentum on the transition amplitude. In the ϕ rest frame, $p^\mu = (M, 0)$ which says the amplitude is only nonzero when $\vec{q}_1 = -\vec{q}_2$ and when $M = 2\sqrt{|q_1|^2 + m^2}$. Notice that this can only happen if $M \geq 2m$.

How do we get from this amplitude to a probability? We have to square it:

$$P_{fi} \sim |S_{fi}|^2 = g^2 (\delta^{d+1}(p_f - p_i))^2.$$

The square of a delta function is infinity. What did we do wrong?

Not so much, we just asked a dumb question. Here is where it helps to be a physicist. Consider:

$$(\delta^{d+1}(p))^2 = \delta^{d+1}(p)\delta^{d+1}(0) = \delta^{d+1}(p) \int d^{d+1}x e^{i0x} = \delta^{d+1}(p)VT$$

where VT is the *volume of spacetime* – the size of the box times how long we’re willing to wait. There is a nonzero probability per unit time per unit volume that a ϕ particle in a plane wave state will decay. We’ll get its lifetime out momentarily.

On the homework you’ll do a few more calculations of amplitudes by this artisanal method. For more complicated examples, it will help to streamline this process, which is the job of §4.3. [End of Lecture 11]

4.3 Time-ordered equals normal-ordered plus contractions

[Peskin §4.3] We have an expression for \hat{S} in (4.9) involving only time-ordered products of operators. If we stick this between states with just a few particles, the annihilation operators in there very much want to move to the right so they can get at the vacuum and annihilate it, as is their wont. Wick’s theorem tells us how to help them do this, along the following lines:

$$\text{Wick: } \underbrace{\mathcal{T}(\phi \dots \phi)}_{\text{have}} = \underbrace{:\phi \dots \phi:}_{\text{want}} + ? \quad (4.11)$$

In the previous schematic non-equation, I introduced a notation for a *normal-ordered product* $:\phi \dots \phi:$ which means each term has all the annihilation operators to the right of all the creation operators, for example

$$:\phi(x)\phi(y): \equiv \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^+(x)\phi^+(y) + \phi^-(x)\phi^-(y). \quad (4.12)$$

Normal-ordering difficulty. There is a sticky point in the definition of normal ordering. The issue is: what is the result of normal-ordering a c-number? We are defining normal ordering as a lexicographic operation of moving symbols around, not as a linear operator. If we tried to define the normal-ordering operation to be linear, then we would need to have

$$: \left(\underbrace{\mathbf{a}^\dagger \mathbf{a}}_{=\mathbf{a}\mathbf{a}^\dagger-1} \right) \stackrel{?}{=} : \mathbf{a}\mathbf{a}^\dagger : - : 1 : .$$

But $\langle 0 | : \mathbf{a}^\dagger \mathbf{a} : | 0 \rangle = \langle 0 | : \mathbf{a}\mathbf{a}^\dagger : | 0 \rangle = 0$ so we would need to have $\langle 0 | : 1 : | 0 \rangle \stackrel{?}{=} 0$, which means we would require the shocking-looking equation:

$$: 1 : \stackrel{?}{=} 0$$

that is: the normal-ordered product of a c-number would be zero. This definition (which, beware, differs from Peskin's) would have the advantage that the vacuum expectation value (VEV) of *any* normal ordered product is zero (with no exceptions for c-numbers). The price is that we wouldn't be able to put the normal-ordering symbol around the c-number bits, as Peskin does.

Instead we'll define $: c := c$ for c a c-number. More generally, let A, B, C be the positive- and negative-frequency bits of some fields. Then

$$: ABC \cdots : \equiv \left(\underbrace{A'B'C' \cdots}_{\text{only } \mathbf{a}^\dagger\text{s}} \underbrace{\cdots}_{\text{only } \mathbf{a}\text{s}} \right)$$

Peskin writes $N(\cdots) \equiv : \cdots : .$ Notice that with our definition (and Peskin's)

$$\langle 0 | : \text{anything except a c-number} : | 0 \rangle = 0.$$

A comment about fermions. Later we will use *anticommuting* operators, which have

$$\mathbf{c}_k \mathbf{c}_p^\dagger + \mathbf{c}_p^\dagger \mathbf{c}_k \equiv \{\mathbf{c}_k, \mathbf{c}_p^\dagger\} = \delta(k-p), \quad \{\mathbf{c}_k^\dagger, \mathbf{c}_p^\dagger\} = 0.$$

In particular, the equation $(\mathbf{c}_p^\dagger)^2 = 0$ is an algebraic realization of the Pauli principle. The cost is that even the ϕ^- bits generate signs when they move through each other. In that case, we define the normal ordered product as

$$: ABC \cdots : \equiv \left(\underbrace{A'B'C' \cdots}_{\text{only } \mathbf{a}^\dagger\text{s}} \underbrace{\cdots}_{\text{only } \mathbf{a}\text{s}} \right) (-1)^P$$

where P is the number of fermion interchanges required to get from $ABC\dots$ to $A'B'C'\dots$. Keeping track of these signs, and replacing commutators with anticommutators, everything below goes through for fermion fields.

Let's go back to (4.12) and compare with (4.11). Because $[\phi^\pm, \phi^\pm] = 0$, the order in the last two terms doesn't matter. So (4.12) can differ from the time-ordered product only in the first or second term. If $y^0 > x^0$, it differs by $[\phi^+(y), \phi^-(x)] = \Delta^+(y-x) = -\Delta^-(x-y)$, and if $x^0 > y^0$, it differs by $[\phi^+(x), \phi^-(y)] = \Delta^+(x-y)$. Altogether this produces the Feynman propagator $\Delta_F(x-y)$, which we'll denote as $\Delta_F(x-y) = \overline{\phi(x)\phi(y)}$, and call a *contraction*:

$$:\phi(x)\phi(y): = \mathcal{T}(\phi(x)\phi(y)) - \Delta_F(x-y) \equiv \mathcal{T}(\phi(x)\phi(y)) - \overline{\phi(x)\phi(y)}. \quad (4.13)$$

More generally, writing $\phi_a \equiv \phi(x_a)$, Wick's theorem says

$$:\phi_1 \cdots \phi_n: = \mathcal{T}(\phi_1 \cdots \phi_n) - (\text{all contractions}) \quad (4.14)$$

where a *contraction* is defined as the price for moving a pair of operators through each other to repair the time ordering, as in (4.13), and denoted by the symbol in (4.13). For a pair $\phi(x)\phi(y)$, this price is $\Delta_F(x-y)$.

For example, for four fields, the theorem says

$$\mathcal{T}(\phi_1 \cdots \phi_4) =: \phi_1 \cdots \phi_4 : + \left(\overline{\phi_1 \phi_2} : \phi_3 \phi_4 : + 5 \text{ more} \right) + \left(\overline{\phi_1 \phi_2} \overline{\phi_3 \phi_4} + 2 \text{ more} \right)$$

Notes: The fully-contracted bits are numbers, so (with Peskin's convention) it doesn't matter if they are inside the normal-ordering symbol. For a product of n fields, there are $\binom{n}{2} + \binom{n}{4} \binom{4}{2} + \binom{n}{6} \binom{6}{4} \binom{4}{2} \cdots + \binom{n}{\lfloor n/2 \rfloor}$ (= many) contractions. But if we take the vacuum expectation value (VEV) of the BHS, all the terms that aren't c-numbers go away.

Here's the idea of the proof of (4.14) [Peskin page 90], which is by induction on the number of fields m in the product. We showed $m = 2$ above. Assume WLOG that $x_1^0 \geq \cdots \geq x_m^0$, or else relabel so that this is the case. Wick for $\phi_2 \cdots \phi_m$ says

$$\mathcal{T}(\phi_1 \cdots \phi_m) = \underbrace{\phi_1}_{=\phi_1^+ + \phi_1^-} : \phi_2 \cdots \phi_m : + (\text{all contractions w/o } \phi_1)$$

The ϕ_1^- term is already in the right place and can slip for free inside the normal-ordering sign. The ϕ_1^+ needs to move past all the uncontracted $\phi_{j \geq 2}^-$ s; this process will add a term for every possible contraction involving ϕ_1 . ■

4.4 Time-ordered correlation functions by diagrams

[Peskin §4.4] Time-ordered correlation (or Green's) functions of local operators will be useful:

$$G^{(n)}(x_1 \cdots x_n) \equiv \langle \Omega | \mathcal{T} (\phi_1^H(x_1) \cdots \phi_n^H(x_n)) | \Omega \rangle .$$

Here, the operators are in Heisenberg picture for the full hamiltonian $H = H_0 + V$, and Ω is its actual lowest-energy eigenstate, $H | \Omega \rangle = E_0 | \Omega \rangle$.

The fourier transform is also useful:

$$\tilde{G}^{(n)}(p_1 \cdots p_n) \equiv \int d^{d+1}x_1 \cdots \int d^{d+1}x_n e^{-i \sum_i^n p_i x_i} G^{(n)}(x_1 \cdots x_n) .$$

Notice that there is no need to restrict the momenta on which \tilde{G} depends to the mass shell $p_i^2 = m^2$, that is, $\tilde{G}(p)$ is a function of the p_i even when p_i are off-shell: these Green's functions contain "off-shell" information, more information than is available in just the scattering matrix. However, something special will happen when the external momenta are on-shell. As you can see from the free two-point function, (4.16), they *blow up* on the mass-shell. The existence of a singularity of \tilde{G} on the mass-shell is a general fact, and their residues give the S -matrix elements:

$$\tilde{G}(p_1 \cdots p_n) \stackrel{p_i^2 \rightarrow m^2}{\sim} \prod_i \frac{\mathbf{i}}{p_i^2 - m^2 + \mathbf{i}\epsilon} S(p_1 \cdots p_n). \quad (4.15)$$

The precise version of this is the content of the LSZ theorem, about which more later.
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In the free theory of a real scalar, we know something about these:

$$G_{\text{free}}^{(2)}(x_1, x_2) = \Delta_F(x_1 - x_2) = \overline{x_1 \text{ --- } x_2}$$

$$\tilde{G}_{\text{free}}^{(2)}(p_1, p_2) = \delta^{d+1}(p_1 + p_2) \frac{\mathbf{i}}{p_1^2 - m^2 + \mathbf{i}\epsilon} = \delta^{d+1}(p_1 + p_2) \cdot \overline{p_1 \text{ --- } p_2} \quad (4.16)$$

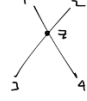
The higher correlations of the free field are Gaussian, in the sense that they are sums of products of the two point functions:

$$G_{\text{free}}^{(4)}(x_1 \cdots x_4) \stackrel{\text{Wick}}{=} \Delta_F(12)\Delta_F(34) + \Delta_F(13)\Delta_F(24) + \Delta_F(14)\Delta_F(23)$$

$$= \overline{1 \text{ --- } 2} \overline{3 \text{ --- } 4} + \overline{1 \text{ --- } 3} \overline{2 \text{ --- } 4} + \overline{1 \text{ --- } 4} \overline{2 \text{ --- } 3} . \quad (4.17)$$

⁴⁶Big picture comment: This is a long chapter. We are working our way towards a useful and correct perturbative expansion of the S -matrix, from which we can extract observable physics. So (4.15) explains a reason to care about time-ordered Green's functions.

Expectations. Our next goal is to construct a perturbative expansion in the case of $V = \int d^d z \frac{\lambda}{4!} \phi^4(z)$. We expect a correction to $G^{(4)}$ of order λ of the form:



Perturbative expansion of time-ordered correlators. We'll do this in three steps: (1) Relate $|\Omega\rangle$ to $|0\rangle$. (2) Relate ϕ_H to ϕ_I . (3) Wick expand and organize the diagrams.

Step (1): [Peskin page 86-87] Some preparations:

- Fix the additive normalization of the hamiltonian by $H_0 |0\rangle = 0$.
- Label the spectrum of H by $|n\rangle$, so

$$\mathbb{1} = \sum_n |n\rangle\langle n| = |\Omega\rangle\langle\Omega| + \sum_{n \neq \Omega} |n\rangle\langle n|. \quad (4.18)$$

This is a very scary sum over the whole QFT Hilbert space, really an integral.

- Assume that $\langle\Omega|0\rangle \neq 0$, where $|0\rangle$ is the free vacuum. A necessary condition for this is that the actual Hamiltonian is in the same phase of matter as the H_0 . Let's keep the volume of space finite for awhile.

Now consider

$$\langle 0| e^{-iHT} \stackrel{(4.18)}{=} \sum_n \langle 0|n\rangle \langle n| e^{-iHT} = \sum_{n \neq \Omega} \langle 0|n\rangle \langle n| e^{-iE_n T} + \langle 0|\Omega\rangle \langle\Omega| e^{-iE_0 T}.$$

Now imagine giving T a large negative imaginary part, $T \rightarrow \infty(1 - i\epsilon)$ Since $E_0 < E_n$ for all other n , we can make the contribution of Ω arbitrarily larger than the others. Multiplying by $e^{iE_0 T} / \langle 0|\Omega\rangle$ gives

$$\langle\Omega| = \lim_{T \rightarrow \infty(1-i\epsilon)} \left(\frac{\langle 0| e^{-iHT} e^{iE_0 T}}{\langle 0|\Omega\rangle} \right) \stackrel{H_0=0}{=} \lim_{T \rightarrow \infty(1-i\epsilon)} \left(\frac{\langle 0| e^{iH_0 T} e^{-iHT} e^{iE_0 T}}{\langle 0|\Omega\rangle} \right)$$

Since T is infinite anyway, we can shift it to $T \rightarrow T - t_0$ without change:

$$\langle\Omega| = \lim_{T \rightarrow \infty(1-i\epsilon)} \left(\frac{\langle 0| e^{iH_0(T-t_0)} e^{-iH(T-t_0)} e^{iE_0(T-t_0)}}{\langle 0|\Omega\rangle} \right) \stackrel{(4.5)}{=} \lim_{T \rightarrow \infty(1-i\epsilon)} \left(\frac{\langle 0| U_I(T, t_0) e^{iE_0(T-t_0)}}{\langle 0|\Omega\rangle} \right). \quad (4.19)$$

Do the same for $|\Omega\rangle$ (rather than just taking the dagger of (4.19)):

$$|\Omega\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{U_I(t_0, -T) |0\rangle}{e^{-iE_0(T+t_0)} \langle\Omega|0\rangle}. \quad (4.20)$$

Now step (2), relate ϕ_H and ϕ_I : in terms of the field ϕ at some reference time t_0 (could be zero), we have $\phi_I = U_0^\dagger \phi U_0$, $\phi_H = U_H^\dagger \phi U_H$ and therefore $\phi_H = U_H^\dagger U_0 \phi_I U_0^\dagger U_H = U_I^\dagger \phi_I U_I$, using (4.5). Now put together the Green's function. First assume $x^0 > y^0$:

$$\begin{aligned}
G^{(2)}(x, y) &= \langle \Omega | \mathcal{T}(\phi_H(x)\phi_H(y)) | \Omega \rangle \\
&\stackrel{(4.19)}{=} \lim_{T \rightarrow \infty(1-i\epsilon)} \left[(e^{-iE_0(T-t_0)} \langle 0 | \Omega \rangle)^{-1} \langle 0 | U_I(T, t_0) \right. && \langle \Omega | \\
&\cdot U_I^\dagger(x^0, t_0)\phi_I(x)U_I(x^0, t_0) \cdot U_I^\dagger(y^0, t_0)\phi_I(y)U_I(y^0, t_0) && \phi_H\phi_H \\
&\cdot U_I(t_0, -T) | 0 \rangle (e^{-iE_0(T+t_0)} \langle \Omega | 0 \rangle)^{-1} \left. \right] && | \Omega \rangle \\
&= \lim_{T \rightarrow \infty(1-i\epsilon)} (e^{-i2E_0T} |\langle 0 | \Omega \rangle|^2)^{-1} \langle 0 | \underbrace{U(T, x^0)\phi(x)U(x^0, y^0)\phi(y)U(y^0, -T)}_{=\mathcal{T}(\phi(x)\phi(y)U(T, -T))} | 0 \rangle
\end{aligned}$$

In the last expression I've gone back to implying the I subscripts on the interaction picture fields. In observing that the big underbraced product is time ordered we are appealing to the Dyson formula for the interaction-picture evolution operators, *e.g.* $U_I(t, t') = \mathcal{T} \left(e^{-i \int_{t'}^t dt'' V(t'')} \right)$ – so it *is* a sum of time-ordered products, evaluated between the times in the argument. Notice that I did something slippery in the first step by combining the factors into one big limit; this is OK if each factor converges separately. If $y^0 > x^0$, the relation between the first and last expressions is unchanged.

What's the denominator? The norm of the vacuum is one, but we can assemble it from these same ingredients, (4.19) and (4.20):

$$1 = \langle \Omega | \Omega \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} (e^{-i2E_0T} |\langle 0 | \Omega \rangle|^2)^{-1} \langle 0 | \underbrace{U(T, t_0)U(t_0, -T)}_{=U(T, -T)} | 0 \rangle$$

Therefore

$$G^{(2)}(x, y) = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | \mathcal{T} \left(\phi(x)\phi(y)e^{-i \int_{-T}^T dt' V(t')} \right) | 0 \rangle}{\langle 0 | \mathcal{T} \left(e^{-i \int_{-T}^T dt' V(t')} \right) | 0 \rangle}$$

The same methods give the analogous formula for $G^{(n)}(x_1 \cdots x_n)$ for any number of any local operators. Now we can immediately perturbate to our hearts' content by expanding the exponentials. Let's do some examples, then I will comment on the familiarity of the prescription for T , and we will see that the denominator is our friend because it cancels annoying (disconnected) contributions in the numerator.

Examples. For $V = \frac{\lambda}{4!} \phi^4$, let's study the numerator of $G^{(2)}(x, y)$ in the first few

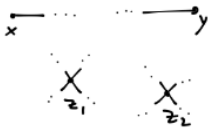
orders of λ :

$$\begin{aligned}
 & G_{\text{num}}^{(2)}(x, y) \\
 & \equiv \langle 0 | \mathcal{T} \left(\phi(x) \phi(y) e^{-i \int_{-T}^T d^{d+1}z \frac{\lambda}{4!} \phi^4(z)} \right) | 0 \rangle \\
 & = \langle 0 | \mathcal{T} \phi(x) \phi(y) | 0 \rangle + \frac{-i\lambda}{4!} \int d^{d+1}z \langle 0 | \mathcal{T} (\phi(x) \phi(y) \phi(z) \phi(z) \phi(z) \phi(z)) | 0 \rangle + \mathcal{O}(\lambda^2) \\
 & = \Delta_F(x - y) + \frac{-i\lambda}{4!} \int d^{d+1}z \left(3 \overbrace{\phi(x) \phi(y) \phi(z) \phi(z)} + 4 \cdot 3 \overbrace{\phi(x) \phi(z) \phi(y) \phi(z)} \overbrace{\phi(z) \phi(z)} \right) + \mathcal{O}(\lambda^2) \\
 & = \text{---} \text{---} + \left(\text{---} \text{---} \text{---} \right) + \text{---} \text{---} + \mathcal{O}(\lambda^2)
 \end{aligned} \tag{4.21}$$

The $\mathcal{O}(\lambda^2)$ contribution is

$$\frac{1}{2!} \left(\frac{-i\lambda}{4!} \right)^2 \int d^{d+1}z_1 d^{d+1}z_2 \langle 0 | \mathcal{T} (\phi(x) \phi(y) \phi(z_1)^4 \phi(z_2)^4) | 0 \rangle$$

With ten fields, there will be five propagators in each diagram. The ingredients that

we must connect together are:  The answer is

$$(\text{---} \text{---} \text{---}) + (\text{---} \text{---} \text{---}) + (\text{---} \text{---}) + (\text{---} \text{---} \text{---}) + \text{---} \text{---} + \text{---} \text{---} + \text{---}$$

For example,


$$\overbrace{\phi(x) \phi(z_1) \phi(y) \phi(z_1)} \left(\overbrace{\phi(z_1) \phi(z_2)} \right)^2 \overbrace{\phi(z_2) \phi(z_2)} \propto \text{---} \text{---}$$

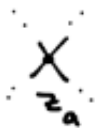
up to the symmetry factor.

Feynman rules for ϕ^4 theory in position space. The set of diagrams at order λ^m is made by drawing one external vertex for each x_i , and m internal vertices, and connecting them in all possible ways with edges (propagators),

$$\{\text{diagrams}\} \equiv \{A\} = \{A_0\} \cup \{A_1\} \cup \dots$$

where A_m gives contributions proportional to λ^m . Let's call \mathcal{M}_A the amplitude associated to diagram A , and the Green's function is $G^{(n)}(x_1 \cdots x_n) = \sum_A \mathcal{M}_A$. (Warning: I won't always be so careful about distinguishing between a diagram and its associated amplitude.) To get \mathcal{M}_A ,

- Put a $\Delta_F(y_i - y_j)$ for each edge  where y may be an external point x_i or an internal point z_a . If λ were zero, this would reproduce the diagrams we drew above for the free theory.

- Put a $-i\lambda \int d^{d+1}z_a$ for each vertex  (notice no $\frac{1}{4!}$).

- Multiply by the symmetry factor $s(A)$. The symmetry factor is defined to be $s(A) = |\text{Aut}(A)|^{-1}$, the inverse of the order of the automorphism group of the diagram, that is: the number of ways of permuting the ingredients of the diagram which map the diagram to itself (fixing the external legs). Symmetries of the diagram mean that the sum over contractions fails to completely cancel Dyson's wretched $\frac{1}{n!}$ and the $\frac{1}{4!}$ in the interaction term. For example, in the diagrams in (4.21), the symmetry factors are:

$$s(\text{---} \text{8}) = \frac{1}{4!} \cdot 3 = \frac{1}{8}, \quad s(\text{---} \text{loop}) = \frac{1}{4!} \cdot 4 \cdot 3 = \frac{1}{2}.$$

In the right diagram, we can reflect the loop across the vertical. In the left diagram we can exchange the two loops, and we can independently reflect them across the vertical. Do not get hung up on this right now. Much more important to understand is the structure of the set of diagrams, which we get started on next.

Let's do the numerator of $G^{(4)}$ through order λ^2 :

$$\mathcal{O}(\lambda^0) : \text{---} \text{---} + \text{---} \text{---} + \text{---} \text{---}$$

$$\mathcal{O}(\lambda^1) : (\text{---} \text{8}) + (\text{---} \text{8}) + (\text{---} \text{8}) + (\text{---} \text{loop}) + \dots + \text{---}$$

Notice that only the last term here is "fully connected" in the sense that you can't divide the diagram up into disjoint pieces without cutting propagators. The other diagrams follow a simple pattern: the first three are obtained from the $\mathcal{O}(\lambda^0)$ diagrams by

multiplying by a figure-eight bubble; we can denote this as $G_0^{(4)}G_1^{(0)}$, where $G_m^{(n)}$ denotes the order- λ^m fully-connected contribution to $G_{\text{num}}^{(n)}$. The second set is a convolution of $G_0^{(2)}$ with $G_1^{(2)}$. The last term is $G_1^{(4)}$. Notice that the numbers in the superscripts in each term add up to the number of external legs, and the numbers in the subscript add up to the number of powers of λ . [End of Lecture 12]

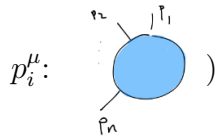
$$\mathcal{O}(\lambda^2): \quad \left(\begin{array}{c} \text{=} \\ \text{8} \end{array} \right) + \dots + \left(\begin{array}{c} \text{=} \\ \text{88} \end{array} \right) + \dots + \left(\begin{array}{c} \text{=} \\ \text{8} \end{array} \right) + \dots + (\text{X8}) + \dots$$

$$+ \begin{array}{c} \text{=} \\ \text{X} \end{array} + \dots + \begin{array}{c} \text{=} \\ \text{X} \end{array} + \begin{array}{c} \text{=} \\ \text{X} \end{array} + \begin{array}{c} \text{=} \\ \text{X} \end{array} + \dots \quad (4.22)$$

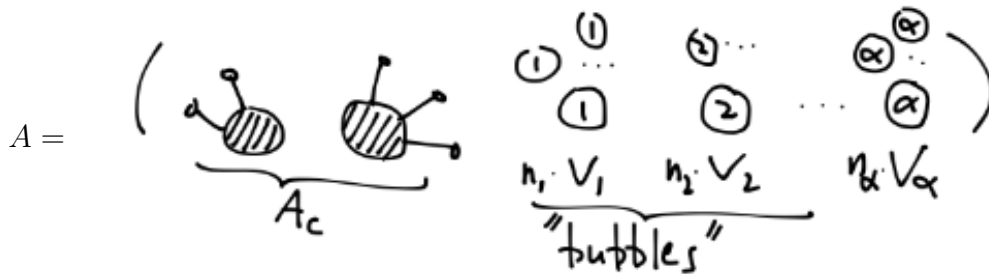
Here we see a new ingredient: there is a term $G_0^{(4)}\frac{1}{2}(G_1^{(0)})^2$. Note that the rules above about adding the numbers still hold. The $\frac{1}{2}$ is part of the symmetry factor associated with the exchange of the two figure-eights.

The exponentiation of the disconnected diagrams. [Peskin page 96] There are some patterns in these sums of diagrams to which it behooves us to attend. (The following discussion transcends the ϕ^4 example.)

To discuss many diagrams at once, it will be useful to introduce the notion of *blobs*: a blob (with some lines sticking out) is the sum of all diagrams with the same external lines. (In momentum space, we'll have blobs with external lines labelled by



The general diagram has the form:



Only some of the components are attached to the external legs; for a given diagram A , call the factor associated with these components A_c (note that A_c need not be *fully* connected). The rest of the diagram is made of a pile of 'bubbles' of various types V_i (each one internally connected, but disconnected from the external lines) and multiplicities n_i (e.g. V_1 could be a figure eight, and there could be $n_1 = 2$ of them, as

(I only drew one kind of bubble in the previous expression since that one was easy to type (it is not the number eight, but the figure-eight diagram!) but it is intended to represent the sum of all connected vacuum bubbles.)

Momentum space Green's functions from Feynman diagrams. In translation-invariant problems, things are usually a little nicer in momentum space. Let's think about

$$\tilde{G}^{(n)}(p_1 \cdots p_n) \equiv \prod_{i=1}^n \int d^{d+1} x_i e^{-i p_i x_i} G^{(n)}(x_1 \cdots x_n).$$

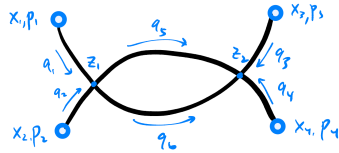
Again, this an off-shell Green's function, a function of general p , not necessarily $p^2 = m^2$. It will, however, vanish unless $\sum_i p_i^\mu = 0$ by translation invariance. Consider a fully-connected contribution to it, at order λ^N . (We'll get the others by multiplying these bits.) In ϕ^4 theory, we need to make a diagram by connecting n external position vertices x_i to N 4-valent vertices z_a using Feynman propagators $\Delta_F(y_A - y_B) = \int d^{d+1} q_r e^{-i(y_A - y_B)q_r} \frac{\mathbf{i}}{q_r^2 - m^2 + i\epsilon}$, where $\{y_A\} = \{z_a, x_i\}$. Since each propagator has two ends, the number of internal lines (by the fully-connected assumption) is

$$N_I = \frac{\# \text{ of ends of lines}}{2} = \frac{n + 4N}{2} = \frac{n}{2} + 2N.$$

The associated amplitude is then

$$\begin{aligned} \mathcal{M}_{FC}^N = & \int d^{d+1} x_1 \cdots d^{d+1} x_n e^{-i \sum_i p_i x_i} (-i\lambda)^N \cdot s(FC) \int d^{d+1} z_1 \cdots \int d^{d+1} z_N \prod_{r=1}^{N_I} \Delta_F(y_A - y_B) = \\ & \int d^{d+1} x_1 \cdots d^{d+1} x_n e^{-i \sum_i p_i x_i} (-i\lambda)^N \cdot s(FC) \int d^{d+1} z_1 \cdots \int d^{d+1} z_N \prod_{r=1}^{N_I} \int d^{d+1} q_r e^{-i(y_A - y_B)q_r} \frac{\mathbf{i}}{q_r^2 - m^2 + i\epsilon}. \end{aligned} \quad (4.25)$$

For example, consider a particular contribution with $n = 4$ external legs and $N = 2$ interaction vertices:



$$\text{has } N_I = \frac{4 + 2 \cdot 4}{2} = 6$$

Notice that we are doing a silly thing here of labelling the momenta of the external lines (the first n momenta $q_{i=1 \dots n}$). Here's why it's silly: Look at the integral over x_1 . Where is the dependence on x_1 ? There is the external factor of $e^{-i p_1 x_1}$ that we put to do the Fourier transform, and there is the propagator taking x_1 to z_1 ,

$\Delta_F(x_1 - z_1) = \int d^{d+1} q_1 e^{-i(x_1 - z_1)q_1} \frac{i}{q_1^2 - m^2 + i\epsilon}$. So the integral over x_1

$$\int d^{d+1} x_1 e^{-ix_1(p_1 - q_1)} = \delta^{d+1}(p_1 - q_1)$$

just sets $p_1 = q_1$, and eats the $\int d^{d+1} q_1$. The same thing happens for each external line, and reduces the number of momentum integrals to $N_I - n$.

Where is the dependence on z_2 ?

$$\int d^{d+1} z_2 e^{-iz_2(p_3 + p_4 + q_5 + q_6)} = \delta^{d+1}(p_3 + p_4 + q_5 + q_6).$$

Similarly, the z_1 dependence is all in the exponentials:

$$\int d^{d+1} z_1 e^{-iz_1(-q_5 - q_6 + p_1 + p_2)} = \delta^{d+1}(q_5 + q_6 - p_1 - p_2).$$

These two factors combine to set $p_1 + p_2 = p_3 + p_4 = -q_5 - q_6$: momentum is conserved at the vertices. Notice that in the example $q_5 - q_6$ is *not* determined.

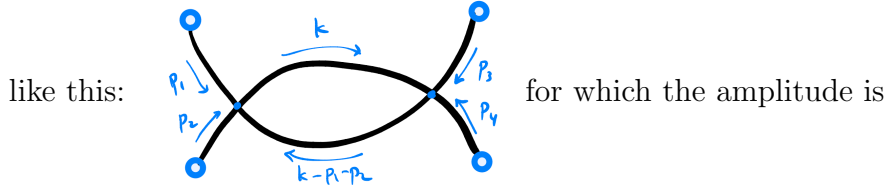
So each internal vertex reduces the number of undetermined momenta by one. One combination of the momenta is fixed by overall momentum conservation. So we have left

$$N_I - n - (N - 1) = N - \frac{n}{2} + 1 \quad (4.26)$$

momentum integrals. This number is ≥ 0 for fully connected diagrams, and it is the number of *loops* in the diagram. (This counting is the same as in a Kirchoff's law resistor network problem.) In the example, $N_L = 2 - 2 + 1 = 1$ which agrees with one undetermined momentum integral.

Here's a proof that (4.26) is the number of loops in the diagram: place the $N + n$ internal and external vertices on the page. Add the propagators one at a time. You must add $N + n - 1$ just to make the diagram fully connected. After that, each line you add makes a new loop. So the number of loops is $N_I - (N + n - 1)$, which is the same as the number of momentum integrals (4.26).

In practice now, we need not introduce all those extra q s. Label the external lines by $p_1 \cdots p_n$, and the loop momenta by $k_\alpha, \alpha = 1..N_L$. In the example, we might do it




$$\mathcal{M}_{FC}(p_1 \cdots p_n) = (-i\lambda)^N \cdot s(FC) \delta^{(d+1)}(\sum p_i) \int \prod_{\text{loops}, \alpha=1}^{N_L} d^{d+1} k_\alpha \prod_{\text{lines}, r} \frac{i}{q_r^2 - m^2 + i\epsilon}$$

$$\begin{array}{c} \text{Diagram: a loop with four external lines} \\ \hline \end{array} = \frac{(-i\lambda)^2}{2!} \oint^{d+1} \left(\sum_{i=1}^4 p_i \right) \prod_{i=1}^{n=4} \frac{\mathbf{i}}{p_i^2 - m^2 - i\epsilon} \int d^{d+1}k \frac{\mathbf{i}}{k^2 - m^2 + i\epsilon} \frac{\mathbf{i}}{(p_1 + p_2 + k)^2 - m^2 + i\epsilon}$$

(You might notice that the integral over k is in fact formally infinite, since at large k it goes like $\int^\Lambda \frac{d^4k}{k^2} \sim \log(\Lambda)$. Try to postpone that worry.) The propagators for the external lines just factor out, and can be brought outside the momentum integrals. Let's celebrate my successful prediction, for this particular graph, that there would be poles when the external particles are on-shell, $p_i^2 = m^2$. (It would be more correct to call it Lehmann, Symanzik and Zimmerman's successful prediction.)

So here are the momentum space Feynman rules for Green's function in ϕ^4 theory:

- Every line gives a factor of $\text{---}\leftarrow = \frac{\mathbf{i}}{p^2 - m^2 + i\epsilon} = \tilde{\Delta}_F(p)$. Notice that since $\Delta_F(x - y) = \Delta_F(y - x)$, the choice of how we orient the lines is not so fateful.

- An internal vertex gives  $\rightsquigarrow (-i\lambda) \int d^{d+1}z e^{-i\sum_i p_i z} = (-i\lambda) \oint^{d+1}(\sum_i p_i)$,

momentum conservation at each vertex. So, set $\sum_i p_i = 0$ at each vertex (I've assumed the arrows are all pointing toward the vertex). After imposing momentum conservation, the remaining consequence of the vertex is

$$\text{Diagram: a four-point vertex with arrows pointing inward} = -i\lambda.$$

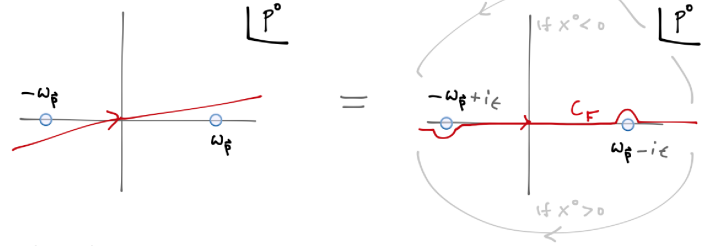
- Integrate over the loop momenta $\prod_{\alpha=1}^{N_L} d^{d+1}q_\alpha$ for each undetermined momentum variable. There is one for each loop in the diagram. You should think of these integrals as just like the Feynman path integral: if there is more than one way to get from here to there, we should sum over the amplitudes.
- Multiply by the wretched symmetry factor $s(A)$.
- For $\tilde{G}(p)$, multiply by an overall $\oint^{d+1}(\sum p)$ in each diagram.
- An external vertex at fixed position, $\text{---}\leftarrow_x = e^{-ipx}$. (Such vertices would arise if we wanted to compute $G(x)$ using momentum-space Feynman rules.) More generally, external vertices are associated with the wavefunctions of the states we are inserting; here they are plane waves.

Comment on $T \rightarrow \infty(1 - i\epsilon)$. [Peskin p. 95] What happened to the limit on T ? It's hidden in the integrals over the vertices:

$$\int d^{d+1}z e^{-iz\sum_i q_i} \dots = \lim_{T \rightarrow \infty(1-i\epsilon)} \int_{-T}^T dz^0 d^d z e^{-i(z^0 \sum_i q_i^0 - \sum_i \vec{z} \cdot \vec{q}_i)} \dots \quad (4.27)$$

One end of the integral $z^0 = \pm\infty$ is going to be infinite unless $\sum_i q_i^0 z^0 \in \mathbb{R}$, in which case it just oscillates. We can fix this problem by replacing each q^0 with $q^0 \rightarrow q^0(1+i\epsilon)$ to cancel the factor of $1-i\epsilon$ (using $\epsilon^2 = 0$). Then the integral (4.27) is just a delta func-

tion. This means that the integrals will look like:



But this replacement is what we are already doing by the Feynman contour prescription. That is: if we use the Feynman contour for every propagator

$$\Delta_F(x) = \int_{C_F} d^{d+1}q e^{-iqx} \frac{\mathbf{i}}{q^2 - m^2 + \mathbf{i}\epsilon_F}$$

with $\epsilon_F = \epsilon$ then this problem goes away – this discussion forces us to choose the same ϵ in the two places.

The factors of T give another perspective on the exponentiation of the vacuum bubbles. Consider the diagram:

$$= (-i\lambda)^2 \prod_{i=1}^4 \int d^{d+1}p_i \delta^{d+1}(p_1 + p_2) \delta^{d+1}(p_1 + p_2) \dots$$

The two delta functions come from the integrals over the positions z, w , and we can restore sense by remembering this:

$$\left(\delta^{d+1}(p_1 + p_2) \right)^2 = \delta^{d+1}(p_1 + p_2) \int d^{d+1}z_2 = \delta^{d+1}(p_1 + p_2) 2TV$$

where V is the volume of space. This factor arises because this process can happen anywhere, anytime. There is one such factor for each connected component of a collection of vacuum bubbles, so for example the diagram $(\text{chain of bubbles})$ is proportional to $(VT)^2$. But the free energy $\propto \log Z = \log G^{(0)}$ should be extensive, $\propto VT$. Therefore, the vacuum bubbles must exponentiate.

The whole two point function in momentum space is then (through order λ^2) :

$$\tilde{G}^{(2)} = \dots + \mathcal{O}(\lambda^3) \tag{4.28}$$

I draw the blue dots to emphasize the external propagators.

Here's an easy one: in the ϕ^4 theory, $G^{(n)} = 0$ (in either position or momentum space) when n is odd. Technically, we can see this from the fact that there is always a ϕ left over after all contractions, and $\langle 0|\phi|0\rangle = 0$. Slightly more deeply, this is because of the $\phi \rightarrow -\phi$ symmetry of the action.

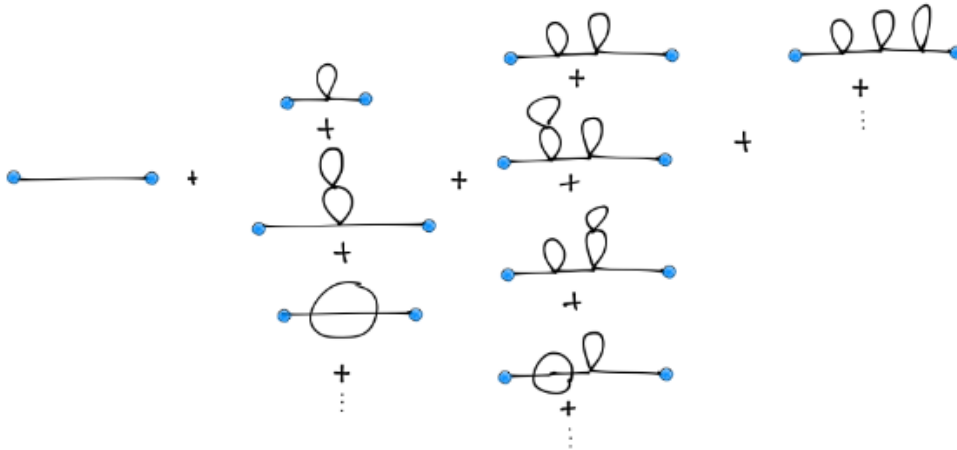
4.5 From correlation functions to the S matrix



Next we would like to unpack the physics contained in the correlation functions that we've learned to compute in perturbation theory. The first interesting one is the two-point function.

Recall our expression for the momentum-space two-point function (4.28) in terms of a sum of connected diagrams, ordered by the number of powers of λ . Let's factor out the overall delta function by writing:

$$\tilde{G}^{(2)}(p_1, p_2) \equiv \delta^{d+1}(p_1 + p_2) \tilde{G}^{(2)}(p_1).$$

It will be useful to re-organize this sum, in the following way:



Here's the pattern: we define a diagram to be *one-particle irreducible* (1PI) if it cannot be disconnected by cutting through a single internal propagator. So for example,  is 1PI, but  is not; rather, the latter contributes to the bit with two 1PI insertions. Then

$$\tilde{G}^{(2)}(p) = \text{---} + \text{---} \circlearrowleft \text{---} + \text{---} \circlearrowleft \circlearrowleft \text{---} + \text{---} \circlearrowleft \circlearrowleft \circlearrowleft \text{---} + \dots$$

So that we may write equations without pictures, let

$$-i\Sigma(p) \equiv \text{---} \circlearrowleft \text{---}$$

denote the 1PI two-point function. Σ being 1PI means that the external lines sticking out of it are ‘nubbins,’ placeholders where propagators may be attached. That’s why there are no blue dots at the ends.

Now suppose we know Σ . It is known as the *self-energy*, for reasons we will see next. Then we can write

$$\begin{aligned} \tilde{G}^{(2)}(p) &= \frac{\mathbf{i}}{p^2 - m_0^2} + \frac{\mathbf{i}}{p^2 - m_0^2} (-\mathbf{i}\Sigma(p)) \frac{\mathbf{i}}{p^2 - m_0^2} + \frac{\mathbf{i}}{p^2 - m_0^2} (-\mathbf{i}\Sigma(p)) \frac{\mathbf{i}}{p^2 - m_0^2} (-\mathbf{i}\Sigma(p)) \frac{\mathbf{i}}{p^2 - m_0^2} + \dots \\ &= \frac{\mathbf{i}}{p^2 - m_0^2} \left(1 + \frac{\Sigma}{p^2 - m_0^2} + \left(\frac{\Sigma}{p^2 - m_0^2} \right)^2 + \dots \right) \\ &= \frac{\mathbf{i}}{p^2 - m_0^2} \frac{1}{1 - \frac{\Sigma}{p^2 - m_0^2}} = \frac{\mathbf{i}}{p^2 - m_0^2 - \Sigma(p)}. \end{aligned} \tag{4.29}$$

We see that the self-energy shifts the m^2 of the particle – it moves the location of the pole in the propagator. In the interacting theory, $m_0^2 + \Sigma(p)|_{\text{pole}}$ is the physical mass, while m_0 (what we’ve been calling m until just now) is deprecatingly called the ‘bare mass’.


[\[End of Lecture 13\]](#)

For $p^2 \sim m^2$, we will write

$$\tilde{G}^{(2)}(p) \equiv \left(\frac{\mathbf{i}Z}{p^2 - m^2} + \text{regular bits} \right) \tag{4.30}$$

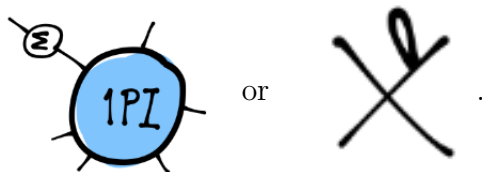
This equation defines the residue of the pole, Z , which is called the ‘wavefunction renormalization factor’. It is 1 in the free theory, and represents the amplitude for the field to create a particle, and the other terms, which are not singular at $p^2 = m^2$, represent the amplitude for the field to do something else (such as create multiparticle states), and are absent in the free theory. Unitarity (positivity of norms of states) requires $Z < 1$. (Perhaps you will see this in 215B. See [here](#) for an argument.) Notice that if we know Σ only to some order in perturbation theory, then (4.29) is still true, up to corrections at higher order.

The notion of 1PI extends to diagrams for $\tilde{G}^{(n>2)}(p_1 \dots p_n)$. Let

$$\tilde{G}_{1PI}^{(n)}(p_1 \dots p_n) \equiv \text{blob with 1PI and } n \text{ nubbins}$$


where the blob indicates the sum over all 1PI diagrams with n external *nubbins* (notice that these do not have the blue circles that were present before). This means G_{1PI}

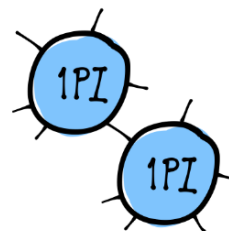
does not include diagrams like:



Notice that 1PI diagrams are *amputated* – their external limbs have been cut off, leaving just nubbins.

This is almost what we need to make S -matrix elements. If we multiply the n -point function by $\prod_{i=1}^n \frac{p_i^2 - m^2}{\sqrt{Z}}$ we cancel out the propagators from the external legs. This object is naturally called the *amputated* n -point function. (It differs from the

1PI n -point Green's function because of diagrams like this one



which is amputated but not 1PI.) If we then take $p_i^2 \rightarrow m^2$, we keep only the part of \tilde{G} which is singular on the mass-shell. And here's why we care about that:

Claim (the LSZ reduction formula):

$$S_{fi} \equiv \langle \vec{p}_1 \cdots \vec{p}_n | S | \vec{k}_1 \cdots \vec{k}_m \rangle = \prod_{a=1}^{n+m} \left(\lim_{P_a^0 \rightarrow E_{\vec{P}_a}} \frac{P_a^2 - m^2}{i\sqrt{Z}} \right) \tilde{G}^{(n+m)}(k_1 \cdots k_m, -p_1 \cdots -p_n) \quad (4.31)$$

where $P_a \in \{p_i, k_i\}$. In words: the S -matrix elements are obtained from Green's functions by amputating the external legs, and putting the momenta on-shell. Notice that choosing all the final momenta p_i different from all the initial momenta k_i goes a long way towards eliminating diagrams that are not fully connected.

This formula provides the bridge from time-ordered Green's functions (which we know how to compute in perturbation theory now) and the S -matrix, which collects probability amplitudes for things to happen to particles, in terms of which we may compute cross sections and lifetimes. Let us spend just another moment inspecting the construction of this fine conveyance.

Why is LSZ true? Here's the argument I've found which best combines concision and truthiness. [It is mainly from the nice book by Maggiore §5.2; I also like Schwartz']

chapter 6; Peskin's argument is in section 4.6.] The argument has several steps. The field operators in this discussion are all in Heisenberg picture.

1. First, for a free field, the mode expansion implies that we can extract the ladder operators by:

$$\begin{aligned}\sqrt{2\omega_k}a_k &= \mathbf{i} \int d^d x e^{\mathbf{i}kx} (-\mathbf{i}\omega_k + \partial_t) \phi_{\text{free}}(x) \\ \sqrt{2\omega_k}a_k^\dagger &= -\mathbf{i} \int d^d x e^{-\mathbf{i}kx} (+\mathbf{i}\omega_k + \partial_t) \phi_{\text{free}}(x).\end{aligned}\quad (4.32)$$

Notice that the LHS is independent of time, but the integrand of the RHS is not.

2. Now, recall our brontosaurus expedient (introduced previously after (4.9)): turn the interactions off at $t = \pm\infty$.⁴⁷ This allows us to write the field in terms of some pretend free fields of mass m (not m_0 !)

$$\phi(x) \begin{cases} \xrightarrow[t \rightarrow -\infty]{} & Z^{\frac{1}{2}} \phi_{\text{in}}(x) \\ \xrightarrow[t \rightarrow +\infty]{} & Z^{\frac{1}{2}} \phi_{\text{out}}(x) \end{cases}.$$

The factors of \sqrt{Z} are required to get the correct two point functions (4.30) near the mass shell. The mode operators for ϕ_{in} are called $a^{(\text{in})}$ etc. $\phi_{\text{in, out}}$ are free fields: their full hamiltonian is H_0 . They are in Heisenberg picture, and the reference time for $\phi_{\text{in, out}}$ is $\pm\infty$ respectively. Since they are free fields, we can use (4.32) to write

$$\sqrt{2\omega_k}a^{(\text{in})\dagger} = -\mathbf{i} \int d^d x e^{-\mathbf{i}kx} (+\mathbf{i}\omega_k + \partial_t) \phi_{\text{in}}(x) = -\mathbf{i}Z^{-1/2} \int d^d x e^{-\mathbf{i}kx} (+\mathbf{i}\omega_k + \partial_t) \phi(x)|^{t \rightarrow -\infty}$$

where in the second step we used the independence on time in (4.32), even though $\phi(x)$ is not a free field. An expression for $a^{(\text{out})\dagger}$ obtains if we take $t \rightarrow +\infty$ instead.

3. Now make this expression covariant using the fundamental theorem of calculus:

$$\begin{aligned}\sqrt{2\omega_k} (a^{(\text{in})\dagger} - a^{(\text{out})\dagger}) &= \mathbf{i}Z^{-1/2} \int_{-\infty}^{\infty} dt \partial_t \left(\int d^d x e^{-\mathbf{i}kx} (\mathbf{i}\omega_k + \partial_t) \phi(x) \right) \\ &= \mathbf{i}Z^{-1/2} \int d^{d+1} x \left(e^{-\mathbf{i}kx} \partial_t^2 \phi - \underbrace{\phi \cdot \partial_t^2 e^{-\mathbf{i}k_\mu x^\mu}}_{(\vec{\nabla}^2 - m^2)e^{-\mathbf{i}kx}} \right)\end{aligned}$$

⁴⁷Here's why this is really bad: nearly everything we might scatter is a boundstate. For example: atoms, nuclei, nucleons etc... But if there are no interactions there are no boundstates. Please see the comments below this list for the explanation of why this works.

$$\stackrel{\text{IBP}}{=} \mathbf{i}Z^{-1/2} \int d^{d+1}x e^{-\mathbf{i}kx} (\square + m^2) \phi(x) \quad (4.33)$$

In the last step we made a promise to only use wavepackets for external states, so that we can do IBP in space.

4. Now, here's where the S -matrix enters. Assume no subset of the incoming momenta $\sum_i k_i$ is the same as any subset of the outgoing momentum $\sum_j p_j$.

$$\begin{aligned} & \langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle \\ &= \prod_{p,k} \sqrt{2\omega} \langle \Omega | \prod a_p^{\text{out}} S \prod a_k^{\text{in}\dagger} | \Omega \rangle \\ &= \prod_{p,k} \sqrt{2\omega} \langle \Omega | \mathcal{T} \left(\prod a_p^{\text{out}} S \prod a_k^{\text{in}\dagger} \right) | \Omega \rangle \quad a^{\text{out}} \text{ lives at } t = +\infty \\ &= \prod_{p,k} \sqrt{2\omega} \langle \Omega | \mathcal{T} \left(\prod a_p^{\text{out}} S \left(a_{k_1}^{\text{in}\dagger} - a_{k_1}^{\text{out}\dagger} \right) \prod_2^m a_k^{\text{in}\dagger} \right) | \Omega \rangle \quad \text{since } p_i \neq k_j, \text{ use } \langle 0 | a^{\text{out}\dagger} = 0 \\ &\stackrel{(4.33)}{=} \prod_{p,k \setminus k_1} \sqrt{2\omega} \mathbf{i}Z^{-1/2} \int d^{d+1}x_1 e^{-\mathbf{i}k_1 x_1} \langle \Omega | \mathcal{T} \left(\prod a_p^{\text{out}} S (\square + m^2) \phi(x_1) \prod_2^m a_k^{\text{in}\dagger} \right) | \Omega \rangle \\ &= \prod_{p,k \setminus k_1} \sqrt{2\omega} \mathbf{i}Z^{-1/2} \int d^{d+1}x_1 e^{-\mathbf{i}k_1 x_1} (\square + m^2) \langle \Omega | \mathcal{T} \left(\prod a_p^{\text{out}} S \phi(x_1) \prod_2^m a_k^{\text{in}\dagger} \right) | \Omega \rangle + X \end{aligned}$$

In the last step, X comes from where the \square_{x_1} hits the time ordering symbol. This gives terms that will not matter when we take $k^2 \rightarrow m^2$, I promise.

5. Now do this for every particle to get

$$\langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle = \prod_{j=1}^m \int d^{d+1}y_j e^{+\mathbf{i}p_j y_j} \mathbf{i}Z^{-1/2} (\square_j + m^2) \prod_{i=1}^n \int d^{d+1}x_i e^{-\mathbf{i}k_i x_i} \mathbf{i}Z^{-1/2} (\square_i + m^2) \langle \Omega | \mathcal{T} \phi(x_i) \cdots \phi(y_j) S | \Omega \rangle + X'$$

The x and y integrals are just Fourier transforms, and this says that near the mass shell,

$$\tilde{G}^{(n+m)}(k_1 \cdots k_m, -p_1 \cdots -p_n) = \prod_a^{n+m} \frac{\mathbf{i}\sqrt{Z}}{P_a^2 - m^2} \langle p_1 \cdots p_n | S | k_1 \cdots k_m \rangle + \text{regular}$$

(where $P_a \in \{p_j, k_i\}$) which is the same as (4.31).

Comments about LSZ.

- An apparently important role is played in the derivation of LSZ by our brontosaurus expedient (after (4.9)). The reason this crazy-seeming trick works is

the locality of interactions. In a scattering experiment, we scatter well-localized wavepackets that start and end far away from each other, and therefore do not interact at the beginning and the end. Although we calculate S -matrix elements of plane waves (not localized at all), this is also just an expedient – if you forced me we could figure out how to write the formula for wavepackets instead. No physics conclusion would be different.

- The real content of the $\phi_{\text{in}}, \phi_{\text{out}}$ business is *adiabatic continuation* – we are labelling the (low-lying) eigenstates of the full Hamiltonian by those of the free theory. This does not always work! There are at least two failure modes.

The most dramatic is when the eigenstates are not particle states at all, but rather some kind of soup of excitations. This is what happens in a conformal field theory (CFT), and is sometimes called *unparticles*. In this case, not only does our formula for the S -matrix not work, but the S -matrix itself is not so well-defined.

Less dramatic is when there are asymptotic particle states, but they are not simply related to the quanta of the fields in the Lagrangian. This happens any time there are boundstates, like positronium in QED. But it happens even more dramatically in QCD, where our microscopic Lagrangian description is in terms of quarks and gluons, but the quarks and gluons are *confined*, and the asymptotic particle states are instead all colorless composites called hadrons (protons, neutrons, pions ...). The next item shows that LSZ also teaches us how to deal with this.

- In our discussion of QFT, a special role has been played by certain fields called ϕ . Suppose we have some other (say hermitian) local operator \mathcal{O} such that

$$\langle p | \mathcal{O}(x) | \Omega \rangle = Z_{\mathcal{O}} e^{ipx}$$

where $\langle p |$ is a one-particle state made by our friend ϕ (we could put some labels, *e.g.* for spin or polarization or flavor, on both the operator and the state, but let's not). Such an \mathcal{O} is called an ‘interpolating field’ or ‘interpolating operator’. And suppose we have information about the correlation functions of \mathcal{O} :

$$G_{\mathcal{O}}^{(n)}(1 \cdots n) \equiv \langle \Omega | \mathcal{T}(\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)) | \Omega \rangle .$$

In this case, there is a more general statement of LSZ:

$$\prod_{a \in i} \left(Z_a^{-1/2} \mathbf{i} \int d^{d+1} x_a e^{-ip_a x_a} (\square_a + m_a^2) \right) \prod_{b \in f} \left(Z_b^{-1/2} \mathbf{i} \int d^{d+1} x_b e^{+ip_b x_b} (\square_b + m_b^2) \right) G_{\mathcal{O}}^{(n)}(1 \cdots n)$$

$$= \langle \{p_f\} | S | \{p_i\} \rangle \quad (4.34)$$

This more general statement follows as above if we can write $\mathcal{O}_a \xrightarrow{t \rightarrow -\infty} \sqrt{Z_a} \phi_{\text{in}}$. This more general formula allows us to scatter particles that are not ‘elementary’ in the sense that they are just the quanta of the fields in terms of which we write our Lagrangian.

(So, for the example of positronium, which is a boundstate of an electron and a positron, we can take $\mathcal{O}(x) = \bar{\psi}_e(x) \psi_e(x)$ where ψ_e is the field that annihilates an electron. This operator has some nonzero amplitude to create a state with a positronium atom from the vacuum: $\langle \text{positronium}, q | \mathcal{O}(x) | \Omega \rangle = Z e^{iqx}$.)

4.6 Interlude: old-fashioned perturbation theory

I'm skipping this section in lecture. You might enjoy reading it anyway.

[Schwartz, chapter 4] I want to take a brief break from the inexorable building of theoretical machines to demonstrate some virtues of those machines. It will explain what I was really mumbling about when I said that the Feynman propagator involves antiparticles going backwards in time.

Consider a system which is a small perturbation of a solvable system $H = H_0 + V$. Suppose that the initial system H_0 has a continuous spectrum, so that there are eigenstates at every nearby energy. Then given an eigenstate of H_0 , $H_0 |\varphi\rangle = E |\varphi\rangle$, we expect an eigenstate of H with the same energy $H |\psi\rangle = E |\psi\rangle$. Palpating the previous equation appropriately gives

$$|\psi\rangle = |\varphi\rangle + \underbrace{\frac{1}{E - H_0 + i\epsilon}}_{\equiv \Pi} V |\psi\rangle$$

(the Lippmann-Schwinger equation). This represents the perturbed eigenstate as the free one plus a scattering term, in terms of the ‘propagator’ Π . The $i\epsilon$ is a safety factor which helps us negotiate the fact that $E - H_0$ is not actually invertible. To write this entirely in terms of the known free state $|\varphi\rangle$, iterate. Let $V |\psi\rangle \equiv T |\varphi\rangle$ where T is the *transfer matrix*:

$$|\psi\rangle = |\varphi\rangle + \Pi T |\varphi\rangle.$$

Now act on both sides with V to get $V |\psi\rangle = T |\varphi\rangle = V |\varphi\rangle + V \Pi T |\varphi\rangle$, which will hold if

$$T = V + V \Pi T = V + V \Pi (V + V \Pi T) = V + V \Pi V + V \Pi V + V \Pi V \Pi V + \dots = \left(\frac{1}{1 - V \Pi} \right) V$$

Given a complete set of eigenstates of H_0 , with $\sum_i |\varphi_i\rangle \langle \varphi_i| = \mathbb{1}$,

$$T_{fi} \equiv \langle \varphi_f | T | \varphi_i \rangle = V_{fi} + V_{fj} \Pi(j) V_{ji} + V_{fj} \Pi(j) V_{jk} \Pi(k) V_{ki} + \dots$$

where $V_{fi} \equiv \langle \varphi_f | V | \varphi_i \rangle$ gives the first Born approximation, and $\Pi(j) \equiv \frac{1}{E - E_j}$, and $E = E_i = E_f$, energy is conserved.

For a vivid example, consider the mediation of a force by a boson field. Let

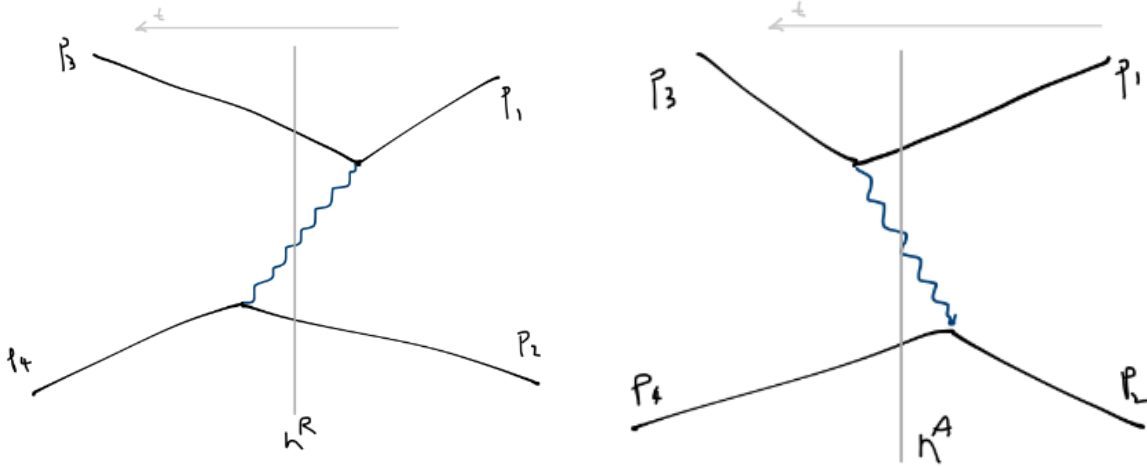
$$V = \frac{1}{2} e \int d^d x (\Psi_e(x) \phi(x) \Psi_e(x) + \Psi_\mu(x) \phi(x) \Psi_\mu(x))$$

where ‘ ϕ ’ is for ‘photon’ and ‘ e ’ is for ‘electron’ and ‘ μ ’ is for ‘muon’ but we’ve omitted spin and polarization information, and got the statistics and charge of the electron and

proton wrong, for simplicity. Consider the free eigenstates $|i\rangle = |\vec{p}_1, \vec{p}_2\rangle$, $\langle f| = \langle \vec{p}_3, \vec{p}_4|$ where 1,3 represent electron states and 2,4 represent muon states respectively. (I introduce two kinds of fields here to avoid the presence of some extra diagrams.) Then,

$$T_{fi} = \underbrace{V_{fi}}_{\substack{p_{1,2} \neq p_{3,4} \\ = 0}} + \sum_n V_{fn} \frac{1}{E_i - E_n} V_{ni} + \dots$$

What are the possible intermediate states $|n\rangle$? It has to be an e , a μ and one ϕ , as in the following visualization (*not* a Feynman diagram in the same sense we've been discussing):



Time goes to the left, as always. The wiggly line represents the quantum of ϕ . You see that there are two classes of possibilities: $|n_R\rangle = |p_3, p_\gamma; p_2\rangle$, $|n_A\rangle = |p_1; p_\gamma, p_4\rangle$. Consider them from the point of view of particle 2. In the first (R) case, μ_2 absorbs a photon emitted by particle 1, after the emission happens:

$$\begin{aligned} V_{ni}^R &= \langle p_3, p_\gamma, p_2 | V | p_1, p_2 \rangle = \langle p_3, p_\gamma | V | p_1 \rangle \underbrace{\langle p_2 | p_2 \rangle}_{=1} \\ &= \frac{e}{2} \int d^d x \langle p^3, p_\gamma | \Psi_e(x) \phi(x) \Psi_e(x) | p^1 \rangle = \frac{e}{2} \int d^d x \underbrace{\langle p_\gamma | \phi(x) | 0 \rangle}_{=e^{-i\vec{p}_\gamma \cdot \vec{x}}} \underbrace{\langle p^3 | \Psi(x)_e^2 | p^1 \rangle}_{=2e^{-i(\vec{p}_3 - \vec{p}_1) \cdot x}} \\ &= e\delta^d(\vec{p}_1 - \vec{p}_3 - \vec{p}_\gamma) \end{aligned} \quad (4.35)$$

– momentum is conserved. Note that energy is not, $E_i \neq E_n$ (or else the denominator is zero).

The other possibility is $|n_A\rangle = |p_1; p_\gamma, p_4\rangle$, which means μ_2 feels the effects of a photon it emitted, which is *later* absorbed by e_1 (!).

$$V_{ni}^A = \langle p_4 p_\gamma | V | p^2 \rangle = e\delta^d(\vec{p}_2 - \vec{p}_4 - \vec{p}_\gamma).$$

Altogether, to leading nonzero order,

$$T_{fi} = \sum_n V_{fn} \frac{1}{E_i - E_n} V_{ni} = \sum_{n=R,A} \int d^d p_\gamma \not{\epsilon}^d(p_1 - p_3 - p_\gamma) \not{\epsilon}^d(p_2 - p_4 + p_\gamma) \frac{e^2}{E_i - E_n}.$$

A bit of kinematics: let the ϕ have mass m_γ , so for a given \vec{p}_γ , $E_\gamma = \sqrt{|\vec{p}_\gamma|^2 + m_\gamma^2}$. Notice that these are *real* particles, they satisfy the equations of motion. For the R case, the intermediate energy is

$$E_n^R = E_3 + E_\gamma^R + E_2 = E_3 + \sqrt{|\vec{p}_1 - \vec{p}_3|^2 + m_\gamma^2} + E_2$$

so

$$E_i - E_n^R = E_1 + E_2 - (E_3 + E_\gamma^R + E_2) = E_1 - E_3 - E_\gamma^R = -\Delta E - E_\gamma$$

where $\Delta E \equiv E_3 - E_1 = E_2 - E_4$ (by overall energy conservation). Momentum conservation means $\vec{p}_\gamma = \vec{p}_2 - \vec{p}_4 = \vec{p}_3 - \vec{p}_1$ so

$$E_\gamma^A = \sqrt{|\vec{p}_1 - \vec{p}_3|^2 + m_\gamma^2} = \sqrt{|\vec{p}_2 - \vec{p}_4|^2 + m_\gamma^2} = E_\gamma^R \equiv E_\gamma.$$

Therefore

$$E_i - E_n^A = E_1 + E_2 - (E_1 + E_4 + E_\gamma) = +\Delta E - E_\gamma.$$

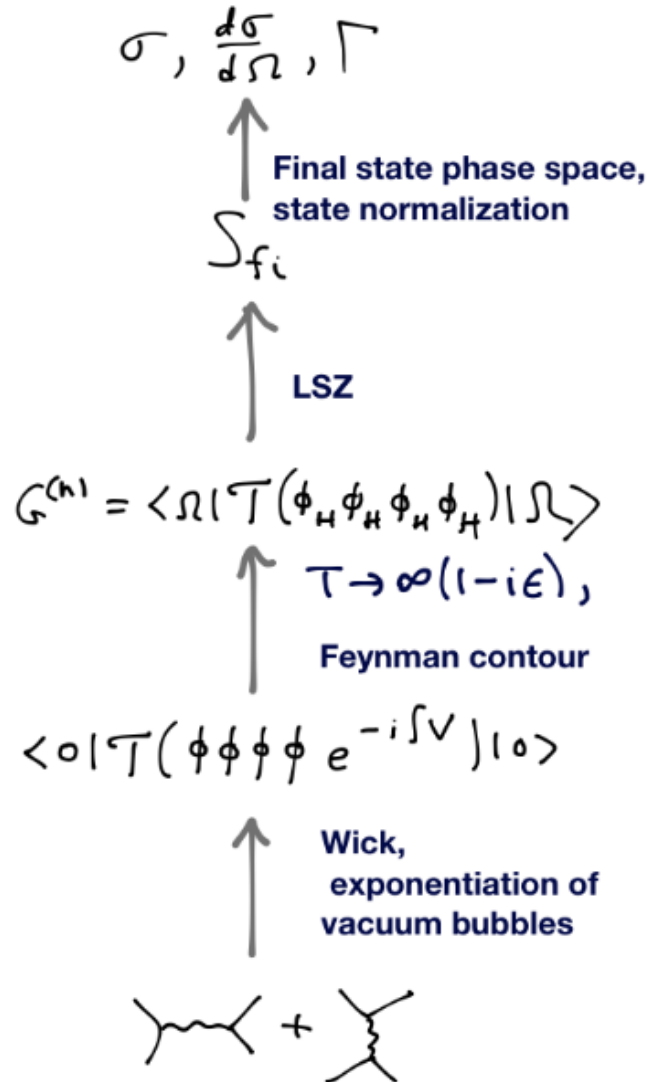
The sum of these factors is

$$\sum_{n=R,A} \frac{e^2}{E_i - E_n} = \frac{e^2}{-\Delta E - E_\gamma} + \frac{e^2}{\Delta E - E_\gamma} = \frac{2E_\gamma e^2}{\Delta E^2 - E_\gamma^2} = 2E_\gamma \frac{e^2}{k^2 - m_\gamma^2}.$$

Here we defined $k^\mu = p_3^\mu - p_1^\mu = (\Delta E, \vec{p}_\gamma)^\mu$, and $k^2 = k_\mu k^\mu$ is a Lorentz-invariant inner product. Ignoring the normalization factor $2E_\gamma$, this is the Lorentz-invariant momentum-space propagator for the ϕ particle *with four-momentum* k^μ . Notice that there is no actual particle with that four-momentum! It is a superposition of a real particle going forward in time and its (also real) antiparticle going backward in time. If we followed the $i\epsilon$ that would work out, too, to give the Feynman propagator.

4.7 S -matrix from Feynman diagrams

Here is a summary of the long logical route connecting Feynman diagrams to measurable quantities:



One step is left.


S -matrix from Feynman diagrams. Now we resume our inexorable progress towards observable physics (such as cross sections and lifetimes). The end result of the previous discussion is a prescription to compute S -matrix elements from Feynman diagrams. In a translation-invariant system, the S matrix always has a delta function outside of it. Also we are not so interested in the diagonal elements of the S matrix where nothing happens. So more useful than the S matrix itself are the scattering amplitudes \mathcal{M} defined by

$$\langle f | (S - \mathbb{1}) | i \rangle \equiv (2\pi)^{d+1} \delta^{(d+1)} \left(\sum_f p_f - \sum_i p_i \right) \mathbf{i} \mathcal{M}_{fi} . \quad (4.36)$$

(The object $\mathbf{i} \mathcal{M}_{\phi^{d+1}}(\sum p)$ is sometimes called the transfer matrix. The \mathbf{i} is a convention.)

The rules for the Feynman diagram calculation of \mathcal{M} (for ϕ^4 theory, as a representative example) are:

1. Draw all amputated diagrams with appropriate external nubbins for the initial and final states. For a diagram with N_L loops think of N_L letters that are like k or q or p to call the undetermined loop momenta.
2. For each vertex, impose momentum conservation and multiply by the coupling ($-\mathbf{i}\lambda$). Label each (oriented) edge with a momentum variable.
3. For each *internal* line, put a propagator.
4. For each loop, integrate over the associated momentum $\int \mathbf{d}^{d+1}k$.

A comment about rule 1: For tree-level diagrams (diagrams with no loops), ‘amputate’ just means leave off the propagators for the external lines. More generally, it means leave off the resummed propagator (4.29). For example, a diagram like  is already included by using the correct Z and the correct m . Note that in all of the diagrams to follow, the external lines are actually just nubbins, in this sense.

Example: snucleon scattering. [Here we follow Tong §3.5 very closely] Let’s return to the example with a complex scalar field Φ and a real scalar field ϕ with Lagrangian (4.10). Relative to ϕ^4 theory, the differences are: we have two kinds of propagators, one of which is oriented (because Φ is complex), and instead of a 4-point vertex that costs $-\mathbf{i}\lambda$, we have a 3-point vertex for $\phi\Phi^*\Phi$ that costs $-\mathbf{i}g$.

Let’s consider $2 \rightarrow 2$ scattering of Φ particles [recall HW 5 or see Tong §3.3.3 for the artisanal version of this calculation], so

$$|i\rangle = |\vec{p}_1, \vec{p}_2\rangle, |f\rangle = |\vec{p}_3, \vec{p}_4\rangle \quad \text{with} \quad |\vec{p}_i, \vec{p}_j\rangle \equiv \sqrt{2E_{\vec{p}_i}} \sqrt{2E_{\vec{p}_j}} \mathbf{b}_{\vec{p}_i}^\dagger \mathbf{b}_{\vec{p}_j}^\dagger |0\rangle .$$

The Feynman rules above give, to leading nonzero order in g ,

$$\begin{aligned}
 i\mathcal{M}_{\Phi\Phi\leftarrow\Phi\Phi} = & \quad \text{[Diagram 1]} + \text{[Diagram 2]} \\
 = & (-ig)^2 \left(\frac{i}{(p_1 - p_3)^2 - M^2 + i\epsilon} + \frac{i}{(p_1 - p_4)^2 - M^2 + i\epsilon} \right). \quad (4.37)
 \end{aligned}$$

The diagrams depict two ‘snucleons’ Φ (solid lines with arrows indicating snucleons versus antinucleons) exchanging a meson ϕ (double gray line, with no arrow) with momentum $k \equiv p_1 - p_3 = p_4 - p_2$ (first term) or $k \equiv p_4 - p_1 = p_2 - p_3$ (second term). Time goes to the left as always. Notice that here I am being careful about using arrows on the lines to indicate flow of particle number through the diagram, while the extra (light blue) arrows indicate momentum flow.

The two diagrams included in (4.37) make the amplitude symmetric under interchanging the two particles in the initial or final state, as it must be because they are indistinguishable bosons.

Some useful names for the Lorentz-invariant kinematic variables in a $2 \leftarrow 2$ process: the diagrams in (4.37) are called the t -channel and u -channel diagrams, according to the momentum flowing in the internal line. They are named after the Mandelstam variables $t \equiv (p_1 - p_3)^2$ and $u \equiv (p_1 - p_4)^2$. I claim that when p_1 and p_2 are on-shell ($p_1^2 = m^2 = p_2^2$), both t and u are strictly negative so that the $i\epsilon$ s in (4.37) can be ignored. There is one more Mandelstam variable for 2-to-2 scattering, $s \equiv (p_1 + p_2)^2$. They are not all independent, but satisfy $s + t + u = \sum_{i=1}^4 m_i^2$.

The meson in these diagrams is a “virtual”, or off-shell, particle in the sense that it does not satisfy its equation of motion $k^2 \neq M^2$. As we saw in §4.6, each of these diagrams is actually the sum of retarded and advanced exchange of real on-shell particles. I think the notion of virtual particle is just a book-keeping device and should not be taken too seriously. Notice that the more virtual is the intermediate state (*i.e.* the more k^2 differs from M^2), the smaller is the amplitude.

Mediation of forces. Consider the non-relativistic (NR) limit of the snucleon-snucleon scattering amplitude (4.37). In the center-of-mass frame $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$ and $\vec{p}' \equiv \vec{p}_3 = -\vec{p}_4$. In the NR limit, $|\vec{p}| \ll m$, and so $p_1^0 = m(1 + \frac{1}{2}(\frac{|\vec{p}|}{m})^2 + \dots)$. Energy-momentum conservation says $p_1 + p_2 = p_3 + p_4$, so $|\vec{p}'| = |\vec{p}| \ll m$ as well. In this limit, the meson propagator (in the first diagram) depends on $(p_1 - p_3)^2 =$

$(p_1^0 - p_3^0)^2 - (\vec{p} - \vec{p}')^2 \simeq -(\vec{p} - \vec{p}')^2$, so the amplitude reduces to

$$\mathbf{i}\mathcal{M} = +\mathbf{i}g^2 \left(\frac{1}{(\vec{p} - \vec{p}')^2 + M^2} + \frac{1}{(\vec{p} + \vec{p}')^2 + M^2} \right).$$

Now compare to NR QM. The scattering amplitude in the COM frame for two particles with relative position \vec{r} and potential $U(\vec{r})$ is, in the first Born approximation,

$$\mathbf{i}\mathcal{A}_{\text{Born}}(\vec{p} \rightarrow \vec{p}') = -\mathbf{i} \text{NR} \langle \vec{p}' | U(\vec{r}) | \vec{p} \rangle_{\text{NR}} = -\mathbf{i} \int d^d r U(\vec{r}) e^{-\mathbf{i}(\vec{p} - \vec{p}') \cdot \vec{r}}$$

where the two-particle state with NR normalization is

$$|\vec{p}\rangle_{\text{NR}} = \frac{1}{\sqrt{2E_1}\sqrt{2E_2}} |p_1, p_2\rangle = \frac{1}{2m} |p_1, p_2\rangle.$$

The two diagrams in the relativistic answer come from Bose statistics, which means we can't distinguish $\vec{p} \rightarrow \pm\vec{p}'$ from each other; to infer the potential we can just compare the first diagram, $(2m)^2 \mathbf{i}\mathcal{A}_{\text{Born}}(\vec{p} \rightarrow \vec{p}') = +\mathbf{i}g^2 \frac{1}{(\vec{p} - \vec{p}')^2 + M^2}$ to find:

$$\int d^d r U(\vec{r}) e^{-\mathbf{i}(\vec{p} - \vec{p}') \cdot \vec{r}} = -\mathcal{M}_{\text{NR limit}} = -\frac{\left(\frac{g}{2m}\right)^2}{(\vec{p} - \vec{p}')^2 + M^2}$$

which means, in $d = 3$, (see footnote 33 for the integral)

$$U(\vec{r}) = -\frac{\left(\frac{g}{2m}\right)^2}{4\pi r} e^{-Mr}.$$

This is the Yukawa potential again. It has a range, M^{-1} , determined by the mass of the exchanged particle. If we take $M \rightarrow 0$, it becomes the Coulomb potential. The sign means that it is attractive, even though this is the potential between particle and particle; this is a general consequence of scalar exchange. Notice that in $d = 3$, the Yukawa coupling between scalars has $0 \stackrel{!}{=} [\int d^4 x g \phi \Phi^2] = -4 + 3 + [g]$ so g/m is dimensionless.

A brief warning: while it is satisfying to make contact with something familiar here, the way we actually measure any such potential is by scattering the particles and measuring cross-sections.

[End of Lecture 14]

Two more examples with the same ingredients are useful for comparison. If we instead scatter a nucleon and an *anti-nucleon*, so $|i\rangle = \sqrt{2E_{\vec{p}_1}} \sqrt{2E_{\vec{p}_2}} \mathbf{b}_{\vec{p}_1}^\dagger \mathbf{c}_{\vec{p}_2}^\dagger |0\rangle$, then

the leading diagrams are

$$\begin{aligned}
 i\mathcal{M}_{\Phi\bar{\Phi}\leftarrow\Phi\bar{\Phi}} &= \text{Diagram 1} + \text{Diagram 2} \\
 &= (-ig)^2 \left(\frac{i}{(p_1 + p_2)^2 - M^2 + i\epsilon} + \frac{i}{(p_1 - p_3)^2 - M^2 + i\epsilon} \right). \quad (4.38)
 \end{aligned}$$

This one has a new ingredient: in the first diagram, (which is called the s -channel diagram after $s \equiv (p_1 + p_2)^2 = E_{\text{CoM}}^2$) the meson momentum is $k = p_1 + p_2$, which *can* be on-shell, and the $i\epsilon$ matters. This will produce a big bump, a *resonance*, in the answer as a function of the incoming center-of-mass energy $\sqrt{s} \equiv \sqrt{(p_1 + p_2)^2}$.

Finally, we can scatter a meson and a nucleon:

$$\begin{aligned}
 i\mathcal{M}_{\Phi\phi\leftarrow\Phi\phi} &= \text{Diagram 1} + \text{Diagram 2} \\
 &= (-ig)^2 \left(\frac{i}{(p + k)^2 - m^2 + i\epsilon} + \frac{i}{(p - k')^2 - m^2 + i\epsilon} \right). \quad (4.39)
 \end{aligned}$$

Now the intermediate state is a nucleon.

4.8 From the S -matrix to observable physics

Now, finally, we extract some physics that can be measured from all the machinery we've built.

Lifetimes. [Schwartz, chapter 4; Peskin §4.5] How do we compute the lifetime of an unstable particle in QFT? Consider such a particle in its rest frame, $p^\mu = (M, \vec{0})^\mu$. Let dP be the probability that the particle decays (into some set of final states f) during a time T . The *decay rate* is then $d\Gamma \equiv \frac{1}{T}dP$, the probability per unit time. I put a $d\Gamma$ to indicate a *differential* decay rate into some particular set of final states. If we sum over all possible final states, we can make a practical, frequentist definition of the decay rate, with the idea that we have a big pile of particles and we just count how many go away in some time window:

$$\Gamma \equiv \frac{\# \text{ of decays per unit time}}{\# \text{ of particles}} \equiv \frac{1}{\tau} \quad (4.40)$$

where τ is the lifetime.

Fortunately for us, particles that are stable in the free theory can decay because of weak interactions; in such a case, we can relate dP to an S matrix element for a process that takes one particle to n particles, $\langle f | \hat{S} | i \rangle = S_{n \leftarrow 1} \left(\{p_j\}_{j=1}^n \leftarrow (M, \vec{0}) \right)$. (To get the full decay rate Γ we would have to sum over n and the possible final states of each particle.) So:

$$d\Gamma \equiv \frac{1}{T} dP = \frac{1}{T} \frac{|\langle f | S | i \rangle|^2}{\langle f | f \rangle \langle i | i \rangle} d\Pi_f \quad (4.41)$$

Here are two new ingredients:

- (1) $d\Pi_f$ is the volume of the region of final-state phase space, $d\Pi_f \propto \prod_{j=1}^n d^d p_j$. We are allowing, as we must, for imperfect measurements. We will normalize the density of final states so that $\int d\Pi = 1$. Putting back the IR and UV walls of our padded room as in (1.4), we take the continuum limit ($N \rightarrow \infty$) of

$$x_i = \frac{i}{N} L, \quad p^i = \frac{2\pi}{L} \frac{i}{N}, \quad i = 1 \cdots N$$

which requires, for each spatial dimension,

$$\Delta x \sum_i = \frac{L}{N} \sum_i \xrightarrow{N \rightarrow \infty, L \text{ fixed}} \int dx = L \quad \text{and} \quad \frac{1}{2\pi} \Delta p \sum_i = \frac{1}{2\pi} \frac{2\pi}{LN} \sum_i \xrightarrow{N \rightarrow \infty, L \text{ fixed}} \int dp = L^{-1}.$$

This gives

$$d\Pi = \prod_{j=1}^n V d^d p_j,$$

a factor of the volume of space $V = L^d$ for each final-state particle.

- (2) The normalization factors $\langle f | f \rangle \langle i | i \rangle$ are not so innocent as they look, partly because of our relativistic state normalization. Recall that $|\vec{p}\rangle = \sqrt{2\omega_{\vec{p}}} \mathbf{a}_{\vec{p}}^\dagger |0\rangle$, the price for the relativistic invariance of which is

$$\begin{aligned} \langle \vec{k} | \vec{p} \rangle &= \sqrt{2\omega_{\vec{p}} 2\omega_{\vec{k}}} \cdot \underbrace{\langle 0 | \mathbf{a}_{\vec{k}} \mathbf{a}_{\vec{p}}^\dagger | 0 \rangle}_{= \langle 0 | [\mathbf{a}_{\vec{k}}, \mathbf{a}_{\vec{p}}^\dagger] | 0 \rangle = \delta^d(\vec{k} - \vec{p})} = 2\omega_{\vec{p}} \delta^d(\vec{p} - \vec{k}) \end{aligned}$$

Therefore,

$$\langle \vec{p} | \vec{p} \rangle = 2\omega_{\vec{p}} \delta^d(0) = 2\omega_{\vec{p}} \left(\int dx e^{i(p=0)x} \right)^d = 2\omega_{\vec{p}} V.$$

Therefore, for a $1 \rightarrow n$ process in the rest frame of the initial particle,

$$|i\rangle = \sqrt{2M} \mathbf{a}_0^\dagger |0\rangle \quad \implies \quad \langle i | i \rangle = 2MV$$

$$|f\rangle = |\{\vec{p}_j\}\rangle \quad \Longrightarrow \quad \langle f|f\rangle = \prod_j^n (2\omega_j V) \quad (4.42)$$

where I've abbreviated $\omega_j \equiv \omega_{\vec{p}_j}$ and evaluated things in the rest frame of the decaying particle.

Now it is time to square the quantum amplitude

$$\langle f|(S - \mathbb{1})|i\rangle = \mathbf{i}\delta^{d+1}(p_T) \langle f|\mathcal{M}|i\rangle$$

($p_T = \sum p_i - \sum p_f$ is the total momentum change) to get the probability (4.41). Again we encounter a δ^2 , and again we use $(2\pi)^{d+1}\delta^{d+1}(0) = TV$. So we have

$$|\langle f|(S - \mathbb{1})|i\rangle|^2 = \delta^{d+1}(0)\delta^{d+1}(p_T) |\langle f|\mathcal{M}|i\rangle|^2 = VT\delta^{d+1}(p_T)|\mathcal{M}|^2$$

and therefore (as long as $f \neq i$),

$$\begin{aligned} dP &= TV\delta^{d+1}(p_T) \frac{1}{2MV \prod_j^n (2\omega_j V)} |\mathcal{M}|^2 \prod_j^n V d^d p_j \\ &= \frac{T}{2M} |\mathcal{M}|^2 d\Pi_{LI} \end{aligned} \quad (4.43)$$

where all the factors of V went away (!), and

$$d\Pi_{LI} \equiv \prod_{\text{final state}, j} \frac{d^d p_j}{2\omega_j} \delta^{d+1}(p_T)$$

is a Lorentz-invariant measure on the allowed final-state phase space. You can see that this is indeed Lorentz-invariant by the same calculation that led us to stick those $2\omega_j$ s in the states. One more step to physics:

$$d\Gamma = \frac{1}{T} dP = \frac{1}{T} \frac{T}{2M} |\mathcal{M}|^2 d\Pi_{LI} = \underbrace{|\mathcal{M}|^2}_{\text{dynamics}} \underbrace{\frac{1}{2M} d\Pi_{LI}}_{\text{kinematics}}$$

$$\boxed{d\Gamma = |\mathcal{M}|^2 \frac{1}{2M} d\Pi_{LI}}.$$

On the RHS is all stuff we know how to calculate (recall the Feynman rules for \mathcal{M} that we listed after (4.36)), and on the LHS is a particle decay rate.

The boxed formula gives the decay rate in the rest frame of the unstable particle. In other frames, the lifetime gets time-dilated. This must be true on general grounds

of special relativity, but we can see this directly: in a general frame, the normalization of the initial state is not $\langle i|i \rangle_{\text{rest frame}} = 2M$ but $\langle i|i \rangle = 2E$. Therefore

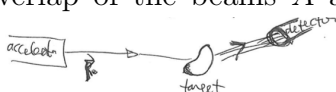
$$\frac{\Gamma^{\text{rest frame}}}{\Gamma} = \frac{E}{M} = \gamma \geq 1$$

and $\tau = \tau^{\text{rest frame}} \gamma \geq \tau^{\text{rest frame}}$.

Cross sections. If we are not in the convenient situation of having in our hands a big pile of particles that are stable in the free theory and decay because of not-too-strong interactions, we need to be more proactive to get physics to come out: we have to smash the particles together. When doing this, we send *beams* of particles at each other and see what comes out. We will treat these beams as perfectly collimated momentum eigenstates; if something goes wrong, we'll make a more accurate representation and put them in better-localized wavepackets. A quantity that is good because it is intrinsic to the particles composing the beams is the *scattering cross section*, σ , defined by

$$\text{Number of events of interest} \equiv \frac{N_A N_B}{A} d\sigma$$

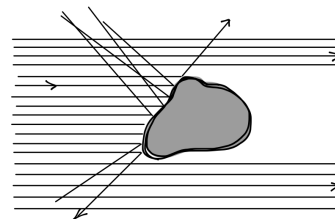
where A is the common area of overlap of the beams A and B , and $N_{A,B}$ are the number of particles in each beam.



(Peskin does a bit more

worrying at this point, for example, about whether the beams have constant density of particles.) By ‘events of interest’ I mean for example those particles that end up going in a particular direction, for example in a solid angle $d\Omega(\theta, \varphi)$. Restricting to events of interest in a particular direction gives the *differential cross section*, $\frac{d\sigma}{d\Omega}$. The notation is motivated by the idea that $\sigma = \int_{\text{all directions}} d\Omega \frac{d\sigma}{d\Omega}$.

The cross-section is the effective cross-sectional area of the beam taken out of the beam and put into the particular state of interest. Here is a picture (adapted from Schwartz’ book) which I think makes vivid the idea behind the definition of a cross section:



Now we relate σ to the S -matrix. The scattering rate $dw_{fi} \equiv \frac{dP_{fi}}{T}$ is the scattering probability per unit time, for some fixed initial and final particle states. In a beam, this is related to the cross section by

$$d\sigma = \frac{dw}{j} \tag{4.44}$$

where j is the particle current density (aka the particle flux), which for the case of scattering from an initial state with two particles $A + B \rightarrow \dots$ is

$$j = \frac{\text{relative velocity of } A \text{ and } B}{\text{volume}} = \frac{|\vec{v}_A - \vec{v}_B|}{V}.$$

The number of particles in each beam does not appear in (4.44) because the BHS is intensive, and this is the point of introducing σ : it's independent of details of the experiment like how many opportunities the particles have to interact. Putting together these statements, we can relate the cross section to the scattering probability:

$$d\sigma = \frac{dw_{fi}}{j} = \frac{dP_{fi}}{Tj} = \frac{V}{T} \frac{1}{|\vec{v}_A - \vec{v}_B|} dP_{fi}. \quad (4.45)$$

In (4.45) we used a practical frequentist expression for (4.44), analogous to (4.40) for decay rates. And just as in the discussion of lifetimes above,

$$dP_{fi} = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} d\Pi_f.$$

Everything is as before except for the different initial state:

$$|i\rangle = |\vec{p}_A, \vec{p}_B\rangle \quad \implies \quad \langle i|i\rangle = (2\omega_A V)(2\omega_B V).$$

Squaring the amplitude gives

$$dP = \frac{T}{V} \frac{1}{2\omega_A 2\omega_B} |\mathcal{M}|^2 d\Pi_{LI};$$

the only difference is that we replace $\frac{1}{2M}$ with the factors for the 2-particle initial state. Finally, $d\sigma = \frac{V}{T} \frac{1}{|\vec{v}_A - \vec{v}_B|} dP$ gives

$$\boxed{d\sigma = \frac{1}{2\omega_A 2\omega_B} \frac{1}{|\vec{v}_A - \vec{v}_B|} |\mathcal{M}|^2 d\Pi_{LI}}.$$

Again all the IR-divergent factors of V and T went away in the intrinsic physical quantity, as they must.

Two-body phase space. [Schwartz §5.1] To make the preceding formulae more concrete, let's simplify them for the case of $n = 2$: two particles in the final state, whose momenta we'll call p_1, p_2 . Note that overall momentum conservation implies $p_1 + p_2 = p_{CM}$; we can use this to eliminate p_2 . In that case

$$\begin{aligned} d\Pi_{LI} &= \delta^{d+1}(p_T) \frac{d^d p_1 d^d p_2}{2E_1 2E_2} & E_i &\equiv \sqrt{|p_i|^2 + m_i^2} \\ &= \frac{1}{4(2\pi)^{2d-(d+1)}} \frac{1}{E_1 E_2} \delta(E_1 + E_2 - E_{CM}) \underbrace{d^d p_1}_{=d^{d-1}\Omega p_1^{d-1} dp_1} & p_1 &\equiv |p_1| > 0 \\ &= \frac{1}{4(2\pi)^{d-1}} \frac{d^{d-1}\Omega p_1^{d-1} dp_1}{E_1 E_2} \theta(p_1) \delta(x) & x(p_1) &\equiv E_1(p_1) + E_2(p_2 = p_{CM} - p_1) - E_{CM} \\ &\stackrel{\text{CoM}}{=} \frac{1}{4(2\pi)^{d-1}} \frac{d^{d-1}\Omega p_1^{d-2}}{E_1 E_2} \frac{E_1 E_2}{E_{CM}} \underbrace{dx \delta(x)}_{=1} \theta(E_{CM} - m_1 - m_2) & dp_1 &= \frac{dp_1}{dx} dx \stackrel{\text{CoM}}{=} \frac{E_1 E_2}{E_1 + E_2} \frac{dx}{p_1} \end{aligned}$$

In the last step, we went to the center-of-momentum frame where $\vec{p}_{CM} = 0$, and we used the fact that $p_1 \geq 0$ means $E_1(p_1) \geq m_1, E_2(p_2 = p_{CM} - p_1) \geq m_2$.

2 → 2 scattering in $d = 3$. Let's be completely explicit in the special case of 3 spatial dimensions, which is sometimes of interest. Let the initial momenta be k_A, k_B . In particular, the relative velocity factor is

$$|\vec{v}_A - \vec{v}_B|_{\text{CoM: } \vec{k}_A = -\vec{k}_B} = |k_A| \frac{E_{CM}}{E_{k_A} E_{k_B}} \left(\frac{|k_A|}{E_{k_A}} + \frac{|k_B|}{E_{k_B}} \right)$$

Therefore

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{CoM}} = \frac{1}{64\pi^2 E_{CM}^2} \frac{|\vec{p}_1|}{|\vec{k}_A|} |\mathcal{M}|^2 \theta(E_{CM} - m_1 - m_2). \quad (4.46)$$

Warning: for identical particles in the final state, one must be careful about over-counting in the integral over angles, since a rotation by π exchanges them. In this case $\sigma = \frac{1}{2} \int_{4\pi} d\Omega \frac{d\sigma}{d\Omega}$.

4.9 Why care about G_R ?

[Hartnoll, 0903.3246, §3.1; Fradkin, *QFT*, §10.9] From discussions of QFT in particle physics, you might be hard-pressed to understand why in the world anyone would care about real-time Green's functions other than the Feynman Green's function. Here I would like to try to ameliorate that situation. I will describe another way to get observable physics from QFT.

Recall that the retarded Green's function for two observables \mathbf{A} and \mathbf{B} is

$$G_{\mathbf{A}\mathbf{B}}^R(\omega, \vec{k}) = -\mathbf{i} \int d^{D-1}x dt e^{i\omega t - i\vec{k}\cdot\vec{x}} \theta(t) \langle [\mathbf{A}(t, \vec{x}), \mathbf{B}(0, 0)] \rangle$$

$\theta(t) = 1$ for $t > 0$, else zero. As I will explain now, we care about this because it determines what $\langle \mathbf{A} \rangle$ does if we kick the system via \mathbf{B} .

What I mean by 'kick the system via \mathbf{B} ' is perturb the Hamiltonian. Consider a time dependent perturbation to the Hamiltonian:

$$\delta H(t) = \int d^{D-1}x f(t, \vec{x}) \mathbf{B}(\vec{x}).$$

And we'll be interested in the resulting change in the expectation value of \mathbf{A} :

$$\begin{aligned} \langle \mathbf{A}(\vec{x}) \rangle(t) &\equiv \text{Tr } \rho(t) \mathbf{A}(\vec{x}) \\ &= \text{Tr } \rho_0 U^{-1}(t) \mathbf{A}(t, \vec{x}) U(t). \end{aligned}$$

Here we can take the density matrix ρ_0 to be $\rho_0 = e^{-\beta H_0}/Z$, the thermal state, or its limit when $\beta \rightarrow \infty$, $\lim_{\beta \rightarrow \infty} e^{-\beta H}/Z = |\Omega\rangle\langle\Omega|$, the projector onto the groundstate. In interaction picture with $H_{\text{total}} = H + \delta H(t)$, the time evolution operator is $U(t) = \mathcal{T}e^{-i\int^t \delta H(t')dt'}$, and $\mathbf{A}(t, \vec{x})$ evolves by the unperturbed Hamiltonian, H . (Note that H may be of the form $H_0 + V$ with H_0 solvable, but this assumption is not necessary for what follows.)

But suppose that δH is small, so we can linearize in the small perturbation:

$$\begin{aligned}\delta\langle\mathbf{A}\rangle(t, \vec{x}) &= -i\text{Tr} \rho_0 \int^t dt' [\mathbf{A}(t, \vec{x}), \delta H(t')] \\ &= -i \int^t d^{D-1}x' dt' \langle [\mathbf{A}(t, \vec{x}), \mathbf{B}(t', \vec{x}')] \rangle f(t', \vec{x}') \\ &= \int d^D x' G_{\mathbf{AB}}^R(x, x') f(x') .\end{aligned}$$

Fourier transforming the preceding equation, we find

$$\delta\langle\mathbf{A}\rangle(\omega, \vec{k}) = G_{\mathbf{AB}}^R(\omega, \vec{k}) f(\omega, \vec{k}) . \quad (4.47)$$

These equations are sometimes written in terms of the *susceptibility matrix* χ_{AB}

$$\delta\langle\mathbf{A}\rangle = \chi_{AB} f$$

(in position space this is a convolution, and in momentum space it is just a product).

Linear response, an example. Consider a system with a conserved charge. Take as the perturbation an external electric field, $E_x = i\omega A_x$, which couples via $\delta H = A_x \mathbf{J}^x$ where J is the electric current. That is: $\mathbf{B} = \mathbf{J}_x$.

And we'll be interested in the response of the electric current, so $\mathbf{A} = \mathbf{J}_x$ as well. It's safe to assume $\langle\mathbf{J}\rangle_{E=0} = 0$ (this is called Bloch's Other Theorem), so (4.47) specializes to

$$\langle\mathbf{J}(\omega, \vec{k})\rangle = G_{\mathbf{JJ}}^R(\omega, \vec{k}) A_x(\omega, \vec{k}) = G_{\mathbf{JJ}}^R(\omega, \vec{k}) \frac{E_x(\omega, \vec{k})}{i\omega}$$

Recall that Ohm's law $J = \sigma E$ defines the conductivity σ . (Really it is a tensor since J and E are both vectors, but don't worry about that now.) Therefore we have the

$$\text{Kubo formula: } \boxed{\sigma(\omega, \vec{k}) = \frac{G_{\mathbf{JJ}}^R(\omega, \vec{k})}{i\omega}} .$$

I have glossed over some important subtleties in this derivation in order to quickly convey the idea; for a more complete treatment, please see Fradkin §10.10.

There is a lot more to say about the analytic structure of G_R (because of causality (the factor of $\theta(t)$), it is analytic in the upper-half frequency plane) and its relation to the time-ordered Green's function. In fact, they are both determined by the same information, namely the spectral density (for which see *e.g.* [here](#)).

[End of Lecture 15]

5 Spinor fields and fermions

[Peskin chapter 3] Now we need to confront the possibility of fields that transform in more interesting ways under Lorentz transformations. To do so, let's back up a bit.

5.1 More on symmetries in QFT

Lightning summary of group theory.⁴⁸ A *group* $G = \{g_i\}$ is a set of abstract elements,

1. two of which can be multiplied to give a third $g_1g_2 \in G$.
2. The product is associative $(g_1g_2)g_3 = g_1(g_2g_3)$
3. and has an identity element $g_0g_i = g_i$ for all g_i
4. and every element has an inverse, $\forall i, \exists g_i^{-1}$ such that $g_i g_i^{-1} = g_0$.

The *order* of G , denoted $|G|$, is the number of elements of the group. G is *abelian* if the product is commutative.

A *Lie group* is a group whose elements depend smoothly on continuous parameters, $g(\theta)$. These then provide local coordinates on the group manifold. The dimension of a Lie group is the number of coordinates (to be distinguished from $|G|$, which is continuously infinite for a Lie group).

A (linear) *representation* R of a group assigns to each abstract element g of the group a linear operator $\hat{D}_R(g)$ on some vector space \mathcal{H} , $R : g \mapsto \hat{D}_R(g)$ in a way that respects the group law (it is a group homomorphism): meaning that $\hat{D}_R(g_0) = \mathbb{1}$ and $\hat{D}_R(g_1)\hat{D}_R(g_2) = \hat{D}_R(g_1g_2)$.⁴⁹ If we choose a basis of the vector space, then $\hat{D}_R(g)$ determines a matrix. Two representations R and R' are regarded as the same $R \simeq R'$ if they are related by a change of basis on \mathcal{H} , $D_R(g) = S^{-1}D_{R'}(g)S$ (with S independent of g !). A *reducible representation* is one for which the matrices can be made block diagonal by a basis change. A reducible representation is equivalent to $R_1 \oplus R_2 \oplus \dots$ a

direct sum of irreducible representations, $D_R \simeq \begin{pmatrix} D_{R_1} & 0 & & \\ 0 & D_{R_2} & & \\ & & \ddots & \end{pmatrix}$.

⁴⁸This is a big subject, which I'm going to try to present efficiently. A whole quarter on the subject is [here](#).

⁴⁹Since the overall phase of a vector in \mathcal{H} is unphysical, quantum mechanics allows for *projective representations* where the group law is only satisfied up to phases. We'll see an example below.

The dimension of R is the dimension of \mathcal{H} as a vector space. Notice that different representations of the same group G can have different dimensions!

What properties of G are inherent in all of its representations? For the case of Lie groups, one answer is the *Lie algebra* relations. Consider a (say n -dimensional) representation of a group element near the identity (which let's label the identity element $g_0 \equiv e \equiv g(\theta = 0)$ by the coordinate value $\theta = 0$):

$$D_R(g(\theta \sim 0)) = \mathbb{1} + \mathbf{i}\theta_a T_R^a + \mathcal{O}(\theta^2), \quad \text{i.e. } T_R^a \equiv -\mathbf{i}\partial_{\theta_a} D_R(g(\theta))|_{\theta=0}$$

where T_R^a are the *generators* of G in the representation R . In a basis for the vector space, they are $n \times n$ matrices.

The generators T^a determine a basis of the tangent space of G at the identity, $T_e G$ (or equivalently, by the group action, at any other point). A finite transformation (in the component of the Lie group that is continuously connected to the identity element) can be written as

$$D_R(g(\theta)) = e^{-\mathbf{i}\theta_a T_R^a}$$

which is unitary if $T = T^\dagger$.

Given two such elements $D_R(g(\theta_1)) = e^{-\mathbf{i}\theta_1^a T_R^a}$ and $D_R(g(\theta_2)) = e^{-\mathbf{i}\theta_2^a T_R^a}$, their product must give a third:

$$D_R(g_1)D_R(g_2) = D_R(g_1g_2) = e^{-\mathbf{i}\theta^a T_R^a} \quad (5.1)$$

for some θ^3 . Expanding the log of the BHS of (5.1) to second order in the θ s (see Maggiore chapter 2.1 for more detail), we learn that we must have

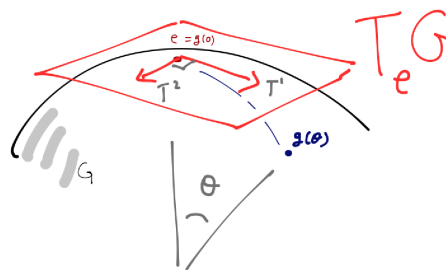
$$\theta_a^3 = \theta_a^1 + \theta_a^2 - \frac{1}{2}\theta_b^1\theta_c^2 f^{bc}_a + \mathcal{O}(\theta)^3$$

which implies that

$$[T^a, T^b] = \mathbf{i}f_c^{ab}T^c \quad (5.2)$$

which relation is called the Lie algebra \mathfrak{g} of G , and the f s are called *structure constants* of \mathfrak{g} or G . f does not depend on the representation. For those of you comfortable with differential geometry, an easy way to see this is that the commutator is the Lie bracket between two tangent vectors (which gives another tangent vector). Note that the normalization of the T^a is ambiguous, and rescaling T rescales f . A common convention is to choose an orthonormal basis

$$\text{tr}T^a T^b = \frac{1}{2}\delta^{ab}. \quad (5.3)$$



Notice that we often use lowercase letters to denote the algebra \mathfrak{g} and uppercase letters to denote the group G , which is natural since the algebra generates small group transformations. The Lie algebra is defined in the neighborhood of the identity element, but by conjugating by finite transformations, the tangent space to any point on the group has the same structure, so the Lie algebra determines the local structure everywhere in the group. But it doesn't know about global, discrete issues, like disconnected components, so different groups can have the same Lie algebra.

Any set of matrices obeying (5.2) provides a *representation* of the Lie algebra. By exponentiating them, we get a representation of the Lie group. This is the best way to think about representations of Lie groups.

A *casimir* of the algebra is an operator made from the generators that commutes with all of them. Acting on an irreducible representation (\equiv one which is not reducible \equiv irrep), where all the states can be made from each other by the action of products of generators, it is proportional to the identity.

Example: representations of the rotation group. This will be a fancy packaging of familiar stuff which will make the step to Lorentz transformations painless (I hope). Recall from QM that generators of rotations about the axes $x, y, z = 1, 2, 3$, $\mathbf{J}^{i=1,2,3}$, satisfy the Lie algebra $\mathfrak{so}(3) = \mathfrak{su}(2)$:

$$[\mathbf{J}^i, \mathbf{J}^j] = \mathbf{i}\epsilon^{ijk}\mathbf{J}^k. \quad (5.4)$$

So the structure constants are $f_k^{ij} = \epsilon^{ijl}\delta_{lk}$. A Casimir of this algebra (meaning that it commutes with all the generators) is $\mathbf{J}^2 = \sum_i (\mathbf{J}^i)^2$, which acts as $j(j+1)\mathbb{1}$ on the spin- j representation, whose dimension is $2j+1$, any non-negative integer. A finite rotation on \mathcal{H} is

$$D(\hat{n}, \theta) = e^{-i\theta\hat{n}\cdot\mathbf{J}}$$

where \hat{n} is a unit vector and θ is an angle, so three real parameters. Familiar matrix solutions of (5.4) are its action on vectors, where the generators are 3×3 matrices:

$$(J_{(j=1)}^i)^j{}_k = -\mathbf{i}\epsilon^{ijk}$$

(with $\epsilon^{123} = 1$) and its $2d$ representation on the Hilbert space of a spin- $\frac{1}{2}$ object:

$$J_{(j=\frac{1}{2})}^i = \frac{1}{2}\sigma^i.$$

Also, its one-dimensional representation, on a scalar, has $J_{(j=0)}^i = 0$, so $e^{i\theta J_{(j=0)}} = \mathbb{1}$.

More generally, the $2j+1$ dimensional representation is $D_{(j)}(\theta) = e^{-i\theta^a J^a}$ with

$$(J^3)_{mm'} = \delta_{mm'}m, \quad (J^\pm)_{mm'} \equiv (J^1 \pm \mathbf{i}J^2)_{mm'} = \delta_{m',m\pm 1}\sqrt{(j \mp m)(j \pm m + 1)}, \quad (5.5)$$

with the basis labels taking the $2j + 1$ values $m, m' \in \{-j, -j + 1 \cdots j - 1, j\}$. You can check that these matrix elements reduce to the above three examples for $j = 0, 1/2, 1$.

Notice that the rotation algebra (5.4) is the statement that \mathbf{J}^i itself transforms as a vector ($j = 1$) under infinitesimal rotations. What I mean by this is: the action of G on \mathcal{H} by $|\psi\rangle \rightarrow D_R |\psi\rangle$ implies an action on linear operators on \mathcal{H} by $\mathcal{O} \mapsto D_R \mathcal{O} D_R^\dagger$. Relabelling the reference axes x, y, z that we used to label \mathbf{J}^i by a rotation g produces a rotation by the same angle in the 3d representation. The reference axes themselves transform in the spin-1 representation:

$$(D_{(j=1)}(g))_j^k \mathbf{J}^j = D_R(g)^\dagger \mathbf{J}^k D_R(g), \quad (5.6)$$

the infinitesimal version of which is (5.4) (or maybe its complex conjugate). More generally, the equation

$$[J^i, K^j] = \mathbf{i} \epsilon^{ijk} K^k$$

is the statement that K transforms as a vector.

O(d) for general d . Some of what I have said so far about rotations is special to rotations in $d = 3$. In particular, the notion of “axis of rotation” is ($d = 3$)-centric. More generally, a rotation is specified by a (2d) *plane* of rotation; in $d = 3$ we can specify a plane by its normal direction, the one that’s left out $J^i \equiv \frac{1}{2} \epsilon^{ijk} J^{jk}$. Notice that only the bit of J^{jk} antisymmetric in $j \leftarrow k$ appears here, so we assume it is antisymmetric. In terms of these generators, the $\mathfrak{so}(3)$ lie algebra is (using ϵ identities)

$$[J^{ij}, J^{kl}] = \mathbf{i} (\delta^{jk} J^{il} + \delta^{il} J^{jk} - \delta^{ik} J^{jl} - \delta^{jl} J^{ik}). \quad (5.7)$$

The RHS here must be antisymmetric under exchange $i \leftarrow j$ or $k \leftarrow l$ or $ij \leftarrow kl$, and is largely determined by these properties. The vector representation is

$$\left(J_{(1)}^{ij} \right)_l^k = -\mathbf{i} (\delta^{ik} \delta_l^j - \delta^{jk} \delta_l^i) \quad (5.8)$$

(that is, there is a $-\mathbf{i}$ in the ij entry and an \mathbf{i} in the ji entry and zeros everywhere else). In $d = 3$, the spinor representation is

$$J_{(\frac{1}{2})}^{ij} = \epsilon^{ijk} \frac{1}{2} \sigma^k = \frac{\mathbf{i}}{4} [\sigma^i, \sigma^j]. \quad (5.9)$$

Define the group $O(d)$ by its action on the d -dimensional vector representation: $d \times d$ real matrices \mathcal{O} preserving lengths of vectors under $n_i \mapsto \mathcal{O}_i^j n_j$: $|\mathcal{O}n|^2 \stackrel{!}{=} |n|^2 \equiv n^i n^j \delta_{ij}, \forall n^i$:

$$\mathcal{O}^t \mathcal{O} = \mathbb{1} \quad \text{or, in a basis,} \quad (\mathcal{O}^t)_i^j \delta_{jk} \mathcal{O}^k_l = \delta_{il}. \quad (5.10)$$

In words: $O(d)$ transformations preserve the bilinear form δ_{ij} . Looking in the connected component with the identity⁵⁰,

$$\mathcal{O} = e^{-i\theta^{ij}J^{ij}}, \quad (5.11)$$

(5.10) implies that the generators J^{ij} , $i, j = 1..d$ satisfy (5.7). Reality of \mathcal{O} for all θ^{ij} says that for each ij , J^{ij} is a pure imaginary matrix, and $\mathcal{O}^t\mathcal{O} = \mathbb{1}$ says that each J^{ij} is an antisymmetric matrix: $(J^{ij})^t = -J^{ij}$, or more explicitly, $(J^{ij})_{kl} = -(J^{ij})_{lk}$. There is a $\frac{d(d-1)}{2}$ -dimensional space of such things, and one good basis is given by (5.8) for general d . This agrees with the $d = 3$ case above where there are $\frac{3 \cdot 2}{2} = 3$ such generators.

A special case is $SO(2)$ where the one generator is $T = \begin{pmatrix} 0 & -\mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} = \sigma^2$, and the finite transformation is

$$e^{i\beta T} = \mathbb{1} \cos \beta + \mathbf{i}\sigma^2 \sin \beta = \begin{pmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{pmatrix}.$$

As we reviewed above, $SO(3)$ has a representation of every positive integer dimension. Similarly, $SO(d)$ has representations besides the d -dimensional, vector representation (5.8). For example, for general d , we can make a spinor representation of dimension k if we find d $k \times k$ matrices γ^i that satisfy the *Clifford algebra* $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ (as the Paulis do for $k = 2, d = 3$). Then I claim that the objects $J_{\text{spinor}}^{ij} \equiv \frac{i}{4}[\gamma^i, \gamma^j]$ also satisfy the $\mathfrak{so}(d)$ Lie algebra, (5.7). More on this soon.

U(N). Another important example is the Lie group $U(N)$ defined by its N -dimensional representation as $N \times N$ complex unitary matrices $\mathbb{1} = M^\dagger M = M M^\dagger$. This one doesn't arise as a spacetime symmetry, but is crucial in the study of gauge theory, and already arose as an example of a global symmetry on the homework. We can generate these

$$M = e^{-i\beta^a T^a} \quad (5.12)$$

by any *hermitian* $N \times N$ matrices T^a , since $T = T^\dagger$ in (5.12) implies $M^\dagger = M^{-1}$. A

⁵⁰For real matrices $\mathcal{O}^t\mathcal{O} = \mathbb{1}$ says $1 = \det \mathcal{O}^t\mathcal{O} = (\det \mathcal{O})^2$, so $\det \mathcal{O} = \pm 1$ gives two disconnected components. The component with $\det \mathcal{O} = 1$ (containing the identity) is called $SO(d)$. The other component is obtained by multiplying by a diagonal matrix with an odd number of minus signs on the diagonal, parity transformation.

basis is given by the following set of generators satisfying (5.3):

$$T^1 = \frac{1}{2} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}, \quad T^2 = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -2 & \\ & & & 0 \\ & & & & \ddots \end{pmatrix}, \quad T^3 = \frac{1}{\sqrt{24}} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3 & \\ & & & & 0 \\ & & & & & \ddots \end{pmatrix}, \dots$$

$$\text{for } i \neq j: \quad (T_x^{ij})^k{}_l = \frac{1}{2} (\delta^{ik} \delta_l^j + \delta^{jk} \delta_l^i), \quad (T_y^{ij})^k{}_l = \frac{\mathbf{i}}{2} (\delta^{ik} \delta_l^j - \delta^{jk} \delta_l^i), \quad T^{N^2} = \frac{1}{\sqrt{2N}} \mathbb{1}_{N \times N}$$

Altogether there are $\frac{N(N-1)}{2} \cdot 2 + N = N^2$ of these. Only the last one has a nonzero trace. The ones called T_x^{ij} and T_y^{ij} only have nonzero entries in the ij and ji place, and are like σ^x and σ^y respectively. A finite transformation is

$$e^{-\mathbf{i}\beta^a T^a} = \underbrace{e^{-\mathbf{i}\sum_{a=1}^{N^2-1} \beta^a T^a}}_{=M} \underbrace{e^{-\mathbf{i}\beta^{N^2} T^{N^2}}}_{=e^{-\mathbf{i}\beta^{N^2}/\sqrt{2N}}}$$

where the first factor has $\det M = 1$ (since $\log \det M = \text{tr} \log M = -\mathbf{i} \sum_{a=1}^{N^2-1} \beta^a \text{tr}(T^a) = 0$) and the second is just a phase. The subgroup with $\det M = 1$ is called $\text{SU}(N)$. This shows that $\text{U}(N) = (\text{SU}(N) \times \text{U}(1))/\mathbb{Z}_N$. (The quotient by \mathbb{Z}_N is because the matrix $M = \text{diag}(\omega, \dots, \omega), \omega^N = 1$ is an element of $\text{SU}(N)$, so we have two ways of making it as an element of $\text{U}(N)$.)

A special case is $\text{SU}(2)$ which has $N^2 - 1 = 2^2 - 1 = 3$ generators, which are 2×2 and are $T^a = \frac{1}{2} \sigma^a$ Pauli matrices. So $\text{SU}(2)$ and $\text{SO}(3)$ have the same lie algebra. They are not the same group, though, since $\text{SU}(2)$ is twice as big: a 2π rotation is not the identity (but squares to it). Half-integer spin representations of $\text{SU}(2)$ are *projective* representations of $\text{SO}(3)$ – the $\text{SO}(3)$ group law is only satisfied up to a phase (in fact, a sign).

The matrices above (neglecting T^{N^2} , which just commutes with everyone) satisfy the Lie algebra (naturally) called $\mathfrak{su}(N)$. As for $\mathfrak{su}(2)$, this algebra has other representations besides the defining (sometimes called *fundamental*) N -dimensional representation displayed above. [End of Lecture 16]

Lorentz group. The Lorentz group can be defined, like $\text{O}(d)$ above, as the linear transformations preserving (proper) lengths of vectors. This implies

$$\eta = \Lambda^t \eta \Lambda \quad \text{i.e.} \quad \eta_{\mu\nu} = (\Lambda^t)_\mu{}^\rho \eta_{\rho\sigma} \Lambda^\sigma{}_\nu. \quad (5.13)$$

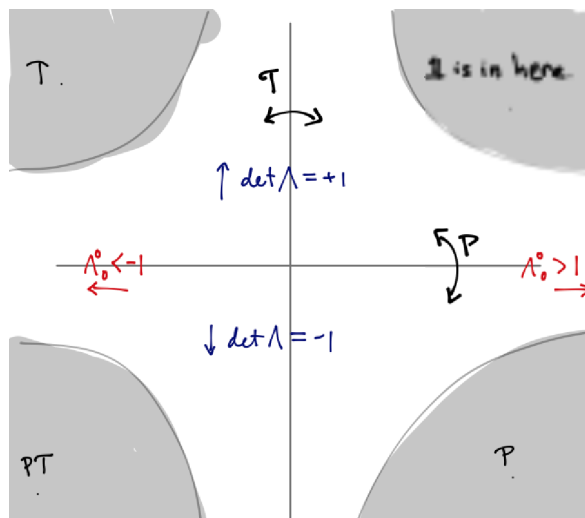
There are four disconnected components of solutions to this condition. As for $\text{O}(d)$, taking the det of both sides of (5.13) implies that such a matrix has $\det \Lambda = \pm 1$; the

two components are called proper and improper Lorentz transformations, respectively. The $\mu\nu = 00$ component of (5.13) says

$$1 = (\Lambda_0^0)^2 - \sum_i (\Lambda_0^i)^2 \implies (\Lambda_0^0)^2 \geq 1$$

which has two components of solutions, $\Lambda_0^0 \geq 1$ (orthochronous) and $\Lambda_0^0 \leq -1$ (not orthochronous).

Below we will focus on the proper, orthochronous component. The other three components are obtained by multiplying one of these by one or both of the following extra discrete symmetries (whose action on (real) vectors is $P = \begin{pmatrix} 1 & \\ & -\mathbb{1}_{3 \times 3} \end{pmatrix}$ and $T = \begin{pmatrix} -1 & \\ & \mathbb{1}_{3 \times 3} \end{pmatrix}$).



(A warning about time reversal: In order to preserve the time evolution operator e^{-iHt} while reversing t and preserving the Hamiltonian, the time reversal transformation T must also be accompanied by complex conjugation $K : \mathbf{i} \rightarrow -\mathbf{i}$, and the combined operation $\mathcal{T} = T \otimes K$ is therefore *antilinear* or *antiunitary*. Such a transformation acts on a linear combination of operators by $c_1\mathcal{O}_1 + c_2\mathcal{O}_2 \mapsto \mathcal{T}(c_1\mathcal{O}_1 + c_2\mathcal{O}_2)\mathcal{T}^{-1} = c_1^*\mathcal{T}\mathcal{O}_1\mathcal{T}^{-1} + c_2^*\mathcal{T}\mathcal{O}_2\mathcal{T}^{-1}$.)

The identity-containing component is called $\text{SO}(1, d)$. More generally $\text{O}(m, n)$ is the group of linear operations preserving the matrix with m -1 s and n $+1$ s on the diagonal, which I will also call $\eta_{\mu\nu}$. For any m, n , all the steps leading to the associated algebra (5.7) (and generators in the $m + n$ dimensional representation (5.8)) are the same as for $\text{SO}(d)$ with the replacement $\delta_{ij} \mapsto \eta_{\mu\nu}$ (and $\delta_i^j \mapsto \delta_\mu^\nu$), with the metric of appropriate signature. We will nevertheless resort to some special features of the case $d = 3$ to build representations.

5.2 Representations of the Lorentz group on fields

Consider a Lorentz-invariant field theory of a (finite) collection of fields $\phi_r = (\phi_1 \dots \phi_n, \psi_\alpha, A_\mu \dots)_r$. Together they form a (in general reducible) representation of the Lorentz group

$$\phi_r(x) \mapsto D_{rs}(\Lambda)\phi_s(\Lambda x)$$

where $D_{rs}(\Lambda)$ is some (finite-dimensional) matrix representation, and Λ specifies an element of $\text{SO}(1, d)$ in the defining representation. So far we know two possibilities: the scalar (one-dimensional) representation, where $D(\Lambda) = 1$, and the vector ($d + 1$ -dimensional) representation, where $D(\Lambda) = \Lambda$ is a $(d+1) \times (d+1)$ matrix in the defining rep. (We can also take direct sums of these to make reducible representations.)

But there are other irreps. To find more, let's think about the algebra in more detail by extracting it from the representation on 4-vectors

$$V^\mu \rightarrow V'^\mu = \Lambda^\mu{}_\nu V^\nu, \quad \Lambda(\theta^a, \beta^a) = \exp \left(-\mathbf{i}\theta^a \underbrace{T_{\text{rot}}^a}_{\equiv J^a} - \mathbf{i}\beta^a \underbrace{T_{\text{boost}}^a}_{\equiv K^a} \right).$$

Let's find the f_c^{ab} by building the J s and K s: with the time component first, the matrix representations are

$$J^i = \begin{pmatrix} 0 & \\ & \mathbf{J}^i \end{pmatrix} \quad (5.14)$$

where the 3×3 matrix is $(\mathbf{J}^i)^j_k = -\mathbf{i}\epsilon^{ijk}$ (with $\epsilon^{123} = 1$) and

$$(K^i)^j_0 = \mathbf{i}\delta_j^i = (K^i)^0_j \quad (5.15)$$

and other components zero. To check this, consider a boost in the x direction:

$$e^{-\mathbf{i}\beta K^1} = \mathbb{1} - \mathbf{i}\beta K^1 + \mathcal{O}(\beta^2) = \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & \mathbb{1}_{2 \times 2} & \\ & & & \mathbb{1}_{2 \times 2} \end{pmatrix} + \mathcal{O}(\beta^2) = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & \mathbb{1}_{2 \times 2} & \\ & & & \mathbb{1}_{2 \times 2} \end{pmatrix} + \mathcal{O}(\beta^2). \quad (5.16)$$

In the last step we used $\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = 1 + \mathcal{O}(\beta^2)$. That is, $\delta V^0 = \beta V^1, \delta V^1 = \beta V^0, \delta V^{2,3} = 0$. The other boosts are related to this one by a rotation. In (5.16), we only checked the infinitesimal transformation; but this is enough, by the uniqueness of solutions of linear first-order differential equations: $\partial_\beta \Lambda(\beta) = -\mathbf{i}K\Lambda$ with initial condition $\Lambda(\beta = 0) = \mathbb{1}$ has a unique solution, and so our solution must be the correct one. We'll use this strategy several times below. ⁵¹

Notice with slight horror that the boost generators are *not* hermitian, and hence the finite boost operator defined above is *not* unitary. This is a symptom of the fact that the Lorentz group is *non-compact* (in the sense that its group manifold is not compact:

⁵¹In case you are wondering, the finite transformation is $e^{-\mathbf{i}\beta K^1} = \begin{pmatrix} \cosh \beta & \sinh \beta & & \\ \sinh \beta & \cosh \beta & & \\ & & \mathbb{1}_{2 \times 2} & \\ & & & \mathbb{1}_{2 \times 2} \end{pmatrix}$. Note

that the parameter β here is the *rapidity*; it is additive under successive finite boosts, unlike the velocity (though they agree when infinitesimal: $\tanh \beta = \frac{v}{c}$.) I apologize for using the letter β in an unfamiliar way.

think of the orbits of rotations on a 4-vector (a sphere, compact), and the orbits of a boost on a 4-vector (a hyperbola, non-compact)). For (faithful) representations of non-compact groups, ‘unitary’ and ‘finite-dimensional’ are mutually exclusive.

The commutators of these objects are⁵²

$$[J^i, J^j] = \mathbf{i}\epsilon^{ijk} J^k, \quad [J^i, K^j] = \mathbf{i}\epsilon^{ijk} K^k \quad (5.18)$$

(which are respectively the statements that the rotation and boost generators each form a vector) and

$$[K^i, K^j] = -\mathbf{i}\epsilon^{ijk} J^k \quad (5.19)$$

which says two boosts commute to a rotation. Notice that these equations are not changed by the (parity-like) operation $K \rightarrow -K, J \rightarrow J$.

Now consider $\vec{J}^\pm \equiv \frac{1}{2}(\vec{J} \pm \mathbf{i}\vec{K})$. (Don’t confuse these 3+3 operators with the raising and lowering operators for $\mathfrak{so}(3)$ from (5.5).) The observation that $K \rightarrow -K$ changes nothing implies that they satisfy

$$[J_+^i, J_-^j] = 0, \quad [J_\pm^i, J_\pm^j] = \mathbf{i}\epsilon^{ijk} J_\pm^k,$$

two independent $\mathfrak{su}(2)$ algebras, which will be called left and right. Formally, we’ve shown that as algebras over the complex numbers, $\mathfrak{so}(1,3) \simeq \mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$. But we know what the representations of $\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$ are! We just have to specify a representation of each factor, *i.e.* an irrep is labelled by j_L and j_R , two positive semidefinite half-integers. And we can label states in an irrep by (j_L, m_L, j_R, m_R) with $m_{L/R} \in \{-j_{L/R}, \dots, j_{L/R}\}$; this has dimension $(2j_L + 1)(2j_R + 1)$. Thus, we have the following table 1 of possible Lorentz representations of fields:

Let me emphasize here that we are identifying the possible ways that the Lorentz group can act on *fields*, not on the particle excitations of such fields. That’s why it’s not a problem that the action is not unitary. The resulting unitaries on the Fock space will come later.

⁵²Like we did for $O(d)$, we can slick this up, and generalize to other $SO(1,d)$ by collecting the generators into an antisymmetric matrix $J^{\mu\nu}$ with components $J^{ij} = \epsilon^{ijk} J^k, J^{0i} = K^i = -J^{i0}$ (exactly as \vec{E}, \vec{B} are collected into $F^{\mu\nu}$). This object satisfies the direct analog of (5.7)

$$[J^{\mu\nu}, J^{\rho\sigma}] = \mathbf{i}(\eta^{\nu\rho} J^{\mu\sigma} + \eta^{\mu\sigma} J^{\nu\rho} - (\mu \leftrightarrow \nu)) \quad (5.17)$$

and the fundamental ($d+1$ -dimensional vector) representation matrices solving this equation are

$$(J^{\mu\nu})^\rho{}_\sigma = -\mathbf{i}(\eta^{\nu\rho} \delta_\sigma^\mu - (\mu \leftrightarrow \nu)).$$

(j_+, j_-)	dim	Preview of physics
$(0, 0)$	1	scalar
$(\frac{1}{2}, 0)$	2	left-handed Weyl spinor
$(0, \frac{1}{2})$	2	right-handed Weyl spinor
$(\frac{1}{2}, 0) \otimes (0, \frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$	$2 \times 2 = 4$	4-vector
$(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$	$2 + 2 = 4$	Dirac spinor (reducible)
$(1, 0) \oplus (0, 1)$	$3 + 3 = 6$	$V^{\mu\nu} = \pm \epsilon^{\mu\nu\rho\sigma} V^{\rho\sigma}$, $V^{\mu\nu} = -V^{\nu\mu}$, antisymmetric tensor
$(1, 1)$	$3 \times 3 = 9$	$S^{\mu\nu} = S^{\nu\mu}$, $S_\mu^\mu = 0$, traceless symmetric tensor
...

Table 1: Lorentz representations on fields.

Weyl spinors. Let's focus on the first nontrivial entry in the table. This is a 2-component field $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ on which 2×2 Lorentz matrices act as

$$D_{(\frac{1}{2}, 0)}(\theta, \beta) = e^{-i(\theta^i J^i + \beta^i K^i)}.$$

But the fact that it's a singlet of $\text{SU}(2)_R$ means that

$$0 = J_-^i \psi = \frac{1}{2}(J - \mathbf{i}K)^i \psi,$$

that is, $J = \mathbf{i}K$ when acting on ψ , which says that the nontrivial generators act as

$$J_+^i \psi = \frac{1}{2}(J + \mathbf{i}K)^i \psi = \frac{1}{2}(J + J)^i \psi = J^i \psi.$$

But we know a 2×2 representation of this object: $\vec{J}_{(\frac{1}{2})} = \frac{1}{2}\vec{\sigma}$ and hence $\vec{K} = -\mathbf{i}\frac{1}{2}\vec{\sigma}$. You can check that these satisfy the three relations (5.18), (5.19). Therefore

$$\psi_\alpha \mapsto \left(e^{-\mathbf{i}\frac{1}{2}\theta \cdot \sigma - \frac{1}{2}\beta \cdot \sigma} \right)_\alpha^\beta \psi_\beta = \left(e^{-\frac{1}{2}\sigma \cdot (\beta + \mathbf{i}\theta)} \right)_\alpha^\beta \psi_\beta \equiv M_\alpha^\beta \psi_\beta.$$

(Please don't confuse the rapidity β with the spinor index β .) Notice that this matrix M is an ordinary rotation with a complexified angle; it is actually an $\text{SL}(2, \mathbb{C})$ matrix, a general 2×2 complex matrix, with unit determinant. It is common to call the $(\frac{1}{2}, 0)$ representation a *left-handed* (L) Weyl spinor. The nomenclature is motivated by the fact that parity takes $J^\pm = \frac{1}{2}(J \pm \mathbf{i}K) \rightarrow J^\mp$.

For the $(0, \frac{1}{2})$ or right-handed representation, χ , the same story obtains but now $(\vec{J}_+)_\alpha^\beta \chi_\beta = 0$ and hence $J = -\mathbf{i}K$. Note the dotted indices to distinguish reps of the two $\text{SU}(2)$ s. Therefore

$$\chi_{\dot{\alpha}} \mapsto \left(e^{-\mathbf{i}\frac{1}{2}\theta \cdot \sigma + \frac{1}{2}\beta \cdot \sigma} \right)_{\dot{\alpha}}^{\dot{\beta}} \chi_{\dot{\beta}} = \left(e^{+\frac{1}{2}\sigma \cdot (\beta - \mathbf{i}\theta)} \right)_{\dot{\alpha}}^{\dot{\beta}} \chi_{\dot{\beta}} = (\sigma^2 M^* \sigma^2)_{\dot{\alpha}}^{\dot{\beta}} \chi_{\dot{\beta}}. \quad (5.20)$$

In the last step of (5.20) we used the identity

$$\sigma^2 \vec{\sigma}^* \sigma^2 = -\vec{\sigma}. \quad (5.21)$$

Please don't get too hung up on dotting the indices, since as we'll see, there are ways to turn an L spinor into an R spinor. (For example, the parity operation $K \rightarrow -K, J \rightarrow J$ interchanges the two.) Also, notice that (5.20) shows that if a left-handed Weyl spinor transforms as $\psi \rightarrow M\psi$ then

$$\sigma^2 \psi^* \rightarrow \sigma^2 M^* \psi^* = (\sigma^2 M^* \sigma^2) \sigma^2 \psi^*$$

transforms like a right-handed Weyl spinor.

Invariants. In order to write Lorentz-invariant local lagrangians, we need to know how to make Lorentz-invariant quantities out of products of fields and their derivatives. For example, given Lorentz vectors V^μ, U^μ , the object $V^\mu U_\mu = V^\mu U^\nu \eta_{\mu\nu}$ is a Lorentz scalar (by the defining property of the Lorentz matrices). Can we make a singlet from two Weyl spinors, $(\frac{1}{2}, 0) \otimes (\frac{1}{2}, 0)$? Yes: we know (*e.g.* from basic QM) that SU(2) representations combine as $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ where the triplet (spin one) part is symmetric and the singlet (spin 0) is the antisymmetric combination, $\uparrow\downarrow - \downarrow\uparrow$. More explicitly, $\psi_\alpha \xi_\beta \epsilon^{\alpha\beta} \equiv \psi_\alpha \xi^\alpha$ is a singlet. To see this explicitly:

$$\begin{aligned} (\mathbf{i}\sigma^2 \psi) &\mapsto \mathbf{i}\sigma^2 e^{-\frac{1}{2}(\beta + \mathbf{i}\theta) \cdot \sigma} \psi && \text{Insert } \mathbb{1} = \sigma^2 \sigma^2 \text{ before } \psi \\ &= \exp \left(-\frac{1}{2} (\vec{\beta} + \mathbf{i}\vec{\theta}) \cdot \underbrace{(\sigma^2 \vec{\sigma} \sigma^2)}_{\substack{(5.21) \\ -\vec{\sigma}^t}} \right) (\mathbf{i}\sigma^2 \psi) \end{aligned}$$

which means that if $\psi_\alpha \mapsto M_\alpha^\beta \psi_\beta$, then

$$\psi^\alpha \equiv (\mathbf{i}\sigma^2 \psi)^t{}^\alpha \mapsto \psi^\beta \left(e^{+\frac{1}{2}(\beta + \mathbf{i}\theta) \cdot \vec{\sigma}} \right)_\beta{}^\alpha \equiv \psi^\beta (M^{-1})_\beta{}^\alpha$$

so $\psi^\alpha \psi_\alpha$ is invariant. Notice that on Weyl spinors, we raise and lower indices with $\epsilon^{\alpha\beta} \equiv (\mathbf{i}\sigma^2)^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which is a good idea because it is an invariant tensor, as we just showed.

The same story holds for the $(0, \frac{1}{2})$ “right-handed” Weyl representation, that is, we can also make a singlet out of two of them using an epsilon tensor. Writing out the matrices again:

$$\chi^{\dot{\alpha}} \equiv \chi_{\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}} \mapsto \left(e^{\frac{1}{2}(\beta - \mathbf{i}\theta) \cdot \sigma^t} \right)^{\dot{\alpha}}{}_{\dot{\beta}} \chi^{\dot{\beta}} = \chi^{\dot{\beta}} \left(e^{-\frac{1}{2}(\beta - \mathbf{i}\theta) \cdot \sigma} \right)_{\dot{\beta}}{}^{\dot{\alpha}}$$

which is indeed the inverse transformation of (5.20).

$\left(\frac{1}{2}, \frac{1}{2}\right) = \text{vector}$. Next, we will show that the $\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right)$ representation is indeed a 4-vector. Introduce the following ‘intertwiners’:

$$\sigma_{\alpha\dot{\alpha}}^{\mu} \equiv (\mathbb{1}_{\alpha\dot{\alpha}}, \vec{\sigma}_{\alpha\dot{\alpha}})^{\mu}, \quad \bar{\sigma}_{\dot{\alpha}\alpha}^{\mu} \equiv (\mathbb{1}_{\dot{\alpha}\alpha}, -\vec{\sigma}_{\dot{\alpha}\alpha})^{\mu}.$$

Our next job is to show that these objects eat a L and an R Weyl spinor and spit out a vector, or vice versa. That is, for example, I claim that if V_{μ} is a vector then $V_{\mu}\sigma_{\alpha\dot{\alpha}}^{\mu}$ transforms as $\left(\frac{1}{2}, 0\right) \otimes \left(0, \frac{1}{2}\right)$, as the notation strongly suggests. And given any two L and R Weyl spinors ψ, χ , $\psi^{\alpha}\sigma_{\alpha\dot{\alpha}}^{\mu}\chi^{\dot{\alpha}}V_{\mu}$ is a singlet.

To summarize:

$$\begin{aligned} \left(\frac{1}{2}, 0\right) \ni \psi_L &\mapsto M_L \psi_L = e^{\frac{1}{2}(-\mathbf{i}\theta - \beta) \cdot \sigma} \psi_L \\ \left(0, \frac{1}{2}\right) \ni \psi_R &\mapsto M_R \psi_R = e^{\frac{1}{2}(-\mathbf{i}\theta + \beta) \cdot \sigma} \psi_R \end{aligned}$$

You can see from this expression and $\sigma^2 \sigma^i \sigma^2 = -(\sigma^i)^{\star}$ that $\sigma^2 M_L^{\star} \sigma^2 = M_R$ and therefore $\sigma^2 \psi_L^{\star} \in \left(0, \frac{1}{2}\right)$.

Claim: for any two right-handed spinors ξ_R, ψ_R , the object $\xi_R^{\dagger} \sigma^{\mu} \psi_R$ is a (complex) 4-vector. To see this, first notice (using $\sigma^{\dagger} = \sigma$) that $\xi_R^{\dagger} \mapsto \xi_R^{\dagger} e^{+\frac{1}{2}(\mathbf{i}\theta + \beta) \cdot \sigma}$, so

$$\xi_R^{\dagger} \sigma^{\mu} \psi_R \mapsto \xi_R^{\dagger} \underbrace{e^{\frac{1}{2}(\mathbf{i}\theta + \beta) \cdot \sigma} \sigma^{\mu} e^{\frac{1}{2}(-\mathbf{i}\theta + \beta) \cdot \sigma}}_{\stackrel{?}{=} \Lambda(\theta, \beta)^{\mu}_{\nu} \sigma^{\nu}} \psi_R$$

where

$$\Lambda(\theta, \beta)^{\mu}_{\nu} = (e^{-\mathbf{i}(\theta \cdot J + \beta \cdot K)})^{\mu}_{\nu} \quad (5.22)$$

is the vector representation of the Lorentz transformation with rotation $\vec{\theta}$ and boost $\vec{\beta}$. To check this, it suffices to check the infinitesimal version (by the uniqueness of solutions to linear first-order ODEs):

$$\begin{aligned} \delta \left(\xi_R^{\dagger} \sigma^{\mu} \psi_R \right) &= \delta \xi_R^{\dagger} \sigma^{\mu} \psi_R + \xi_R^{\dagger} \sigma^{\mu} \delta \psi_R \\ &= \xi_R^{\dagger} \left(\frac{1}{2} (\mathbf{i}\theta + \beta)^j \sigma^j \sigma^{\mu} + \sigma^{\mu} \frac{1}{2} (-\mathbf{i}\theta + \beta)^j \sigma^j \right) \psi_R \\ &= \begin{cases} \xi_R^{\dagger} \frac{1}{2} 2\beta_j \sigma^j \psi_R & \dots \text{if } \mu = 0 \\ \xi_R^{\dagger} \frac{1}{2} \left(\underbrace{\beta_j (\sigma^j \sigma^i + \sigma^i \sigma^j)}_{=2\delta^{ij}} + \mathbf{i}\theta_j \underbrace{(\sigma^j \sigma^i - \sigma^i \sigma^j)}_{=-2\mathbf{i}\epsilon^{ijk} \sigma^k} \right) \psi_R & \dots \text{if } \mu = i \end{cases} \end{aligned}$$

On the other hand, using the form of the vector Lorentz generators (5.14), (5.15), the transformation of a vector is

$$\delta V^{\mu} = -(\mathbf{i}\beta_j (K^j)^{\mu}_{\nu} + \mathbf{i}\theta_j (J^j)^{\mu}_{\nu}) V^{\nu} = \begin{cases} \beta_j V^j & \dots \text{if } \mu = 0 \\ \beta_i V^0 - \theta_j \epsilon_{jim} V^m & \dots \text{if } \mu = i \end{cases}$$

which is just the form we've found. ■

A few more claims and consequences:

- Since the vector representation matrices K, J in (5.14), (5.15) are pure imaginary, the matrices (5.22) are *real* for any θ, β and we can impose the reality condition $V^\mu = (V^\mu)^*$ consistent with Lorentz orbits. This is not true of the Weyl spinors by themselves.
- Similarly, for any ξ_L, ψ_L $\xi_L^\dagger \bar{\sigma}^\mu \psi_L$ is a (complex) 4-vector. Notice that we need to use the $\bar{\sigma}^\mu$, so that

$$\delta \left(\xi_L^\dagger \bar{\sigma}^\mu \psi_L \right) = - \left(-\beta_i \xi_L^\dagger \sigma_i \psi_L, \beta_i \xi_L^\dagger \psi_L + \epsilon_{ijk} \theta_j \xi_L^\dagger \sigma_k \psi_L \right)^\mu$$

- Our explicit calculation was about $\xi_R^\dagger \sigma^\mu \psi_R$. But we showed that ξ_R^\dagger transforms like a χ_L . So $\chi_L^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \psi_R^{\dot{\alpha}}$ is a vector, too. And $\chi_L^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \psi_R^{\dot{\alpha}} V_\mu$ is Lorentz invariant.
- Notice that $\xi_R^\dagger \psi_R$ is *not* Lorentz invariant (even if $\xi = \psi$), it is rather the 0 component of a four-vector. The dotted index can be a bit helpful in reminding us that $\psi_L^\dagger \psi_L$ is not a singlet, since if the index on $(\psi_L)_\alpha$ is undotted then $(\psi_L^\dagger)_{\dot{\alpha}}$ has a dotted index.

On the other hand, given an R and an L spinor,

$$\delta \xi_L^\dagger = \xi_L^\dagger (+\mathbf{i}\theta - \beta) \cdot \sigma / 2$$

the combination $\xi_L^\dagger \psi_R$ is Lorentz invariant, since

$$\delta \left(\xi_L^\dagger \psi_R \right) = \xi_L^\dagger \left(\frac{1}{2} (\mathbf{i}\theta - \beta) \cdot \sigma + \frac{1}{2} (-\mathbf{i}\theta + \beta) \cdot \sigma \right) \psi_R = 0. \quad (5.23)$$

- Two more occasionally-useful facts:

$$V^\mu \sigma_\mu = \begin{pmatrix} V^0 + V^3 & V^1 - \mathbf{i}V^2 \\ V^1 + \mathbf{i}V^2 & V^0 - V^3 \end{pmatrix} \mapsto M V^\mu \sigma_\mu M^\dagger$$

Also, $\det V^\mu \sigma_\mu = V^\mu V_\mu$ is Lorentz invariant.

5.3 Spinor lagrangians

Given a Weyl spinor field ψ_R , we'd like to make a local lorentz-invariant lagrangian of the form $\mathcal{L} \left(\psi_R, \psi_R^\dagger, \partial_\mu \psi_R, \partial_\mu \psi_R^\dagger \right)$. The sort of obvious generalization of the KG

lagrangian is $\psi_R^\dagger (\square + m^2) \psi_R$, which transforms like $\psi_R^\dagger \psi_R$, which is not boost invariant. On the other hand, the object $\psi_R^\dagger \sigma^\mu \psi_R$ is a vector, and we can find another index with which to contract by taking a derivative:

$$\mathcal{L}_{\text{Weyl}} \equiv \psi_R^\dagger \sigma^\mu \mathbf{i} \partial_\mu \psi_R = \psi_R^\dagger \mathbf{i} \partial_t \psi_R + \psi_R^\dagger \vec{\sigma} \cdot (\mathbf{i} \vec{\nabla}) \psi_R$$

is a nice Lorentz invariant kinetic term. The factor of \mathbf{i} is to make $\mathbf{i} \partial_\mu$ hermitian, so

$$\mathcal{L}_{\text{Weyl}}^\dagger = -\mathbf{i} \left(\partial_\mu \psi_R^\dagger \right) (\sigma^\mu)^\dagger \psi_R \stackrel{\text{IBP}}{=} \psi_R^\dagger \sigma^\mu \mathbf{i} \partial_\mu \psi_R = \mathcal{L}_{\text{Weyl}}.$$

Notice that we neglected the total derivative $\partial_\mu \left(\psi_R^\dagger \sigma^\mu \psi_R \right)$ which does not change the equations of motion.

For a left-handed field, the Lorentz-invariant Weyl lagrangian involves $\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})^\mu$:

$$\mathcal{L}_{\text{Weyl}}(\psi_L) = \psi_L^\dagger \bar{\sigma}^\mu \mathbf{i} \partial_\mu \psi_L.$$

What about mass terms? For a single R Weyl field, we could use the ϵ tensor:

$$\psi_R \mathbf{i} \sigma^2 \psi_R + h.c. = \psi_1 \psi_2 - \psi_2 \psi_1 + h.c. \quad (5.24)$$

is Lorentz invariant. It is not invariant under $\psi_R \rightarrow e^{i\theta} \psi_R$, it violates the particle number. This is called a Majorana mass term. Neutrinos may have such a term, but electrons don't. (Notice that it would be zero if the components ψ_α were commuting objects. We will see that in fact they are *anticommuting* objects and (5.24) is not zero.) [End of Lecture 17]

Dirac spinors. To make a particle-number-conserving Lorentz-invariant mass term, we need one of each L and R , and the *Dirac mass* pairs them up via the invariant $\psi_L^\dagger \psi_R + h.c.$. We can slick this up with some nice packaging, by combining the two 2-component spinors into one 4-component spinor Ψ :

$$\left(\frac{1}{2}, 0 \right) \oplus \left(0, \frac{1}{2} \right) \ni \Psi \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$

Now let

$$\bar{\Psi} \equiv \left(\psi_R^\dagger, \psi_L^\dagger \right) = \Psi^\dagger \begin{pmatrix} 0 & \mathbb{1}_{2 \times 2} \\ \mathbb{1}_{2 \times 2} & 0 \end{pmatrix} \equiv \Psi^\dagger \gamma^0.$$

Then we can package the whole thing beautifully as

$$\mathcal{L}_{\text{Dirac}} = \psi_R^\dagger \mathbf{i} \sigma^\mu \partial_\mu \psi_R + \psi_L^\dagger \mathbf{i} \bar{\sigma}^\mu \partial_\mu \psi_L - m \left(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L \right) = \bar{\Psi} \left(\mathbf{i} \gamma^\mu \partial_\mu - m \right) \Psi$$

with

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \quad (5.25)$$

The equations of motion are

$$0 = \frac{\partial S}{\partial \bar{\Psi}} = (\mathbf{i}\gamma^\mu \partial_\mu - m) \Psi \equiv (\mathbf{i}\not{\partial} - m) \Psi.$$

Being explicit about indices, the Dirac equation is $0 = (\mathbf{i}\gamma_{ab}^\mu \partial_\mu - m\delta_{ab}) \Psi_b$ with $a, b = 1..4$.

Notice that by dimensional analysis of the kinetic terms (that is, demanding that $[S_{\text{kin}} = \int \bar{\Psi} \gamma^\mu \partial_\mu \Psi] = 0$, $[\Psi] = \frac{D-1}{2} \stackrel{D=4}{=} 3/2$, so $[m] = 1$, so m is indeed a mass. Its sign has not been fixed (and I will probably mix up m and $-m$ at various points).

- The Dirac (or gamma) matrices γ^μ satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \quad (5.26)$$

Any set of D matrices satisfying this equation can be combined to form a representation of $\mathfrak{so}(1, D-1)$ in the form

$$J_{\text{Dirac}}^{\mu\nu} \equiv \frac{\mathbf{i}}{4} [\gamma^\mu, \gamma^\nu].$$

That is, these objects satisfy the algebra (5.17). (If you want to see the algebra involved in this statement, see David Tong's notes.) This works in any dimension. The representation is reducible for D even. The size of the matrices required to represent (5.26) grows exponentially with D .

- Notice that the equation (5.26) is not modified by a change of basis in the spinor space. The particular basis of gamma matrices we've chosen in (5.25) is called the Weyl basis. It makes the reducibility of the Dirac rep manifest, since the resulting $\mathfrak{so}(1, 3)$ generators $J^{\mu\nu}$ are block diagonal:

$$\begin{aligned} J_{\text{Dirac}}^{\mu\nu} &\stackrel{\text{Weyl basis}}{=} \frac{\mathbf{i}}{4} \left[\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \right] \\ &= \frac{\mathbf{i}}{4} \begin{pmatrix} \sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu \end{pmatrix} \\ &= \begin{cases} \frac{\mathbf{i}}{4} \begin{pmatrix} -2\sigma^i & 0 \\ 0 & 2\sigma^i \end{pmatrix} = -\frac{\mathbf{i}}{2} \sigma^i \otimes \sigma^3 & \text{if } \mu = 0, \nu = i \\ \frac{\mathbf{i}}{4} \begin{pmatrix} -[\sigma^i, \sigma^j] & 0 \\ 0 & -[\sigma^i, \sigma^j] \end{pmatrix} = +\frac{1}{2} \epsilon^{ijk} \sigma^k \otimes \mathbb{1}_{2 \times 2} & \text{if } \mu = i, \nu = j \end{cases} \end{aligned} \quad (5.27)$$

So you see that in the Weyl basis, we already know that these satisfy the $\mathfrak{so}(1, d)$ algebra since it is just the Lorentz generators for the Weyl_L and Weyl_R representations in blocks. This proves that it works in any basis, since the algebra is basis-independent.

- This 4-dimensional Dirac representation is *not* the 4d vector representation. We can see this in several ways: It is complex (the generators are not pure imaginary), though more on this below. It is reducible (we built it by adding together two irreps!). And it is definitely different because, using (5.27), we have, *e.g.* $J^{12} = \frac{1}{2}\sigma^3 \otimes \mathbb{1}$, so

$$\Lambda_{\text{Dirac}}(\theta = 2\pi\hat{z}) = e^{-i2\pi J^{12}} \stackrel{(5.27)}{=} e^{i\pi\sigma^3 \otimes \mathbb{1}} = \cos \pi \mathbb{1} + \sin \pi \sigma^3 \otimes \mathbb{1} = -\mathbb{1}.$$

A 2π rotation gives minus one. This is just as for the spin- $\frac{1}{2}$ (projective) representation of $\text{SO}(3)$.

- The Weyl spinors ψ_L, ψ_R are irreps. What's the big deal about the Dirac rep? Only that the electron is a Dirac spinor (and some other folks are too). Neutrinos could be Weyl spinors. There are several possibilities: perhaps there is a secret (heavy, non-interactive) partner with whom the neutrinos pair up by a Dirac mass; perhaps lepton number is violated by a Majorana mass term; these possibilities are not mutually exclusive (see the homework).
- Other bases of the gamma matrices are possible and sometimes useful. If we replace γ^μ with

$$\gamma^\mu \mapsto \tilde{\gamma}^\mu = U\gamma^\mu U^\dagger, \quad \Psi \mapsto \tilde{\Psi} = U\Psi$$

for some 4×4 unitary U then this gives an equivalent representation, since $\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu}$ still and hence $\tilde{J}^{\mu\nu} = UJ^{\mu\nu}U^\dagger$ will still solve $\mathfrak{so}(1, d)$.

A particular useful other basis is the *Majorana basis*

$$\gamma_m^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma_m^1 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \gamma_m^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \gamma_m^3 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & -i\sigma^3 \end{pmatrix}.$$

They have the property that they are all imaginary, which means so are the resulting Lorentz generators $J_m^{\mu\nu}$, which means that all the matrix elements of $e^{i\theta_{\mu\nu}J_m^{\mu\nu}}$ are real. So it is consistent to impose a reality condition on the spinors in this basis, $\eta_m = \eta_m^*$. (The reality condition can be imposed in any basis, but in another $\gamma_m^\mu = U\tilde{\gamma}^\mu U^\dagger$, the condition looks like $(U^*)^{-1}U\psi = \psi^*$.) This 4d real representation is still different from the vector; a proof is that a 2π rotation is still -1 . A good analogy is (real scalar):(complex scalar)::(majorana spinor):(Dirac spinor). For example, a Majorana spinor particle will be its own antiparticle, just like for a real scalar.

- The Dirac equation $(i\partial - m)\Psi = 0$ implies the wave equation. Act on the BHS by

$$(i\partial + m)(BHS) \implies 0 = (i\partial + m)(i\partial - m)\Psi$$

$$\begin{aligned}
&= (-\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu - m^2) \Psi \\
&= \left(-\frac{1}{2} \left(\underbrace{[\gamma^\mu, \gamma^\nu]}_{\text{antisymmetric}} + \underbrace{\{\gamma^\mu, \gamma^\nu\}}_{=2\eta^{\mu\nu}} \right) \underbrace{\partial_\mu \partial_\nu}_{\text{symmetric}} - m^2 \right) \Psi \\
&= -(\partial^2 + m^2) \Psi.
\end{aligned}$$

- The equation of motion for $\bar{\Psi}$ can be obtained by taking the dagger⁵³, or by IBP in $\mathcal{L}_{\text{Dirac}}$:

$$\mathcal{L}_{\text{Dirac}} \stackrel{\text{IBP}}{=} \bar{\Psi} \left(-\mathbf{i} \overleftarrow{\partial}_\mu \gamma^\mu - m \right) \Psi + \text{total deriv}$$

$$\text{so } 0 = \frac{\partial S_{\text{Dirac}}}{\partial \bar{\Psi}} = \bar{\Psi} \left(-\mathbf{i} \overleftarrow{\partial}_\mu \gamma^\mu - m \right).$$

- The Dirac lagrangian is real if $m = m^*$, since we already checked the kinetic terms.
- **Lorentz transformations of Dirac spinors.** We have $\Psi(x) \mapsto e^{-\mathbf{i}\theta_{\mu\nu} J_{\text{Dirac}}^{\mu\nu}} \Psi(\Lambda^{-1}x) \equiv$

$\Lambda_{\frac{1}{2}} \Psi(\Lambda^{-1}x)$ with $\Lambda_{\frac{1}{2}} = \begin{pmatrix} M & 0 \\ 0 & \sigma^2 M^* \sigma^2 \end{pmatrix}$. The Dirac conjugate spinor transforms as

$$\bar{\Psi} \mapsto \Psi^\dagger e^{+\mathbf{i}\theta_{\mu\nu} (J_{\text{Dirac}}^{\mu\nu})^\dagger} \gamma^0 = \Psi^\dagger \gamma^0 \Lambda_{\frac{1}{2}}^{-1} = \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1}.$$

Here we used that the ij components of J_{Dirac} are hermitian and commute with γ^0 (same for L, R), while the $0i$ components are antihermitean and anticommute with γ^0 (opposite sign for L, R). This makes it clear that the mass term $\bar{\Psi}\Psi$ is Lorentz invariant.

The gamma matrices also provide nice packaging of the relation we showed above between vectors and bispinors⁵⁴:

$$\Lambda_{\frac{1}{2}}^{-1}(\theta) \gamma^\mu \Lambda_{\frac{1}{2}}(\theta) = \Lambda^\mu{}_\nu(\theta) \gamma^\nu. \quad (5.28)$$

This means that any product of gamma matrices between two spinors $V^{\mu_1 \dots \mu_n} \equiv \bar{\Psi} \gamma^{\mu_1} \dots \gamma^{\mu_n} \Psi$ is a tensor, in the sense that $V^{\mu_1 \dots \mu_n} \mapsto \Lambda^{\mu_1}{}_{\nu_1} \dots \Lambda^{\mu_n}{}_{\nu_n} V^{\nu_1 \dots \nu_n}$. (A *tensor* is defined to be an object that transforms this way under Lorentz

⁵³In doing so, note that $\gamma^0 = (\gamma^0)^\dagger$, but the spatial ones $(\vec{\gamma})^\dagger = -\vec{\gamma}$ are anti-hermitian. This compensates the fact that the $\vec{\gamma}$ s acquire a minus sign in moving through the γ^0 in $\bar{\Psi}$.

⁵⁴This is really the same equation as we showed above. In terms of Dirac fermions, its infinitesimal version is,

$$(1 + \mathbf{i}\theta \cdot J_{\text{Dirac}}) \gamma^\mu (1 - \mathbf{i}\theta \cdot J_{\text{Dirac}}) = (1 - \mathbf{i}\theta \cdot J_{\text{vector}})^\mu{}_\nu \gamma^\nu,$$

which follows from

$$[\gamma^\mu, J_{\text{Dirac}}^{\rho\sigma}] = (J_{\text{vector}}^{\rho\sigma})^\mu{}_\nu \gamma^\nu,$$

the infinitesimal version of the statement that γ^μ is a Lorentz vector.

transformations.) To see this, just use $V^{\mu_1 \dots \mu_n} \mapsto \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1} \gamma^{\mu_1} \dots \gamma^{\mu_n} \Lambda_{\frac{1}{2}} \Psi$, insert $\mathbb{1} = \Lambda_{\frac{1}{2}} \Lambda_{\frac{1}{2}}^{-1}$ in between each pair of gammas, and use (5.28).

Notice that any combination of $A_{\mu_1 \dots \mu_n} \bar{\Psi} \gamma^{\mu_1} \dots \gamma^{\mu_n} \Psi$ which is *symmetric* under interchange of a pair of indices can be related using the Clifford algebra to a tensor with two fewer indices. Let $\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ be just the antisymmetric bit, and similarly for more indices. In fact, any bispinor $\Gamma_{ab} \bar{\Psi}_a \Psi_b$ can be decomposed as a sum of these tensors: $\sum_n A_{\mu_1 \dots \mu_n} \bar{\Psi} \gamma^{\mu_1 \dots \mu_n} \Psi$, where the A s are totally antisymmetric. This follows from counting: $4 \times 4 = 1 + 4 + 6 + 4 + 1$.

- Consider the object $\gamma^5 \equiv \mathbf{i} \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -\frac{\mathbf{i}}{4!} \epsilon_{\mu\nu\rho\sigma} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$. The factor of \mathbf{i} is chosen so that $(\gamma^5)^\dagger = \gamma^5$. Notice that $(\gamma^5)^2 = 1$ so its eigenvalues are ± 1 . Since it contains one of each of the other four gamma matrices, it anticommutes with each of them: $\{\gamma^5, \gamma^\mu\} = 0, \forall \mu$. Since the Lorentz generators $J_{\text{Dirac}}^{\mu\nu}$ are quadratic in γ s, this implies $[\gamma^5, J_{\text{Dirac}}^{\mu\nu}] = 0$, *i.e.* γ^5 is a Casimir, proportional to the identity on irreps, indeed ± 1 . By direct calculation, in the Weyl basis,

$$\gamma^5 \stackrel{\text{Weyl basis}}{=} \begin{pmatrix} -\mathbb{1} & \\ & \mathbb{1} \end{pmatrix}$$

is ± 1 on right-handed and left-handed spinors, respectively. This means that the *chirality projectors* $P_{R/L} \equiv \frac{1 \pm \gamma^5}{2}$ project onto R/L spinors, respectively, $P_{R/L}^2 = P_{R/L}$. Notice that $P_L \gamma^\mu = \gamma^\mu P_R$ and $P_R \gamma^\mu = \gamma^\mu P_L$.

- Our basis of bispinors can be rewritten using γ^5 . Using $\gamma^{\mu\nu\rho\sigma} = -\mathbf{i} \epsilon^{\mu\nu\rho\sigma} \gamma^5$ and $\gamma^{\mu\nu\rho} = +\mathbf{i} \epsilon^{\mu\nu\rho\sigma} \gamma_\sigma \gamma^5$, we can make a basis of (hermitean) bispinors as in the table below. The modifier ‘pseudo’ here refers to the properties under parity; for

Bispinor	multiplicity	representation
$\bar{\Psi} \mathbb{1} \Psi$	1	scalar
$\bar{\Psi} \gamma^\mu \Psi$	4	vector
$\mathbf{i} \bar{\Psi} \gamma^{\mu\nu} \Psi$	6	antisymmetric tensor
$\mathbf{i} \bar{\Psi} \gamma^\mu \gamma^5 \Psi$	4	pseudovector
$\mathbf{i} \bar{\Psi} \gamma^5 \Psi$	1	pseudoscalar

example, in terms of $\Psi = (\psi_L \ \psi_R)$, consider $\mathbf{i} \bar{\Psi} \gamma^5 \Psi = \mathbf{i} (\psi_L^\dagger \psi_R - \psi_R^\dagger \psi_L)$. It is Lorentz invariant (since it is of the form (5.23)), but under parity $P : \psi_L \leftrightarrow \psi_R$ (that is, parity acts by γ^0 on Dirac spinors) it goes to minus itself.

Why care about these bispinors? One reason is that we can make 4-fermion interactions out of them. For example, $\bar{\Psi} \gamma^{\mu\nu} \Psi \cdot \bar{\Psi} \gamma_{\mu\nu} \Psi$ is a Lorentz-invariant local interaction term which we might add to our Lagrangian.

Another reason is that the vector combinations play an important role:

$$j^\mu \equiv \bar{\Psi}\gamma^\mu\Psi = \psi_R^\dagger\sigma^\mu\psi_R + \psi_L^\dagger\bar{\sigma}^\mu\psi_L$$

is the Noether current associated with the symmetry $\Psi \rightarrow e^{-i\alpha}\Psi$ of the Dirac Lagrangian. You can directly check that $\partial_\mu j^\mu = 0$ using the Dirac equation. Similarly, the *axial* current

$$j_5^\mu \equiv \bar{\Psi}\gamma^\mu\gamma^5\Psi = \psi_R^\dagger\sigma^\mu\psi_R - \psi_L^\dagger\bar{\sigma}^\mu\psi_L$$

would be the Noether current associated with the transformation $\Psi \rightarrow e^{-i\alpha\gamma^5}\Psi$. This transformation rotates the L and R bits oppositely, and is only a symmetry if $m = 0$. Indeed, the Dirac equation implies that $\partial_\mu j_5^\mu = 2im\bar{\Psi}\gamma^5\Psi$. For $E \gg m$, the breaking of this symmetry by m can be ignored and it is still useful⁵⁵. The combinations

$$j_{R/L}^\mu \equiv \bar{\Psi}\gamma^\mu\left(\frac{1 \pm \gamma^5}{2}\right)\Psi$$

involve only the Weyl components and are separately conserved if both j^μ and j_5^μ are conserved.

- Notice that gamma matrices have a lot in common with quaternions.

Coupling to the electromagnetic field. Here's another purpose for the current. Suppose our spinor field is propagating in a background electromagnetic field with vector potential A_μ . The whole thing should be Lorentz invariant, so we should be able to couple them via a Lorentz-invariant Lagrangian. How can we resist adding $\mathcal{L}_{EM} = -ej^\mu A_\mu$ for some constant e (blame Ben Franklin for the sign). The full Lagrangian is then

$$\bar{\Psi}[\mathbf{i}(\partial_\mu + \mathbf{i}eA_\mu)\gamma^\mu - m]\Psi$$

and the Dirac equation is modified to

$$0 = (\mathbf{i}\gamma^\mu D_\mu - m)\Psi, \quad \text{where } D_\mu\Psi \equiv (\partial_\mu + \mathbf{i}eA_\mu)\Psi$$

is the gauge covariant derivative in the following sense: $D_\mu\Psi \mapsto e^{-i\alpha(x)}D_\mu\Psi$ under $\Psi \rightarrow e^{-i\alpha(x)}\Psi(x), A_\mu \rightarrow A_\mu + \frac{1}{e}\partial_\mu\alpha$. We could have used the demand that the action respects this transformation to determine the coupling to A_μ to replace $\partial_\mu \rightarrow D_\mu$.

The solutions of the Dirac equation in a background EM field are no longer solutions of the KG equation:

$$0 = (\mathbf{i}\not{D} + m)(\mathbf{i}\not{D} - m)\Psi = (\mathbf{i}D_\mu\mathbf{i}D_\nu\gamma^\mu\gamma^\nu - m^2)\Psi.$$

⁵⁵Beware that the coupling to gauge field also breaks this symmetry, quantum mechanically. This is called the chiral anomaly; more later.

Whereas mixed partials commute, $[\partial_\mu, \partial_\nu] = 0$, the covariant derivatives need not:

$$[D_\mu, D_\nu] = e\mathbf{i}(\partial_\mu A_\nu - \partial_\nu A_\mu) = e\mathbf{i}F_{\mu\nu}$$

so the antisymmetric term matters:

$$0 = \left((\partial_\mu + \mathbf{i}eA_\mu)^2 + e\frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} + m^2 \right) \Psi$$

$$\stackrel{\text{Weyl basis}}{=} \left((\partial_\mu + \mathbf{i}eA_\mu)^2 - e \begin{pmatrix} (\vec{B} + \mathbf{i}\vec{E}) \cdot \vec{\sigma} & \\ & (\vec{B} - \mathbf{i}\vec{E}) \cdot \vec{\sigma} \end{pmatrix} + m^2 \right) \Psi . \quad (5.29)$$

In the last step we used the form of the Lorentz generators $J_{\text{Dirac}}^{\mu\nu} = \frac{\mathbf{i}}{4}[\gamma^\mu, \gamma^\nu]$ in the Weyl basis. This extra term (relative to the gauge covariant scalar wave equation) is an intrinsic magnetic dipole moment of a Dirac particle. This is a consequence of the Dirac equation with implications for the non-relativistic limit.

Notice that we could add extra terms coupling the spin to the EM field strength

$$\mathcal{L}_{\text{DM}} = F_{\mu\nu} (g_m \bar{\Psi} \mathbf{i} \gamma^{\mu\nu} \Psi + g_e \bar{\Psi} \mathbf{i} \gamma^{\mu\nu} \gamma^5 \Psi)$$

but now $[g_{e,m}] = -1$ so these coefficients (which would change the magnetic and electric dipole moments respectively) are suppressed by the inverse of some new mass scale (a priori independent of m), which is presumably large or we would have noticed it, and hence we will ignore such terms for a while.

5.4 Free particle solutions of spinor wave equations

[Peskin §3.3] To understand a quantum scalar field, we had to know that the solutions of the KG equation were plane waves e^{-ipx} with $p^2 = m^2$ (then we associated a mode operator $\mathbf{a}_{\vec{p}}$ with each solution and added them up). To do the analog for spinors, we'll need to know the free particle solutions.

Let's focus on the Dirac equation. This implies the wave equation, so solutions can be made from superpositions of plane waves with $p^2 = m^2$

$$\Psi(x) = e^{-ipx}u(p)$$

but the Dirac equation places a further restriction on the constant spinor $u(p)$:

$$0 = (\gamma_\mu p^\mu - m)u(p).$$

Let's assume $m \neq 0$ and solve this in the rest frame, $p_0 = (m, \vec{0})$. Then we can find the answer for general p^μ (with $p^0 > 0$) by a boost: $u(p) = \Lambda_{\frac{1}{2}}u(p_0)$.

$$0 = (m\gamma^0 - m)u(p_0) = m \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ \mathbb{1} & -\mathbb{1} \end{pmatrix} u(p_0)$$

which is solved by $u(p_0) \propto \begin{pmatrix} \xi \\ \xi \end{pmatrix}$ for any 2-component spinor ξ . The fact that there are two solutions for each p is the "intrinsic two-valuedness" associated with spin $\frac{1}{2}$. It will be convenient to normalize the solutions by

$$u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \quad \xi^\dagger \xi = 1.$$

We can choose a basis for such ξ s which diagonalize σ^3 , e.g. $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in the standard basis for the Paulis. ξ is an ordinary non-relativistic spinor.

Now, under a boost in the z direction (suppressing the x, y components which remain zero),

$$p_0 \mapsto \begin{pmatrix} E \\ p^3 \end{pmatrix} = \exp\left(\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m \cosh \eta \\ m \sinh \eta \end{pmatrix}, \quad (5.30)$$

and the positive-energy solution of the Dirac equation becomes⁵⁶

$$\begin{aligned}
u(p_0) \mapsto \Lambda_{\frac{1}{2}}(\eta)u(p_0) &= \underbrace{\exp\left(-\frac{1}{2}\eta\begin{pmatrix}\sigma^3 & \\ & -\sigma^3\end{pmatrix}\right)}_{\cosh(\eta/2)\mathbb{1}-\sinh(\eta/2)\begin{pmatrix}\sigma^3 & \\ & -\sigma^3\end{pmatrix}} \sqrt{2m}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \\
&= \begin{pmatrix}\left(\begin{matrix}\sqrt{E+p^3}P_+ + \sqrt{E-p^3}P_- \\ \sqrt{E-p^3}P_+ + \sqrt{E+p^3}P_-\end{matrix}\right)\xi \\ \left(\begin{matrix}\sqrt{E+p^3}\sigma^3P_+ + \sqrt{E+p^3}\sigma^3P_- \\ \sqrt{E-p^3}\sigma^3P_+ + \sqrt{E-p^3}\sigma^3P_-\end{matrix}\right)\xi\end{pmatrix} & P_{\pm} \equiv \frac{1}{2}(1 \pm \sigma^3) \\
&= \begin{pmatrix}\left(\begin{matrix}\sqrt{E+p^3}\sigma^3P_+ + \sqrt{E+p^3}\sigma^3P_- \\ \sqrt{E-p^3}\sigma^3P_+ + \sqrt{E-p^3}\sigma^3P_-\end{matrix}\right)\xi \\ \left(\begin{matrix}\sqrt{E+p^3}\sigma^3P_+ + \sqrt{E+p^3}\sigma^3P_- \\ \sqrt{E-p^3}\sigma^3P_+ + \sqrt{E-p^3}\sigma^3P_-\end{matrix}\right)\xi\end{pmatrix} = \begin{pmatrix}\sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi\end{pmatrix}.
\end{aligned}$$

In the final expression, we define the square root of a matrix by its action on eigenstates. The last expression also works for any boost direction since it's rotation invariant. Using the identity

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 \quad (5.34)$$

(check it on the homework) we can check directly that this expression actually solves the Dirac equation for general p :

$$(p_{\mu}\gamma^{\mu} - m)\begin{pmatrix}\sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi\end{pmatrix} = \begin{pmatrix}-m & \sigma \cdot p \\ \bar{\sigma} \cdot p & -m\end{pmatrix}\begin{pmatrix}\sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi\end{pmatrix} = \begin{pmatrix}-m\sqrt{p \cdot \sigma} + \sqrt{p^2}\sqrt{p \cdot \sigma} \\ \sqrt{\bar{\sigma} \cdot p}\sqrt{p^2} - \sqrt{\bar{\sigma} \cdot p}m\end{pmatrix}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \stackrel{p^2=m^2}{=} 0.$$

Negative-energy solutions. So far we've found two independent solutions of the Dirac equation for each \vec{p} . But if we think about Fourier transforming the Dirac equation just in space, we get four linear first-order ODEs (for each \vec{p}), which should have four linear-independent solutions.

Just as for the KG equation, there are also negative-energy solutions with the same \vec{p} (which are not related to the previous by any orthochronous Lorentz transformation):

$$\Psi(x) = v(\vec{p})e^{+ip \cdot x}, \quad p^2 = m^2, p^0 > 0$$

⁵⁶Some useful kinematical relations: adding and subtracting

$$2 \cosh \eta = e^{\eta} + e^{-\eta} = 2E/m \quad (5.31)$$

$$2 \sinh \eta = e^{\eta} - e^{-\eta} = 2p/m \quad (5.32)$$

(with $p \equiv p^3$), we learn that $e^{\pm\eta} = 2\frac{E \pm p}{m}$, and therefore

$$\cosh \eta/2 = \frac{e^{\eta/2} + e^{-\eta/2}}{2} = \sqrt{\frac{E+p}{2m}} + \sqrt{\frac{E-p}{2m}}, \quad \sinh \eta/2 = \frac{e^{\eta/2} - e^{-\eta/2}}{2} = \sqrt{\frac{E+p}{2m}} - \sqrt{\frac{E-p}{2m}}. \quad (5.33)$$

where the Dirac equation further imposes $(\not{p} + m)v(\vec{p}) = 0$, solved by

$$v^s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{pmatrix}, \quad s = 1, 2.$$

(If we wish to compare solutions with the same spatial momentum as we studied for the positive-energy solutions, we should write $\Psi(x) = v(-\vec{q})e^{-iq \cdot x}$, with $q = -p$. Notice that the spinors are a function of just a 3-vector, since p^0 is determined by the wave equation.)

Normalization. A Lorentz-invariant normalization condition is

$$\bar{u}^s u^r = 2m \xi_s^\dagger \xi_r = 2m \delta_{sr} \quad \bar{v}^r v^s = -2m \delta^{rs}.$$

This is equivalent to the following statements about the Lorentz-variant quantities:

$$u_r^\dagger(p) u_s(p) = 2E_p \xi_r^\dagger \xi_s = 2E_p \delta_{rs} \quad v_r^\dagger v_s = +2E_p \eta_r^\dagger \eta_s = 2E_p \delta_{rs}.$$

Notice that for each \vec{p} , $0 = \bar{u}^r(\vec{p})v^s(\vec{p}) = \bar{v}^r(\vec{p})u^s(\vec{p})$ (but $(u^r)^\dagger(\vec{p})v^s(\vec{p}) \neq 0$). It will also be useful below to note that $u_r^\dagger(\vec{p})v_s(-\vec{p}) = 0$.

Completeness relations. Suppose we choose a basis of two-component spinors:

$$\mathbb{1}_{2 \times 2} = \sum_{s=1,2} \xi^s (\xi^s)^\dagger. \quad (5.35)$$

Then

$$\begin{aligned} \sum_{s=1,2} u^s(p) \bar{u}^s(p) &= \sum_s \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \left((\xi^s)^\dagger \sqrt{p \cdot \bar{\sigma}}, (\xi^s)^\dagger \sqrt{p \cdot \sigma} \right) \stackrel{(5.35)}{=} \begin{pmatrix} \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} & (\sqrt{p \cdot \sigma})^2 \\ (\sqrt{p \cdot \bar{\sigma}})^2 & \sqrt{p \cdot \bar{\sigma}} \sqrt{p \cdot \sigma} \end{pmatrix} \\ &\stackrel{(5.34)}{=} \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} = \gamma \cdot p + m \equiv \not{p} + m. \end{aligned} \quad (5.36)$$

Similarly, $\sum_s v \bar{v} = \not{p} - m$. Note that these completeness relations for spinor polarizations are analogous to the relation I quoted for photon polarizations (1.39).

Helicity. Define the *helicity* operator, acting on one-particle states

$$\hat{h} \equiv \hat{p} \cdot \vec{S} \equiv \hat{p}^i \cdot J_{\text{Dirac}}^i = \hat{p}^i \frac{1}{2} \begin{pmatrix} \sigma^i & \\ & \sigma^i \end{pmatrix}.$$

$\hat{p} = \frac{\vec{p}}{|\vec{p}|}$ is a unit vector, so $\hat{h}^2 = 1/4$ and the eigenvalues are $\pm 1/2$, which are called right-handed and left-handed. (Sometimes people normalize the helicity so that $h = \pm 1$.) Naturally, positive-energy (massless) Weyl R/L spinors are helicity eigenstates, with $h = \pm 1/2$ respectively, since the R/L Weyl equation is $0 = p_0 \mathbb{1} \mp p^i \sigma^i$, and

$p_0 = |\vec{p}|$. More generally, consider the ultra-relativistic limit of a Dirac spinor, where $E = \sqrt{\vec{p}^2 + m^2} \rightarrow |\vec{p}|$, with (WLOG) $\vec{p} = \hat{z}p^3$,

$$u(E_p, p^3) = \begin{cases} \begin{pmatrix} \sqrt{E - p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \xrightarrow{E_p \rightarrow |\vec{p}|} \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, & \text{if } \sigma^3 = +1 \\ \begin{pmatrix} \sqrt{E + p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E - p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \xrightarrow{E_p \rightarrow |\vec{p}|} \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & \text{if } \sigma^3 = -1 \end{cases}$$

These are helicity eigenstates. More generally, any ultrarelativistic spinor wavefunction can be decomposed into a linear combination of such helicity eigenstates. For massive particles (away from this limit), you can switch the helicity by outrunning the particle; it is not Lorentz invariant. [\[End of Lecture 18\]](#)

5.5 Quantum spinor fields

What did we need to build the Fock space of a relativistic scalar field? A quick recapitulation: For each mode, we had ladder operators, $\mathbf{a} \neq \mathbf{a}^\dagger$, and a number operator $\mathbf{N} = \mathbf{a}^\dagger \mathbf{a}$ which was hermitian and hence observable. The ladder operators are so called because

$$[\mathbf{N}, \mathbf{a}] = -\mathbf{a}, \quad [\mathbf{N}, \mathbf{a}^\dagger] = \mathbf{a}^\dagger. \quad (5.37)$$

This says that given a number eigenstate $\mathbf{N} |n\rangle = n |n\rangle$, we can make others by $\mathbf{N}(\mathbf{a} |n\rangle) \stackrel{(5.37)}{=} (n-1)(\mathbf{a} |n\rangle)$ and $\mathbf{N}(\mathbf{a}^\dagger |n\rangle) \stackrel{(5.37)}{=} (n+1)(\mathbf{a}^\dagger |n\rangle)$. And we know $n \geq 0$ since $0 \leq \|\mathbf{a} |n\rangle\|^2 = \langle n | \mathbf{N} |n\rangle = n \langle n | n \rangle$, so there must exist a lowest n_0 that we can't lower any further, $\mathbf{a} |n_0\rangle = 0$, but then $n_0 |n_0\rangle = \mathbf{N} |n_0\rangle = \mathbf{a}^\dagger \underbrace{\mathbf{a} |n_0\rangle}_{=0} = 0$, so $n_0 = 0$. Hence, a ladder of eigenstates of \mathbf{N} with eigenvalues $0, 1, 2, 3, \dots$

OK, but here's why I just did that: the necessary equation (5.37) did *not* require that $[\mathbf{a}, \mathbf{a}^\dagger] = 1$. In fact, (5.37) would also follow from the *anticommutation* relation (for each mode)

$$\mathbf{a}\mathbf{a}^\dagger + \mathbf{a}^\dagger\mathbf{a} \equiv \{\mathbf{a}, \mathbf{a}^\dagger\} = 1, \quad \{\mathbf{a}, \mathbf{a}\} = 0 = \{\mathbf{a}^\dagger, \mathbf{a}^\dagger\}.$$

To see this, note the identity $[AB, C] = A\{B, C\} - \{A, C\}B$, so

$$[\mathbf{a}^\dagger\mathbf{a}, \mathbf{a}] = \mathbf{a}^\dagger\{\mathbf{a}, \mathbf{a}\} - \{\mathbf{a}^\dagger, \mathbf{a}\}\mathbf{a} = -\mathbf{a}.$$

But now $0 = \{\mathbf{a}, \mathbf{a}\} = 2\mathbf{a}^2$ means $\mathbf{a}^2 = 0$ and $(\mathbf{a}^\dagger)^2 = 0$: the ladder has only two rungs: $|0\rangle$ with $\mathbf{a} |0\rangle = 0$ and $|1\rangle = \mathbf{a}^\dagger |0\rangle$, and there is no $|2\rangle \propto (\mathbf{a}^\dagger)^2 |0\rangle = 0$. This is Pauli exclusion.

For multiple modes, we can take

$$\{\mathbf{a}_i, \mathbf{a}_j^\dagger\} = \delta_{ij}, \quad \{\mathbf{a}_i, \mathbf{a}_j\} = 0 = \{\mathbf{a}_i^\dagger, \mathbf{a}_j^\dagger\}.$$

And states look like

$$\mathbf{a}_i^\dagger \mathbf{a}_j^\dagger |0\rangle = \pm |00001_i 00\dots 001_j 0000\rangle = -\mathbf{a}_j^\dagger \mathbf{a}_i^\dagger |0\rangle \quad (5.38)$$

we have to remember the order in which the quanta are created to get the sign right (the overall choice is a convention, but relative signs are physics). This is Fermi statistics.

Given a field, how do we know whether to use commutators or anticommutators? A practical answer: try both and one of them will be *bad* somehow. The general answer, with Lorentz invariance, is the *spin-statistics* theorem: fields with half-integer spin are fermionic, and those with integer spin are bosonic. Schwartz, §12 has an illuminating

discussion of many points of view on the connection between spin and statistics. I'll say a little more below.

Anticommuting scalar fields? Consider a real scalar $\phi(x) = \int \frac{d^d p}{\sqrt{2\omega_p}} \mathbf{a}_p e^{-ipx} + h.c.$, with $\pi = \dot{\phi}$ and

$$H = \frac{1}{2} \int \left(\pi^2 + \left(\vec{\nabla} \phi \right)^2 + m^2 \phi^2 \right) = \int d^d p \frac{\omega_p}{2} \left(\mathbf{a}_p^\dagger \mathbf{a}_p + \mathbf{a}_p \mathbf{a}_p^\dagger \right).$$

You see that if $\{\mathbf{a}_p, \mathbf{a}_{p'}^\dagger\} = \delta^d(p-p')$ then we get a hamiltonian which is just an infinite constant, independent of the state. Not such a useful energy functional. We also get $\{\phi(x), \pi(y)\} = 0$.

The situation is even worse for a complex scalar, with $\Phi(x) = \int \frac{d^d p}{\sqrt{2\omega_p}} \left(\mathbf{a}_p e^{-ipx} + \mathbf{b}_p^\dagger e^{ipx} \right)$. Then with anticommutators we get

$$H \stackrel{\text{no assumptions}}{=} \int d^d p \frac{1}{2} \omega_p \left(\mathbf{a}_p^\dagger \mathbf{a}_p + \mathbf{b}_p \mathbf{b}_p^\dagger \right) \stackrel{\{\mathbf{b}, \mathbf{b}^\dagger\} = \delta}{=} \int d^d p \frac{1}{2} \omega_p \left(\mathbf{a}_p^\dagger \mathbf{a}_p - \mathbf{b}_p^\dagger \mathbf{b}_p \right) + \text{cst} \quad (5.39)$$

and the energy is unbounded below. We could alternatively allow negative norm states or non-local anticommutators. Is that any better?

Dirac Hamiltonian. From the Dirac lagrangian, we have the canonical momentum density $\Pi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = \mathbf{i} \bar{\Psi} \gamma^0 = \mathbf{i} \Psi^\dagger$ and the hamiltonian density

$$\mathfrak{h} = \Pi \dot{\Psi} - \mathcal{L} = \underbrace{\mathbf{i} \Psi^\dagger \partial_t \Psi}_{= \mathbf{i} \bar{\Psi} \gamma^0 \partial_t \Psi} - \mathcal{L} = \bar{\Psi} \left(\mathbf{i} \vec{\gamma} \cdot \vec{\nabla} + m \right) \Psi \stackrel{\text{eom}}{=} \Psi^\dagger \mathbf{i} \partial_t \Psi. \quad (5.40)$$

Following our nose and writing the operator-valued field as a sum over all solutions of the eom weighted by ladder operators, we have⁵⁷

$$\begin{aligned} \Psi(x) &= \int \frac{d^3 p}{\sqrt{2\omega_p}} \sum_{s=1,2} \left(u^s(p) e^{-ipx} \mathbf{a}_p^s + v^s(p) e^{ipx} \mathbf{b}_p^{s\dagger} \right) \\ \bar{\Psi}(x) &= \int \frac{d^3 p}{\sqrt{2\omega_p}} \sum_{s=1,2} \left(\bar{u}^s(p) e^{ipx} \mathbf{a}_p^{s\dagger} + \bar{v}^s(p) e^{-ipx} \mathbf{b}_p^s \right) \end{aligned} \quad (5.41)$$

where, as for the scalar, we implicitly set $p^0 = \omega_p$.

The hamiltonian is then (using the last expression in (5.40))

$$H = \int d^3 x \mathfrak{h}$$

⁵⁷If you haven't already, open your copy of Peskin and cross out equation 3.92, which describes 'how not to quantize the Dirac equation'.

$$= \int d^3x \int \frac{d^3p}{\sqrt{2\omega_p}} \int \frac{d^3q}{\sqrt{2\omega_q}} \sum_{ss'} (u^{s\dagger}(p)e^{ipx} \mathbf{a}_p^{s\dagger} + v^{s\dagger}(p)e^{-ipx} \mathbf{b}_p^s) (\omega_q u^s(q)e^{-iqx} \mathbf{a}_q^s - \omega_q v^s(q)e^{iqx} \mathbf{b}_q^{s\dagger})$$

This is of the form $u^\dagger u - v^\dagger v + u^\dagger v - v^\dagger u$. In the first two terms, the x integral is $\int d^3x e^{i(p-q)x} = \delta^{(3)}(p-q)$ while in the mixed uv terms, we have $\vec{q} = -\vec{p}$. We use the spinor identities

$$u_s^\dagger(p)u_{s'}(p) = v_s^\dagger(p)v_{s'}(p) = +2\omega_p \delta_{ss'}, \quad u_s^\dagger(p)v_{s'}(-p) = v_s^\dagger(p)u_{s'}(-p) = 0$$

and get (no assumptions about commutators or anticommutators yet)

$$H = \int d^3p \omega_p \sum_s (\mathbf{a}_p^{s\dagger} \mathbf{a}_p^s - \mathbf{b}_p^s \mathbf{b}_p^{s\dagger}). \quad (5.42)$$

Now, if $[\mathbf{b}, \mathbf{b}^\dagger] = 1$, this is $\sum_p \omega_p (N_p^a - N_p^b) + \text{constant}$ and the world explodes in a spontaneous shower of antiparticles lowering the energy by coming from nowhere. But if instead we have anticommutation relations,

$$\{\mathbf{b}_s(p), \mathbf{b}_{s'}(q)^\dagger\} = \delta^d(p-q),$$

then this is

$$H = \int d^3p \omega_p \sum_s (N_s^a(p) + N_s^b(p)) + \text{const}$$

and all is well, $H - E_0 \geq 0$.⁵⁸

This gives canonical equal time local anticommutators

$$\{\Psi(\vec{x})_a, \Pi(\vec{y})_b\}_{ET} = \mathbf{i} \delta^d(\vec{x} - \vec{y}) \delta_{ab}$$

or

$$\{\Psi(\vec{x})_a, \bar{\Psi}(\vec{y})_b\}_{ET} = \mathbf{i} \gamma_{ab}^0 \delta^d(\vec{x} - \vec{y}).$$

Comments on the spin-statistics connection. [Schwartz's very nice §12] Two more comments about spin-statistics. First, the general pattern seen above continues for all spins. The crucial ingredient is that with Lorentz invariance, integer-spin fields have Lagrangians that involve two time derivatives:

$$\mathcal{L}_{\mathbb{Z}} \ni \phi_{\rho\dots} \partial_\mu \partial^\mu \phi^{\rho\dots} = \phi_{\rho\dots} \ddot{\phi}^\rho + \dots,$$

⁵⁸Here we only used $\{\mathbf{b}, \mathbf{b}^\dagger\} = \delta$, without any assumption about the commutators or anticommutators of \mathbf{a} . You could ask why we can't have $[\mathbf{a}, \mathbf{a}^\dagger] = 1$ still. That would just be gross. Slightly more scientifically, we can show (see below) that the Lorentz invariance of the propagator would fail.

while half-integer-spin fields have Lagrangians with a single time-derivative:

$$\mathcal{L}_{\mathbb{Z}+\frac{1}{2}} \ni \bar{\psi}_{\alpha\rho\dots} \gamma_{\alpha\beta}^{\mu} \partial_{\mu} \psi_{\beta}^{\rho\dots}.$$

In the former case, this means the Hamiltonian density is $\mathfrak{h}_{\mathbb{Z}} = \phi_{\rho\dots} \ddot{\phi}^{\rho\dots}$, while in the other case, $\mathfrak{h}_{\mathbb{Z}+\frac{1}{2}} = \psi_{\alpha\rho\dots}^{\dagger} \partial_t \psi^{\alpha\rho\dots}$, leading to the same relative signs we found in (5.39) and (5.42).

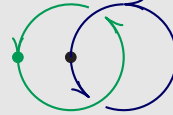
The statistics are defined by the swap operation as in (5.38). More generally, consider a two-particle state $|\phi_1(x_1)\phi_2(x_2)\rangle$ where we temporarily label the particles 1 and 2, and the wavefunctions $\phi_{1,2}$ are localized at positions $x_{1,2}$. The swap operator acts to interchange the positions of the particles

$$\text{SWAP}_{12} |\phi_1(x_1)\phi_2(x_2)\rangle = e^{i\phi_{\kappa}} |\phi_2(x_1)\phi_1(x_2)\rangle.$$

As you can see, we've allowed for a phase to appear; κ is a label depending on the species of particles.

In $d \geq 3$, the phase $e^{i\phi_{\kappa}}$ can only be a sign, ± 1 . To see this, ask: how do we implement the swap operator? More generally, if the particles are identical, what can happen in between two states with a particle at x_1 and x_2 ? We must move the particles on some trajectory which ends up with $1 \rightarrow 1, 2 \rightarrow 2$ or $1 \leftrightarrow 2$ at the end. Let $\varphi \equiv$ the angle by which particle 1 went around 2. φ is a topological property of the trajectory of the two particles. But in $d = 3$, φ is defined only mod 2π , because there is no non-trivial linking of curves in 3+1 dimensions. You can see this by visualizing the

following figure in three plus one dimensions:

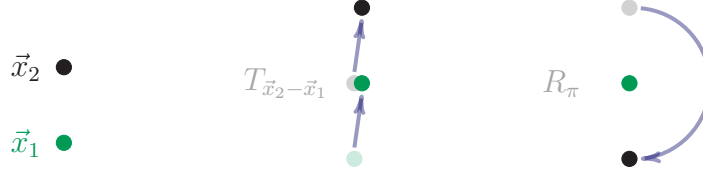


So the phase ϕ_{κ} must be $\phi_{\kappa} = \pi\kappa$ with $\kappa \in \mathbb{Z}$, a property of the particles (in fact, twice the spin), so that in $d \geq 3$,

$$\text{SWAP}_{12} |\phi_1(x_1)\phi_2(x_2)\rangle = (-1)^{\kappa} |\phi_2(x_1)\phi_1(x_2)\rangle.$$

In $d = 2$, there is a lot more to the story of particle statistics (because two curves in 2+1 dimensions can have a topologically-nontrivial linking).

Now, this swap of two particles can be accomplished using Poincaré transformations (Lorentz times translations), as follows. First translate by $x_2 - x_1$. Then rotate around x_1 by π .



Therefore, whatever phase ϕ_κ is going to happen under interchange is already built into the rotation matrices (translations don't produce any phase (at least in the absence of a magnetic field)).

Note that I am demanding that the statistics phase ϕ_κ depends only the topology of the path by which we exchange the particles. So the phase we get from the protocol above is the answer for any other protocol which may not just be a rotation.

On a Dirac spinor particle at rest, the action of a Lorentz transformation is (recall our expression for the Lorentz current)

$$|p^\mu, a\rangle \rightarrow \Lambda_{\frac{1}{2}}(\theta)^a_b |\Lambda^\mu_\nu(\theta)p^\nu, b\rangle, \quad (5.43)$$

with $\Lambda_{\frac{1}{2}}(\theta) = e^{-i\theta_{\mu\nu}J_{\text{Dirac}}^{\mu\nu}}$. (Here $|p, a\rangle = u_a^r(p) |p, r\rangle$, and $|p, r\rangle \equiv \sqrt{2\omega_p} a_{p,r}^\dagger |0\rangle$.) For the particular case of a rotation in the xy -plane, this is a diagonal matrix

$$\Lambda_{\frac{1}{2}}(\vec{\theta} = \theta_z \hat{z}, \beta = 0) = e^{i\theta_z \sigma^3 \otimes \mathbb{1}_{L/R}} = \begin{pmatrix} e^{i\theta_z/2} & & & \\ & e^{-i\theta_z/2} & & \\ & & e^{i\theta_z/2} & \\ & & & e^{-i\theta_z/2} \end{pmatrix} \xrightarrow{\theta^z \rightarrow \pi} \begin{pmatrix} \mathbf{i} & & & \\ & -\mathbf{i} & & \\ & & \mathbf{i} & \\ & & & -\mathbf{i} \end{pmatrix} = (-1)^{s_z} \mathbf{i}.$$

These factors of \mathbf{i} are a direct consequence of the fact that the 2π rotation gives -1 , so a π rotation gives $\pm\mathbf{i}$ (where the sign depends on the spin σ^z).

Now suppose the initial spin of each particle is in the \hat{z} direction (if not, the state will not be an eigenstate of Λ), so that (*e.g.* for the $\sigma_z = +1$ eigenstate) $\psi_1 = (1, 0, 1, 0) \mapsto \Lambda\psi_1 = (\mathbf{i}, 0, \mathbf{i}, 0) = \mathbf{i}\psi_1$. Then the Poincaré transformation that implements the swap of two such particles is

$$T_{\vec{x}_2 - \vec{x}_1} \Lambda(\vec{\theta} = \pi \hat{z}) |\psi_1(\vec{x}_1)\psi_2(\vec{x}_2)\rangle \stackrel{s_1^z = s_2^z}{=} (\pm\mathbf{i})^2 |\psi_1(\vec{x}_2)\psi_2(\vec{x}_1)\rangle = -|\psi_1(\vec{x}_2)\psi_2(\vec{x}_1)\rangle.$$

For either spin (as long as the spins of the two particles are the same), the two π rotations combine to give a (-1) – Fermi statistics follows from the spinor representation matrices.

Dirac propagator. [Peskin §4.7] Now we will efficiently redo the story of interaction picture perturbation theory and Wick's theorem for Dirac spinor fields. The story differs by just a few very important signs.

Time ordering for fermions is defined with an extra minus sign:

$$\mathcal{T}(A_1(x_1) \cdots A_n(x_n)) \equiv (-1)^P A_{1'}(x_{1'}) \cdots A_{n'}(x_{n'}), \quad x_{1'}^0 > x_{2'}^0 > \cdots > x_{n'}^0$$

where $P \equiv$ the number of fermion interchanges required to get from the ordering $1 \dots n$ to the ordering $1' \dots n'$ (mod two). Similarly, normal ordering is

$$: ABC \cdots : \equiv (-1)^P A' B' C' \cdots ,$$

where on the RHS all annihilation operators are to the right of all creation operators, and P is defined the same way.⁵⁹ Wick's theorem is then still

$$\mathcal{T}(ABC \cdots) = : ABC \cdots : + \text{all contractions.}$$

The possible contractions of Dirac fields are:

$$\overbrace{\Psi(x)\Psi(y)} = 0, \quad \overbrace{\bar{\Psi}(x)\bar{\Psi}(y)} = 0, \quad \overbrace{\Psi(x)\bar{\Psi}(y)} = S_F(x-y),$$

where S_F is the Feynman propagator for the Dirac field, *i.e.* the time-ordered vacuum two-point function. It is

$$\begin{aligned} S_F^{ab}(x-y) &= \langle 0 | \mathcal{T}(\Psi^a(x) \bar{\Psi}^b(y)) | 0 \rangle \\ &= \theta(x^0 - y^0) \langle 0 | \Psi^a(x) \bar{\Psi}^b(y) | 0 \rangle - \theta(y^0 - x^0) \langle 0 | \bar{\Psi}^b(y) \Psi^a(x) | 0 \rangle \\ &= \theta(x^0 - y^0) \underbrace{\langle 0 | \Psi_a^{(+)}(x) \bar{\Psi}_b^{(-)}(y) | 0 \rangle}_{= \langle 0 | \{\Psi_a^{(+)}(x), \bar{\Psi}_b^{(-)}(y)\} | 0 \rangle \equiv S^+} - \theta(y^0 - x^0) \underbrace{\langle 0 | \bar{\Psi}_b^{(+)}(y) \Psi_a^{(-)}(x) | 0 \rangle}_{= \langle 0 | \{\bar{\Psi}_b^{(+)}(y), \Psi_a^{(-)}(x)\} | 0 \rangle \equiv S^-} \end{aligned}$$

$$\Psi \equiv \underbrace{\Psi^{(+)}}_a + \underbrace{\Psi^{(-)}}_{b^\dagger}$$

The S^+ bit, made only from \mathbf{a} s, is a c-number

$$\begin{aligned} S_{ab}^+(x-y) &= \{\Psi_a^{(+)}(x), \bar{\Psi}_b^{(-)}(y)\} = \int \frac{d^3p}{\sqrt{2\omega_{\vec{p}}}} e^{-ipx} \sum_{s=1,2} \int \frac{d^3q}{\sqrt{2\omega_{\vec{q}}}} e^{+iqy} \sum_{s'=1,2} u_a^s(p) \bar{u}_b^{s'}(q) \underbrace{\{\mathbf{a}_p^s, \mathbf{a}_q^{s'\dagger}\}}_{= \delta^{sd} \delta_{(p-q)} \delta^{ss'}} \\ &= \int \frac{d^3p}{2\omega_{\vec{p}}} e^{-ip(x-y)} \underbrace{\sum_s u_a^s(p) \bar{u}_b^s(p)}_{\stackrel{(5.36)}{=} (\not{p} + m)_{ab}} \\ &= \int \frac{d^3p}{2\omega_{\vec{p}}} (\mathbf{i}\not{\partial}_x + m)_{ab} e^{-ip(x-y)} \end{aligned}$$

⁵⁹Why the extra signs? One way to see that they must be there is if they weren't everything would be zero. With the sign, the following two choices for a normal ordered product are equivalent:

$$: a_p a_q a_r^\dagger := (-1)^2 a_r^\dagger a_p a_q = (-1)^3 a_r^\dagger a_q a_p,$$

but without it, we would conclude that the LHS would be zero.

$$\begin{aligned}
&= (\mathbf{i}\not{\partial}_x + m)_{ab} \underbrace{\int \frac{\mathrm{d}^3 p}{2\omega_{\vec{p}}} e^{-\mathbf{i}p(x-y)}}_{=\Delta^+(x-y)} \quad \Delta^+(x-y) \stackrel{(3.5)}{=} \int_{C^+} \mathrm{d}^4 p e^{-\mathbf{i}p(x-y)} \frac{\mathbf{i}}{p^2 - m^2} \\
&= \int_{C^+} \mathrm{d}^4 p e^{-\mathbf{i}p(x-y)} \frac{\mathbf{i}(\not{p} + m)_{ab}}{p^2 - m^2}
\end{aligned} \tag{5.44}$$

The same calculation for S^- , the bit involving $\{\mathbf{b}, \mathbf{b}^\dagger\}$, gives the same integrand, the only difference, as for the complex KG field, is the contour $C^+ \rightarrow C^-$. Getting the same integrand with the same sign required the relative minus (the red one above, which came from the fermionic definition of time-ordering) which for bosons came from the sign in the commutator $[\mathbf{b}, \mathbf{b}^\dagger] = \mathbf{b}\mathbf{b}^\dagger - \mathbf{b}^\dagger\mathbf{b}$. Adding the two terms together, we learn that the momentum space Dirac propagator is

$$\tilde{S}(p) = \frac{\mathbf{i}(\not{p} + m)}{p^2 - m^2} = \frac{\mathbf{i}(\not{p} + m)}{(\not{p} + m)(\not{p} - m)} = \frac{\mathbf{i}}{\not{p} - m}. \tag{5.45}$$

(These matrices all commute with each other, so my cavalier manipulation of them can be done in the eigenbasis of \not{p} without trouble.) It is not a coincidence that the numerator of the propagator is the polarization sum: $\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m$.

The position-space Feynman propagator comes from integrating (5.45) over the Feynman contour, as for scalars:

$$S_F(x-y) = \int_{C_F} \mathrm{d}^4 p \frac{\mathbf{i}}{\not{p} - m} e^{-\mathbf{i}p(x-y)}.$$

Fermions and causality. Earlier I made a big deal that we need commutators to vanish outside the lightcone to prevent acausal communication. But the Dirac field $\Psi(x)$ *doesn't* commute with $\bar{\Psi}(y)$ for spacelike $x-y$ (rather, they anticommute). Why is this OK? What saves the day is the fact that we can't measure a single fermion operator. The operators we can measure (such as the number density of fermions $\Psi^\dagger\Psi$, or their momentum density $\Psi^\dagger\mathbf{i}\vec{\nabla}\Psi$) are all made of even powers of Ψ and $\bar{\Psi}$. And these do commute outside the lightcone.

A principle that would make this restriction on what we can measure precisely true and inevitable is if *fermion parity is gauged*. By 'fermion parity' I mean the transformation that takes $\Psi \rightarrow -\Psi$ for every fermionic operator in the world. By 'gauged' I mean that this transformation should be regarded as an equivalence relation, rather than a transformation that relates distinct physical configurations. In that case, a local operator with an odd number of fermions would not be gauge invariant. ⁶⁰

⁶⁰This point of view, that locality should be built into the Hilbert space, and that therefore that

5.6 Feynman rules with spinor fields

[Peskin §4.7] We can now write down Feynman rules for Dirac fermion fields. They follow from an analysis exactly parallel to what we did earlier by interaction-picture perturbation theory for scalars, with a few extra signs, and a few extra matrices in spinor space.

As always in these notes, time goes to the left, so I draw the initial state on the right (like the ket) and the final state on the left (like the bra).

1. An internal fermion line gives

$$\begin{array}{c} \overleftarrow{k} \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} = \frac{i}{\not{k} - m_\Psi}$$

which is a matrix on the spinor indices (not shown).

There are four possibilities for an external fermion line of definite momentum.

$$2. \quad \dots \overleftarrow{k} \longleftarrow \bullet = \overline{\Psi}|k, r\rangle = u^r(k). \quad \text{Here } |k, r\rangle \equiv \sqrt{2\omega_k} a_{k,r}^\dagger |0\rangle, \text{ and this follows just from the mode expansion (5.41).}$$

$$3. \quad \bullet \longleftarrow \overleftarrow{k} \dots = \langle k, r | \overline{\Psi} = \bar{u}^r(k).$$

$$4. \quad \dots \overleftarrow{k} \longrightarrow \bullet = \overline{\Psi}|k, r\rangle = \bar{v}^r(k).$$

$$5. \quad \overleftarrow{k} \longrightarrow \bullet = \langle k, r | \overline{\Psi} = v^r(k).$$

6. Some advice: When evaluating a Feynman diagram with spinor particles, always begin at the head of the particle-number arrows on the fermion lines, and keep going along the fermion line until you can't anymore. This will keep the spinor indices in the form of matrix multiplication. Why: every Lagrangian you'll ever encounter has fermion parity symmetry, under which every fermionic field gets a minus sign; this means fermion lines cannot end, except on external legs. The

fermion parity should be gauged, is advocated forcefully by Wen, *Quantum Field Theory of Many-Body Systems* (Oxford, 2004) in Chapter 10. It is not clear how to include gravitational degrees of freedom in this principle.

result is always of the form of a scalar function (not a matrix or a spinor) made by sandwiching gamma matrices between external spinors:

$$r'p' \leftarrow \text{---} \cdots \text{---} \leftarrow rp = \sum_{a,b,\dots=1..4} \bar{u}^{r'}(p')_a (\text{pile of gamma matrices})_{ab} u^r(p)_b$$

Furthermore, in S-matrix elements the external spinors $u(p), v(p)$ satisfy the equations of motion $(\not{p} - m)u(p) = 0$, a fact that can be used to our advantage to shrink the pile of gammas.

There can also be fermion lines that form internal loops (though not at tree level, by definition). In this case, the spinor indices form a trace,

$$\sum_a (\text{pile of gamma matrices})_{aa} \equiv \text{tr}(\text{pile of gamma matrices}).$$

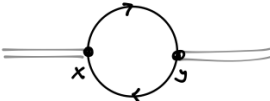
We'll learn to compute such traces below (around (6.3)); in fact, traces appear even in the case with external fermions if we do not measure the spins.

7. Diagrams related by exchanging external fermions have a relative minus sign.
8. Diagrams with an odd number of fermion loops have an extra minus sign.

The last two rules are best understood by looking at an example in detail. A good example of a QFT with interacting fermions (simpler than QED) is the Yukawa theory of a Dirac fermion field plus a scalar ϕ and an interaction density

$$V = g\phi\bar{\Psi}\Psi \quad \Rightarrow \quad \begin{array}{c} p', r' \\ \nearrow \\ \text{---} \\ \searrow \\ p, r \end{array} = -ig\delta^{rr'}. \quad (5.46)$$

Notice that in $3 + 1$ dimensions, $[g] = +4 - [\phi] - 2[\Psi] = 4 - 1 - 2\frac{3}{2} = 0$, the coupling is dimensionless. This describes more realistically the interactions between nucleons (which are fermions, as opposed to snucleons) and scalar pions, which hold together nuclei. It also is a crude sketch of the Higgs coupling to matter; notice that if ϕ is some nonzero constant $\langle\phi\rangle$, then there is a contribution to the mass of the fermions, $g\langle\phi\rangle$.

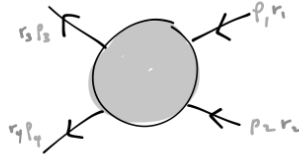
To understand rule 8 consider the following amplitude in the Yukawa theory with interaction (5.46):  It is a contribution to the meson propagator.

It is proportional to

$$\begin{aligned} \sum_{abcd} \delta_{ab}\delta_{cd} \overbrace{\bar{\Psi}_a(x)\Psi_b(x)\Psi_c(y)\Psi_d(y)} &= \sum_{abcd} \delta_{ab}\delta_{cd} (-1)^3 \overbrace{\Psi_b(x)\bar{\Psi}_c(y)\Psi_d(y)\bar{\Psi}_a(x)} \\ &= (-1)\delta_{ab}\delta_{cd} S_F^{bc}(x-y) S_F^{da}(y-x) = -\text{tr} S_F(x-y) S_F(y-x). \end{aligned} \quad (5.47)$$

[Peskin page 119] To understand rule 7 consider $\Psi\Psi \rightarrow \Psi\Psi$ (nucleon) scattering

in the Yukawa theory:



The blob represents the matrix

element

$${}_0 \langle p_3 r_3; p_4 r_4 | \mathcal{T} e^{-i \int V d^4 z} | p_1 r_1; p_2 r_2 \rangle_0$$

(up to a factor of $-i\delta(\sum p)$) where the initial state is

$$|p_1 r_1; p_2 r_2\rangle_0 = +\sqrt{2\omega_1 2\omega_2} \mathbf{a}_{p_1}^{r_1 \dagger} \mathbf{a}_{p_2}^{r_2 \dagger} |0\rangle$$

– note the convention for the order of anticommuting creation operators. The final state is then

$${}_0 \langle p_3 r_3; p_4 r_4 | = (|p_3 r_3; p_4 r_4\rangle_0)^\dagger \propto \langle 0 | \mathbf{a}_{p_4}^{r_4} \mathbf{a}_{p_3}^{r_3} = -\langle 0 | \mathbf{a}_{p_3}^{r_3} \mathbf{a}_{p_4}^{r_4}$$

where note that the dagger reverses the order.

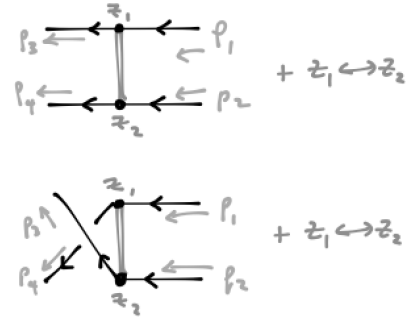
The leading contribution comes at second order in V :

$${}_0 \langle p_3 r_3; p_4 r_4 | \mathcal{T} \left(\frac{1}{2!} (\mathbf{i}g)^2 \int d^4 z_1 \int d^4 z_2 (\bar{\Psi}\Psi\phi)_1 (\bar{\Psi}\Psi\phi)_2 \right) | p_1 r_1; p_2 r_2 \rangle_0$$

To get something nonzero we must contract the ϕ s with each other. The diagrams at right indicate best the possible ways to contract the fermions. Exchanging the roles of z_1 and z_2 interchanges two pairs of fermions so costs no signs and cancels the $\frac{1}{2!}$.

The overall sign is annoying but can be fixed by demanding that the diagonal bit of the S -matrix give

$$\langle p_3 p_4 | (\mathbb{1} + \dots) | p_1 p_2 \rangle = +\delta(p_1 - p_3)\delta(p_2 - p_4) + \dots$$



The relative sign is what we're after, and it comes by comparing the locations of fermion operators in the contractions in the two diagrams at right. In terms of the contractions, these t - and u - channel diagrams are related by leaving the annihilation operators alone and switching the contractions between the creation operators and the final state. Denoting by $\mathbf{a}_{1,2}^\dagger$ the fermion creation operators coming from the vertex at $z_{1,2}$,

$$\begin{aligned} & \langle 0 | \mathbf{a}_{p_4} \underbrace{\mathbf{a}_{p_3} \mathbf{a}_1^\dagger}_{\text{contracted}} \mathbf{a}_2^\dagger \dots + \langle 0 | \mathbf{a}_{p_4} \underbrace{\mathbf{a}_{p_3} \mathbf{a}_1^\dagger}_{\text{contracted}} \mathbf{a}_2^\dagger \dots \\ &= \langle 0 | \mathbf{a}_{p_4} \underbrace{\mathbf{a}_{p_3} \mathbf{a}_1^\dagger}_{\text{contracted}} \mathbf{a}_2^\dagger \dots - \langle 0 | \underbrace{\mathbf{a}_{p_4} \mathbf{a}_1^\dagger}_{\text{contracted}} \underbrace{\mathbf{a}_{p_3} \mathbf{a}_2^\dagger}_{\text{contracted}} \dots \end{aligned}$$

In the last expression the fermion operators to be contracted are all right next to each other and we see the relative minus sign.

While we're at it, let's evaluate this whole amplitude to check the Feynman rules I've claimed and get some physics out. It is

$$S_{fi} = -g^2 \int dz_1 dz_2 \int \bar{d}^4 q \frac{e^{-i q(z_1 - z_2)} \mathbf{i}}{q^2 - m_\phi^2 + i\epsilon} \left(e^{-i z_2(p_1 - p_3)} \bar{u}^{r_3}(p_3) u^{r_1}(p_1) \cdot e^{-i z_1(p_2 - p_4)} \bar{u}^{r_4}(p_4) u^{r_2}(p_2) - (3 \leftrightarrow 4) \right).$$

In the first (t -channel) term, the integrals over $z_{1,2}$ gives $\delta(p_1 - p_3 - q)\delta(p_4 - p_2 - q)$, and the q integral then gives $\delta(p_1 + p_2 - p_3 - p_4)$, overall momentum conservation. In the second (u -channel) term, $q = p_1 - p_4 = p_3 - p_2$. Altogether,

$$(S - \mathbb{1})_{fi} = \delta^4(p_T) \mathbf{i} \mathcal{M}$$

with, to leading order,

$$\mathbf{i} \mathcal{M} = -\mathbf{i} g^2 \left(\frac{1}{t - m_\phi^2} (\bar{u}_3 u_1) (\bar{u}_4 u_2) - \frac{1}{u - m_\phi^2} (\bar{u}_4 u_1) (\bar{u}_3 u_2) \right) \quad (5.48)$$

with $t \equiv (p_1 - p_3)^2$, $u \equiv (p_1 - p_4)^2$. This minus sign implements Fermi statistics.

Yukawa force revisited. In the non-relativistic limit, we can again relate this amplitude to the force between particles, this time with the actual spin and statistics of nucleons. In the COM frame, in the NR limit, $p_1 = (m, \vec{p})$, $p_2 = (m, -\vec{p})$ and $p_3 = (m, \vec{p}')$, $p_4 = (m, -\vec{p}')$. In the non-relativistic limit, the spinors become $u_p^r = \begin{pmatrix} \sqrt{\sigma \cdot \vec{p}} \xi^r \\ \sqrt{\bar{\sigma} \cdot \vec{p}} \xi^r \end{pmatrix} \rightarrow \sqrt{m} \begin{pmatrix} \xi^r \\ \xi^r \end{pmatrix}$ so that $\bar{u}_3 u_1 \equiv \bar{u}(p_3)^{r_3} u(p_1)^{r_1} = 2m \xi_{r_3}^\dagger \xi_{r_1} = 2m \delta_{r_3 r_1}$. Let's simplify our lives and take two *distinguishable* fermions (poetically, they could be proton and neutron, but let's just add a label to our fermion fields; they could have different masses, for example, or different couplings to ϕ , call them g_1, g_2). Then we only get the t -channel diagram. The intermediate scalar momentum is $q = p_1 - p_3 = (0, \vec{p} - \vec{p}')$ so $t = (p_1 - p_3)^2 = -\vec{q}^2 = -(\vec{p} - \vec{p}')^2$ and

$$\mathbf{i} \mathcal{M}_{NR, COM} = \mathbf{i} g_1 g_2 \frac{1}{\vec{q}^2 + m_\phi^2} 4m^2 \delta^{r_1 r_3} \delta^{r_2 r_4}.$$

Recall that this is related to the S -matrix element by $S(34 \leftarrow 12) = \delta^4(\sum p) \mathbf{i} \mathcal{M}$. Compare this to the NR Born approximation matrix element

$$\begin{aligned} 2\pi \delta(E_p - E_{p'}) \left(-\mathbf{i} \tilde{U}(\vec{q}) \right) \delta^{r_1, r_3} &=_{NR} \langle \vec{p}', r_3 | S | \vec{p}, r_1 \rangle_{NR} \\ &= \sum_{r_4} \int \bar{d}^3 p_4 \underbrace{\prod_{i=1}^4 \frac{1}{\sqrt{2E_i}}}_{= \frac{1}{\sqrt{2m^4}}} S(34 \leftarrow 12) \end{aligned}$$

$$= 2\pi\delta(E_p - E_{p'})\delta^{r_1 r_3} \frac{\mathbf{i}g_1 g_2}{\vec{q}^2 + m_\phi^2}$$

where in the second line we summed over possible final states of the second (target) particle, and corrected the relativistic normalization, so that ${}_{NR}\langle\vec{p}'|\vec{p}\rangle_{NR} = \delta^3(p - p')$. This is completely independent of the properties of the second particle. We infer that the scalar mediates a force with potential $U(x) = -\frac{g_1 g_2 e^{-m_\phi r}}{4\pi r}$. It is attractive if $g_1 g_2 > 0$.

6 Quantum electrodynamics

6.1 Feynman rules for QED

The Lagrangian for QED is

$$\mathcal{L}_{QED} = \bar{\Psi} (\mathbf{i}\not{\partial} - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + e \bar{\Psi} \gamma^\mu A_\mu \Psi = \bar{\Psi} (\mathbf{i}\not{D} - m) \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (6.1)$$

Why is this the form of the interaction? It is the simplest gauge-invariant and Lorentz-invariant we can make out of a spinor and a vector field. Notice that Ψ has dimension 3 and A has dimension 1 so the coupling e is dimensionless. The theory is classically scale invariant (and therefore has a chance to be renormalizable). There are other interactions we can add, but they all involve more derivatives or more fields and therefore are irrelevant, in the sense that they are suppressed by some new mass scale M , and their effects therefore become unimportant for questions about energies $E \ll M$. A good example is

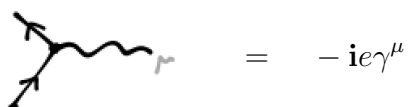
$$\delta L = \frac{c}{M} F_{\mu\nu} \bar{\Psi} \gamma^{\mu\nu} \Psi. \quad (6.2)$$

This term shifts the magnetic dipole moment of the Ψ particle. You may have heard about the high-precision measurements of $g - 2$ of the muon that people have done. One way to describe what they are doing is measuring the coupling c .

For efficiency, I'm going to just write down the Feynman rules for QED. Then we'll come back and understand the form of the photon propagator in the following subsection.

The new ingredients in QED (relative to field theories we've studied above) are the propagating vectors, and the interaction hamiltonian density $V = e \bar{\Psi} \gamma^\mu \Psi A_\mu$. The rest of the Feynman rules are

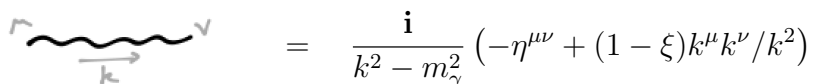
9. The interaction vertex gets a



$$= -ie\gamma^\mu$$

10. An external photon in the initial state gets a $\epsilon^\mu(p)$, and in the final state gets a $\epsilon^{\mu*}(p)$.

11. An internal photon line gets a



$$= \frac{i}{k^2 - m_\gamma^2} (-\eta^{\mu\nu} + (1 - \xi)k^\mu k^\nu / k^2)$$

where $m_\gamma = 0$ (it's sometimes useful to keep it in there for a while as an IR regulator) and the value of ξ is up to you (meaning that your answers for physical quantities should be independent of ξ).

Spinor trace ninjutsu.

$$\text{The trace is cyclic: } \quad \text{tr}(AB \cdots C) = \text{tr}(CAB \cdots). \quad (6.3)$$

Our gamma matrices are 4×4 , so $\text{tr} \mathbb{1} = 4$.

$$\text{tr} \gamma^\mu = \text{tr} (\gamma^5)^2 \gamma^\mu \stackrel{(6.3)}{=} \text{tr} \gamma^5 \gamma^\mu \gamma^5 \stackrel{\{\gamma^5, \gamma^\mu\}=0}{=} -\text{tr} \gamma^\mu = 0. \quad (6.4)$$

The same trick shows that the trace of any odd number of gammas vanishes. The idea is that an odd number of gammas is a map between the L and R subspaces, so it has only off-diagonal terms in the Weyl basis.

$$\text{tr} \gamma^\mu \gamma^\nu \stackrel{\text{clifford}}{=} -\text{tr} \gamma^\nu \gamma^\mu + 2\eta^{\mu\nu} \text{tr} \mathbb{1} \stackrel{(6.3)}{=} -\text{tr} \gamma^\mu \gamma^\nu + 8\eta^{\mu\nu} \implies \text{tr} \gamma^\mu \gamma^\nu = 4\eta^{\mu\nu}. \quad (6.5)$$

$$\text{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = 4(\eta^{\mu\nu} \eta^{\rho\sigma} + \eta^{\sigma\mu} \eta^{\nu\rho} - \eta^{\mu\rho} \eta^{\nu\sigma}). \quad (6.6)$$

Why is this? The completely antisymmetric bit vanishes because it is proportional to γ^5 which is traceless (by the same argument as (6.4)). If any pair of indices is the same then the other two must be too by (6.5). If adjacent pairs are the same they can just square to one and we get +1; if alternating pairs are the same (and different from each other) then we must move them through each other with the anticommutator. If they are all the same we get 4.

$$\text{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5 = -4i\epsilon^{\mu\nu\rho\sigma}.$$

6.2 QED processes at leading order

Now we are ready to do *lots* of examples, nearly all of which (when pushed to the end) predict cross sections which are verified by experiments to about one part in 137.⁶¹ Here $\frac{1}{137} \approx \alpha \equiv \frac{e^2}{4\pi}$ is the small number by which the next order corrections are suppressed.⁶²

⁶¹I guess it is this overabundance of scientific victory in this area that leads to the intrusion of so many names of physicists in the following discussion.

⁶²This statement is true naively (in the sense that the next diagrams which are nonzero come with two more powers of e), and also true in fact, but in between naiveté and the truth is a long road of renormalization, which begins next quarter.

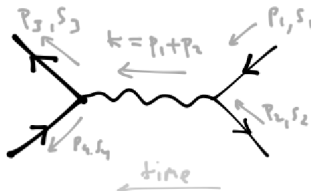
Did I mention that the antiparticle of the electron, predicted by the quantum Dirac theory (*i.e.* by Dirac), is the positron? It has the same mass as the electron and the opposite electromagnetic charge, since the charge density is the 0 component of the electromagnetic current, $j^\mu = \bar{\Psi}\gamma^\mu\Psi$, so the charge is

$$\int d^3x j^0(x) = \int \bar{\Psi}\gamma^0\Psi = \int \Psi^\dagger\Psi = \int d^3p \sum_s (\mathbf{a}_{p,s}^\dagger \mathbf{a}_{ps} - \mathbf{b}_{p,s}^\dagger \mathbf{b}_{ps}).$$

So \mathbf{b}^\dagger creates a positron. Notice that such an antiparticle is a logical consequence of demanding a Lorentz-invariant theory of a massive spin-half particle with a conserved particle number.

[Schwarz §13.3, Peskin §5.1] Perhaps the simplest to start with is scattering of electrons and positrons. We can make things even simpler (one diagram instead of two) by including also the muon, which is a heavy version of the electron⁶³, and asking about the process $\mu^+\mu^- \leftarrow e^+e^-$. At leading order in e , this comes from

$i\mathcal{M}_{\mu^+\mu^-\leftarrow e^+e^-} =$



$$= (-ie\bar{u}^{s_3}(p_3)\gamma^\mu v^{s_4}(p_4))_{\text{muons}} \frac{-i\left(\eta_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2}\right)}{k^2} (-ie\bar{v}^{s_2}(p_2)\gamma^\nu u^{s_1}(p_1))_{\text{electrons}} \quad (6.7)$$

with $k \equiv p_1 + p_2 = p_3 + p_4$ by momentum conservation at each vertex. I've labelled the spinors according to the particle types, since they depend on the mass.

Ward identity in action. What about the $k_\mu k_\nu$ term in the photon propagator? I claimed that physics does not depend on ξ in its coefficient – this is called the Ward identity. The spinors satisfy their equations of motion, $\not{p}_1 u_1 = m_e u_1$ (where $u_1 \equiv u_{p_1}^{s_1}$ for short) and $\bar{v}_2 \not{p}_2 = -m_e \bar{v}_2$. The k_ν appears in the combination

$$k_\nu \bar{v}_2 \gamma^\nu u_1 = \bar{v}_2 (\not{p}_1 + \not{p}_2) u_1 = \bar{v}_2 \not{p}_1 u_1 + \bar{v}_2 \not{p}_2 u_1 = (m - m)\bar{v}u = 0.$$

(The other factor is also zero, but one factor of zero is enough.) Therefore

$$\mathcal{M} = \frac{e^2}{s} \bar{u}_3 \gamma_\mu v_4 \cdot \bar{v}_2 \gamma^\mu u_1$$

⁶³Who ordered *that*? (I. Rabi's reaction to learning about the muon.) I hope you don't find it too jarring that the number of 'elementary' particles in our discussion increased by three in the last two paragraphs. People used to get really disgruntled about this kind of thing. But here we have, at last, uncovered the true purpose of the muon, which is to halve the number of Feynman diagrams in this calculation (compare (6.17)).

where $s \equiv k^2 = (p_1 + p_2)^2 = E_{CoM}^2$ is the Mandelstam variable. And I am relying on you to remember which spinors refer to muons (3,4) and which to electrons (1,2).

Squaring the amplitude. We need to find \mathcal{M}^\dagger (the dagger here really just means complex conjugate, but let's put dagger to remind ourselves to transpose and reverse the order of all the matrices). Recall the special role of γ^0 here:

$$\gamma_\mu^\dagger \gamma_0 = \gamma_0 \gamma_\mu, \quad \gamma_0^\dagger = \gamma_0.$$

This means that for any two Dirac spinors,

$$(\bar{\Psi}_1 \gamma^\mu \Psi_2)^\dagger = \bar{\Psi}_2 \gamma^\mu \Psi_1.$$

(This is the same manipulation that showed that the Dirac Lagrangian was hermitian.)
So

$$\mathcal{M}^\dagger = \frac{e^2}{s} (\bar{v}_4 \gamma^\mu u_3) (\bar{u}_1 \gamma_\mu v_2).$$

and therefore

$$|\mathcal{M}_{\mu^+\mu^-\leftarrow e^+e^-}|^2 = \frac{e^4}{s^2} \underbrace{(\bar{v}_4 \gamma^\mu u_3) (\bar{u}_3 \gamma^\nu v_4)}_{\text{out}} \cdot \underbrace{(\bar{u}_1 \gamma_\mu v_2) (\bar{v}_2 \gamma_\nu u_1)}_{\text{in}}. \quad (6.8)$$

These objects in parentheses are just c-numbers, so we can move them around, no problem. I've grouped them into a bit depending only on the initial state (the electron stuff 1, 2) and a bit depending only on the final state (the muon stuff 3,4).

Average over initial, sum over final. In the amplitude above, we have fixed the spin states of all the particles. Only very sophisticated experiments are able to discern this information. So suppose we wish to predict the outcome of an experiment that does not measure the spins of the fermions involved. We must sum over the final-state spins using

$$\sum_{s_4} v_a^{s_4}(p_4) \bar{v}_b^{s_4}(p_4) = (\not{p}_4 - m_\mu)_{ab} = \sum_{s_4} \bar{v}_b^{s_4}(p_4) v_a^{s_4}(p_4)$$

(where I wrote the last expression to emphasize that these are just c-numbers) and

$$\sum_{s_3} u_a^{s_3}(p_3) \bar{u}_b^{s_3}(p_3) = (\not{p}_3 + m_\mu)_{ab}.$$

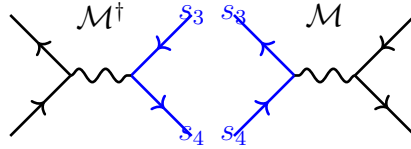
Looking at just the 'out' factor of $|\mathcal{M}|^2$ in (6.8), we see that putting these together

produces a spinor trace, as promised:

$$\begin{aligned}
& \sum_{s_3, s_4} (\bar{u}(p_3)_a^{s_3} \gamma_{ab}^\mu \underbrace{v(p_4)_b^{s_4}}_{(\not{p}_4 - m_\mu)_{bc}}) (\bar{v}(p_4)_c^{s_4} \gamma_{cd}^\nu u^{s_3}(p_3)_d) \\
&= \gamma_{ab}^\mu (\not{p}_4 - m_\mu)_{bc} \gamma_{cd}^\nu (\not{p}_3 + m_\mu)_{da} \\
&= \text{tr} \left(\gamma^\mu (\not{p}_4 - m_\mu) \gamma^\nu (\not{p}_3 + m_\mu) \right) \tag{6.9}
\end{aligned}$$

$$\begin{aligned}
&= p_{4\rho} p_{3\sigma} \text{tr} \gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma - m_\mu^2 \text{tr} \gamma^\mu \gamma^\nu \\
&\stackrel{(6.5), (6.6)}{=} 4 \left(p_4^\mu p_3^\nu + p_4^\nu p_3^\mu - \underbrace{p_3 \cdot p_4}_{\equiv p_{34}} \eta^{\mu\nu} - m_\mu^2 \eta^{\mu\nu} \right) \tag{6.10}
\end{aligned}$$

Diagrammatically what just happened was that we took a copy of \mathcal{M}^\dagger and a copy of \mathcal{M} ; summing over s_3, s_4 glued the muon lines together to make a loop (at least for the purposes of the spinor index sums):



(Muons are in blue in this figure.)

[End of Lecture 20]

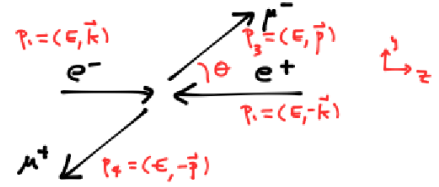
If also we don't know the initial (electron) spins, then the outcome of our experiment is the *average* over the initial spins, of which there are four possibilities. This is because, in this circumstance, the initial state is an incoherent mixture of the available possibilities: $\rho = \sum_{s_1, s_2} p_{s_1, s_2} \rho_{s_1, s_2}$, a density matrix rather than a pure state, with ρ_{s_1, s_2} the projector onto the pure state with the indicated spins, and in this case $p_{s_1, s_2} = \frac{1}{4}$. Therefore, the relevant probability for unpolarized scattering is

$$\begin{aligned}
\frac{1}{4} \sum_{s_1, 2, 3, 4} |\mathcal{M}|^2 &\stackrel{(6.9) \text{ twice}}{=} \frac{e^4}{4s^2} \text{tr} \left(\gamma^\mu (\not{p}_4 - m_\mu) \gamma^\nu (\not{p}_3 + m_\mu) \right) \text{tr} \left(\gamma_\nu (\not{p}_2 - m_e) \gamma_\mu (\not{p}_1 + m_e) \right) \\
&\stackrel{(6.10) \text{ twice}}{=} \frac{8e^4}{s^2} (p_{13}p_{24} + p_{14}p_{23} + m_\mu^2 p_{12} + m_e^2 p_{34} + 2m_e^2 m_\mu^2) \\
&\stackrel{\text{algebra}}{=} \frac{2e^4}{s^2} (t^2 + u^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2) \tag{6.11}
\end{aligned}$$

Here $p_{ij} \equiv p_i \cdot p_j$. In the second step of (6.11) the $p_{12}p_{34}$ terms cancel. In the last step of (6.11) we used all the Mandelstam variables:

$$\begin{aligned} s &\equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 = E_{CoM}^2 = 4E^2 \\ t &\equiv (p_1 - p_3)^2 = (p_2 - p_4)^2 = m_e^2 + m_\mu^2 - 2E^2 + 2\vec{k} \cdot \vec{p} \\ u &\equiv (p_1 - p_4)^2 = (p_2 - p_3)^2 = m_e^2 + m_\mu^2 - 2E^2 - 2\vec{k} \cdot \vec{p} \end{aligned}$$

where the particular kinematic variables (in the rightmost equalities) are special to this problem, in the center of mass frame (CoM), and are defined in the figure at right. Really there are only two independent Lorentz-invariant kinematical variables, since $s + t + u = \sum_i m_i^2$.



Now we can use the formula (4.46) that we found for a differential cross section with a two-body final state, in the CoM frame:

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{CoM} &= \frac{1}{64\pi^2 E_{CoM}^2 |\vec{k}|} \left(\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2\right) \\ &= \frac{\alpha^2}{16E^6} \frac{|\vec{p}|}{|\vec{k}|} \left(E^4 + |\vec{k}|^2 |\vec{p}|^2 \cos^2 \theta + E^2(m_e^2 + m_\mu^2)\right) \end{aligned} \quad (6.12)$$

where $\alpha \equiv \frac{e^2}{4\pi}$ is the fine structure constant. This can be boiled a bit with kinematical relations $|\vec{k}| = \sqrt{E^2 - m_e^2}$, $|\vec{p}| = \sqrt{E^2 - m_\mu^2}$ to make manifest that it depends only on two independent kinematical variables, which we can take to be the energy E and the scattering angle θ in $\vec{k} \cdot \vec{p} = |\vec{k}| |\vec{p}| \cos \theta$ (best understood from the figure). It simplifies a bit if we take $E \gg m_e$, and more if we take $E \gg m_\mu \sim 200m_e$, to

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{CoM}^2} (1 + \cos^2 \theta). \quad (6.13)$$

In fact, the two terms here come respectively from spins transverse to the scattering plane and in the scattering plane; see Schwartz §5.3 for an explanation.

You may be surprised that the cross section (6.13) decreases with energy. Mechanically this comes mainly from the $1/s^2$ from the photon propagator: as s grows, the intermediate photon is more and more off-shell. But more deeply, it's because above we've studied an *exclusive* cross-section, in the sense that we fixed the final state to be exactly a muon and an antimuon. At higher energies, nothing new happens here, because the final state is fixed, and most of the probability is elsewhere. More about where the probability goes in the next subsection.

It has also been very valuable to think about *inclusive* cross-sections for e^+e^- scattering, because in this way you can make anything that the s -channel photon couples

to, if you put enough energy into it. The inclusive cross section for (e^+e^- goes to anything) does grow with energy, and jumps at energies that are thresholds for new particles in the final state. In this way, for example, we can also make *quarks* (more specifically quark-antiquark pairs) since they also carry electric charge. See Peskin pp 139-140 for a bit more about that, and in particular how this observable gives evidence that there are three colors of quarks.

6.3 Vector fields, quickly

We must fill a small hole in our discussion: in §1.3, we figured out a bit of the quantum theory of the radiation field. A few things we did not do: study the propagator, figure out the data on external states, and the relation between the masslessness of the photon and gauge invariance. After that we will couple electrons and photons and study leading-order (tree-level) processes in the resulting theory of quantum electrodynamics (QED).

[We'll follow Ken Intriligator's efficient strategy for this discussion.] Consider the following Lagrangian for a vector field A_μ (which I claim is the most general quadratic Poincaré-invariant Lagrangian with at most two derivatives in $D = 3 + 1$):

$$\mathcal{L} = -\frac{1}{2} \left(\partial_\mu A^\nu \partial^\mu A_\nu + a \underbrace{\partial_\mu A^\mu \partial_\nu A^\nu}_{=(\partial A)^2} + b A_\mu A^\mu + c \epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma \right).$$

The sign is chosen so that spatial derivatives are suppressed, and the normalization of the first term is fixed by rescaling A . (Another possible-seeming term, $\partial_\mu A^\nu \partial_\nu A^\mu$, is related to the second term by two IBPs.) The last term is a total derivative, $\epsilon^{\mu\nu\rho\sigma} \partial_\mu A_\nu \partial_\rho A_\sigma \propto \partial_\mu (\epsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma)$, and will not affect the EoM or anything at all in perturbation theory; it is called a θ term.

The EoM are

$$0 = \frac{\delta}{\delta A^\nu(x)} \int \mathcal{L} = -\partial^2 A_\nu - a \partial_\nu (\partial \cdot A) + b A_\nu \quad (6.14)$$

which (like any translation-invariant linear equation) is solved by Fourier transforms $A_\mu(x) = \epsilon_\mu e^{-ikx}$, if

$$k^2 \epsilon_\mu + a k_\mu (k \cdot \epsilon) + b \epsilon_\mu = 0.$$

There are two kinds of solutions: ones with $\epsilon_\mu \propto k_\mu$ (for which the dispersion relation is $k^2 = -\frac{b}{1+a}$, and which is called the *longitudinal* polarization), and solutions $\epsilon \cdot k = 0$ with dispersion $k^2 = -b$. The mode polarized along k^μ may be removed by taking

$b \neq 0$ and $a \rightarrow -1$, which we will do from now on and never speak of it again. This gives the Proca Lagrangian:

$$\mathcal{L}_{a=-1, b=-\mu^2} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu,$$

where as usual $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that the EOM (Proca equation) $0 = \partial^\nu F_{\nu\mu} + \mu^2 A_\mu$ implies $0 = \partial^\nu A_\nu$ by the Bianchi identity $0 = \partial^\mu \partial^\nu F_{\mu\nu}$ (which holds because of the symmetry of mixed partials and the antisymmetry of $F_{\mu\nu}$). So each component of A_μ satisfies (by (6.14)) the KG equation, $k^2 = \mu^2$, and the transverse condition $\epsilon \cdot k = 0$. In the rest frame, $k^\mu = (k^0, \vec{0})^\mu$, we can choose a basis of plane-wave transverse solutions which are eigenstates of the vector rotation generator

$$J^z = \mathbf{i} \begin{pmatrix} & & & \\ & +1 & & \\ -1 & & & \\ & & & \end{pmatrix}, \text{ namely, } \epsilon^{(\pm)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \mp \mathbf{i} \\ 0 \end{pmatrix}, \epsilon^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

They are normalized so that $\epsilon^{(r)} \cdot \epsilon^{(s)} = -\delta^{rs}$ and $\sum_{r=\pm 1, 0} \epsilon_\mu^{(r)\star} \epsilon_\nu^{(r)} = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}$ so that they project out $\epsilon \propto k$. More generally, if $k^\mu = (\omega, 0, 0, p^z)$, then the third polarization meeting these demands is

$$\epsilon_\mu^{(0)} = \frac{1}{\mu}(p^z, 0, 0, -\omega)_\mu, \quad \epsilon^{(0)\mu} = \frac{1}{\mu}(p^z, 0, 0, \omega)^\mu$$

Notice that in the massless case, there's no rest frame where $\vec{k} = 0$ and the story can be different, and also that in this case $p^z = E$, so $\epsilon^{(0)\mu} \propto k^\mu$ is longitudinal.

If $\vec{k} \propto \hat{z}$ (for example in the massless case with $k^\mu = (E, 0, 0, E)^\mu$) then these three ϵ are also all helicity eigenstates: $h = \vec{J} \cdot \hat{k} = J^z$.

Canonical stuff: The canonical momenta are $\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} = E^i$ (as for electrodynamics in §1.3) and $\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$. This last bit is a little awkward, but it just means we can solve the equations of motion for A_0 algebraically in terms of the other (real) dofs:

$$0 = \frac{\delta S}{\delta A_0} \propto \vec{\nabla} \cdot \vec{E} - \mu^2 A_0 = (\nabla^2 - \mu^2)A_0 - \vec{\nabla} \cdot \dot{\vec{A}} \implies A_0(\vec{x}) = \int d^3y e^{-\mu|\vec{x}-\vec{y}|} \frac{(-\vec{\nabla} \cdot \dot{\vec{A}})}{4\pi|\vec{x}-\vec{y}|}. \quad (6.15)$$

So at each moment A_0 is determined by A_i . (Notice that this is still true for $\mu \rightarrow 0$.) The hamiltonian density is (after using $\pi^i = F^{0i}$, integration by parts, and the equations of motion for A_0)

$$\mathfrak{h} = \frac{1}{2} \left(\vec{E}^2 + \vec{B}^2 + \mu^2 \vec{A}^2 + \mu^2 A_0^2 \right) \geq 0,$$

where positivity follows from the fact that it is a sum of squares of real things.

The canonical equal-time commutators are then

$$[A_i(t, \vec{x}), F^{j0}(t, \vec{y})] = \mathbf{i} \delta_i^j \delta^{(3)}(\vec{x} - \vec{y})$$

which if we add up the plane wave solutions as

$$A_\mu(x) = \sum_{r=1,2,3} \int \frac{d^3k}{\sqrt{2\omega_k}} \left(e^{-\mathbf{i}kx} \mathbf{a}_k^r \epsilon_\mu^{(r)} + e^{+\mathbf{i}kx} \mathbf{a}_k^{r\dagger} \epsilon_\mu^{(r)\star} \right)$$

give the bosonic ladder algebra for each mode

$$[\mathbf{a}_k^r, \mathbf{a}_p^{s\dagger}] = \delta^{(3)}(\vec{k} - \vec{p}) \delta^{rs}.$$

The normal-ordered hamiltonian is

$$: H := \sum_r \int d^3k \omega_k \mathbf{a}_k^{r\dagger} \mathbf{a}_k^r.$$

The propagator for the $A_\mu(x)$ field is

$$\langle 0 | \mathcal{T} A_\mu(x) A_\nu(y) | 0 \rangle = \int d^4k e^{-\mathbf{i}k(x-y)} \left[\frac{-\mathbf{i}(\eta_{\mu\nu} - k_\mu k_\nu / \mu^2)}{k^2 - \mu^2 + \mathbf{i}\epsilon} \right]. \quad (6.16)$$

Notice that like in the spinor case the polarization sum $\sum_r \epsilon_\mu^{r\star} \epsilon_\nu^r$ appears in the numerator of the propagator. The way to understand the sign is that the spatial components are just like scalars. The quantity in square brackets is then the momentum-space propagator. Since $\langle 0 | A_\mu(x) | k, r \rangle = \epsilon_\mu^r(k) e^{-\mathbf{i}kx}$, a vector in the initial state produces a factor of $\epsilon_\mu^r(k)$, and in the final state gives $\epsilon^{r\star}$.



Massless case. In the limit $\mu \rightarrow 0$ some weird stuff happens. If we couple A_μ to some object j^μ made of other matter, by adding $\Delta\mathcal{L} = j^\mu A_\mu$, then we learn that $\partial_\mu A^\mu = \mu^{-2} \partial_\mu j^\mu$. This means that in order to take $\mu \rightarrow 0$, it will be best if the current is conserved $\partial_\mu j^\mu = 0$.

One example is the QED coupling, $j^\mu = \bar{\Psi} \gamma^\mu \Psi$. We saw that this coupling $A_\mu j^\mu$ arose from the ‘minimal coupling’ prescription of replacing $\partial_\mu \rightarrow D_\mu = \partial_\mu + \mathbf{i}eqA_\mu$ in the Dirac Lagrangian. In that case, the model had a local invariance under $A_\mu \rightarrow A_\mu + \partial_\mu \lambda(x)/e$, $\Psi(x) \rightarrow e^{\mathbf{i}q\lambda(x)} \Psi(x)$. For λ non-constant (and going to zero far away), this is a redundancy of our description rather than a symmetry (for example, two sets of fields related by this transformation have the same values of the physical quantities $\vec{E}, \vec{B}, e^{\mathbf{i}\oint A}$).

That is, configurations related by this gauge transformation should be regarded as equivalent. This gives us a new interpretation of the polarization with $\epsilon^\mu \propto k^\mu$: this


means in position space that $A_\mu = \partial_\mu \lambda$ for some function λ , *i.e.* it is a pure gauge configuration, equivalent to $A_\mu = 0$ by a gauge transformation. See Schwartz §8.6 for a nice diatribe on why we use such a redundant description.

Another example of coupling a gauge field to matter can be obtained by taking a complex scalar and doing the same minimal coupling $\partial \rightarrow D$ replacement: $\mathcal{L} = (D_\mu \Phi)^* D^\mu \Phi + \dots$. Notice that in this case the vertex involves a derivative, so it comes

with a factor of  $= -ieq(p_\Phi + p_{\Phi^*})^\mu$. Also, there is a $A_\mu A_\nu \Phi^* \Phi$ coupling, which gives a vertex  $= -ie^2 q^2 \eta_{\mu\nu}$.

How do I know that configurations related by a gauge transformation should be regarded as equivalent? If not, the kinetic operator K for the massless vector field $K_{\mu\nu}^\rho \equiv (\eta_{\mu\nu} (\partial^\rho \partial_\rho) - \partial_\mu \partial_\nu) A^\nu = 0$ is not invertible, since it annihilates $A_\nu = \partial_\nu \lambda$. This would be bad because the path integral $\int DA e^{-\int AKA}$ would blow up from the integral over this degenerate mode. So gauge transformations are redundancies of our description of massless vector fields in terms of vector potentials.

What's the propagator, then? One strategy is to simply ignore the gauge equivalence and use the same propagator (6.16) that we found in the massive case with $\mu \rightarrow 0$. Since the dynamics are gauge invariant, it will never make gauge-variant stuff, and the longitudinal bits $\propto k_\mu k_\nu$ in (6.16) (which depend on μ) will just drop out, and we can take $\mu \rightarrow 0$ in the denominator at the end. This actually works. The guarantee that it works is the QED Ward identity: any amplitude with an external vector $\epsilon(k)_\mu$ is of the form



$$= i\mathcal{M} = i\mathcal{M}^\mu(k) \epsilon_\mu(k)$$

and if all external fermion lines are on-shell then

$$\mathcal{M}^\mu(k) k_\mu = 0.$$

There is a complicated diagrammatic proof of this statement in Peskin; Schwartz §8.4 argues that it is a necessary condition for Lorentz invariance of $\mathcal{M} = \epsilon_\mu \cdot \mathcal{M}^\mu$; and we'll give some illustrations of it below (I also recommend Zee §II.7). But it is basically a statement of current conservation: such an amplitude is made (by LSZ and the photon Schwinger-Dyson equation) from a correlation function involving an insertion of the electromagnetic current $j^\mu(k) = \int d^4x e^{-ikx} j^\mu(x)$, in the form, $\mathcal{M}^\mu \sim \dots \langle \Omega | \dots j^\mu(k) \dots | \Omega \rangle$, and $k_\mu j^\mu(k) = 0$ is current conservation⁶⁴.

⁶⁴Current conservation $\partial_\mu j^\mu$ is a statement that requires the equations of motion (recall the proof of

This property guarantees that we will not emit any longitudinal photons, since the amplitude to do so is the $\mu \rightarrow 0$ limit of

$$\begin{aligned} \mathcal{A} \left(\begin{array}{l} \text{emit } \epsilon_\lambda^L = \frac{1}{\mu}(k, 0, 0, -\omega)_\lambda \\ \text{with } k^\lambda = (\omega, 0, 0, k)^\lambda \end{array} \right) &\propto \epsilon_\mu^L \mathcal{M}^\mu = \frac{1}{\mu} (k\mathcal{M}^0 - \omega\mathcal{M}^3) = \frac{1}{\mu} \left(k\mathcal{M}^0 - \underbrace{\sqrt{k^2 + \mu^2}}_{=k + \frac{\mu^2}{2k} + \dots} \mathcal{M}^3 \right) \\ &= \frac{1}{\mu} \underbrace{k_\mu \mathcal{M}^\mu}_{=0, \text{ by Ward}} - \underbrace{\frac{\mu}{2k} \mathcal{M}^3}_{\rightarrow 0 \text{ as } \mu \rightarrow 0} + \mathcal{O}(\mu^3) \xrightarrow{\mu \rightarrow 0} 0. \end{aligned}$$

Gauge fixing. You might not be happy with the accounting procedure I've advocated above, where unphysical degrees of freedom are floating around in intermediate states and only drop out at the end by some formal trick. In that case, a whole zoo of formal tricks called *gauge fixing* has been prepared for you. Here's a brief summary to hold you over until 215B.

At the price of Lorentz invariance, we can make manifest the physical dofs, by choosing *Coulomb gauge*. That means we restrict $\partial_\mu A^\mu = 0$ (so far, so Lorentz invariant) and also $\vec{\nabla} \cdot \vec{A} = 0$. Looking at (6.15), we see that this kills off the bit of A_0 that depended on \vec{A} . We also lose the helicity-zero polarization $\vec{\nabla} \cdot \vec{A} \propto \epsilon^{(0)}$. But the Coulomb interaction is instantaneous action at a distance.

To keep Lorentz invariance, we can instead merely discourage configurations with $\partial \cdot A \neq 0$ by adding a term to the action

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2$$

for some arbitrary number ξ . Physics should not depend on ξ , and this is a check on calculations. With $\xi < \infty$ the kinetic operator is invertible and its inverse is the propagator:

$$\langle \mathcal{T} A_\mu(x) A_\nu(y) \rangle = \int \mathrm{d}^4 k e^{-ik(x-y)} \left[\frac{-\mathbf{i}(\eta_{\mu\nu} - (1-\xi)k_\mu k_\nu / k^2)}{k^2 - \mu^2 + \mathbf{i}\epsilon} \right]$$

Noether's theorem). Recall that we proved that equations of motion are true in correlation functions, up to contact terms, using the independence of the path integral on choice of integration variables. By contact terms, I mean terms that are only nonzero when two operators are at the same point. So you can worry about the contact terms in the argument for the Ward identity. The reason they do not contribute is that all the operators in the correlation function (using the LSZ formula) correspond to external states. A collision between the operators creating the external particles would lead to a disconnected amplitude, which could only contribute for degenerate kinematical configurations, and we can ignore them. If you would like to read more words about this, look at Schwartz §14.8, or Appendix A.

and again the bit with $k_\mu k_\nu$ must drop out. $\xi = 1$ is called Feynman gauge and makes this explicit. $\xi = 0$ is called Landau gauge and makes the propagator into a projector onto k_\perp .

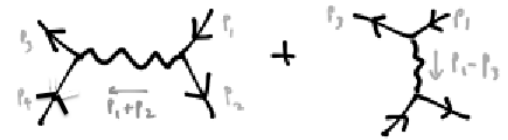
It becomes much more important to be careful about this business in non-Abelian gauge theory.

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6.4 More examples of QED processes at tree level

There is a lot more to say about what happens when we scatter an electron and a positron. Another thing that can happen is that the final state could be an electron and positron again (Bhabha scattering⁶⁶).

In the s -channel diagram they are not the same e^- and e^+ (except in the sense that they are all the same). Another way to get there at tree level is the second, t -channel, diagram, at right, where there is a sense in which the initial particles survive in the final state. The intermediate photon in that diagram has $k_t = (p_1 - p_3)$, so that the denominator of the propagator is $t = k_t^2 = (p_1 - p_3)^2$ instead of s .



Squaring this amplitude gives

$$|\mathcal{M}_s + \mathcal{M}_t|^2 = |\mathcal{M}_s|^2 + |\mathcal{M}_t|^2 + 2\text{Re}(\mathcal{M}_s \mathcal{M}_t^*), \quad (6.17)$$

⁶⁵By the way, you might be bothered that we didn't go back to our table 1 of possible Lorentz representations on fields to think about spin one particles. Indeed, we could start with the $(1, 0) \oplus (0, 1)$ antisymmetric tensor $F_{\mu\nu}$ as our basic object. (See, for example, the book by Haag, *Local Quantum Physics*, page 47.) Indeed, in this way we could construct a theory of a free EM field. But don't we need a vector potential to couple to charged matter? The answer turns out to be 'no,' as explained by Mandelstam [here](#). The price is that the charged fields depend on not just a point, but a choice of *path*; if we did introduce the vector potential, they would be related to our fields by

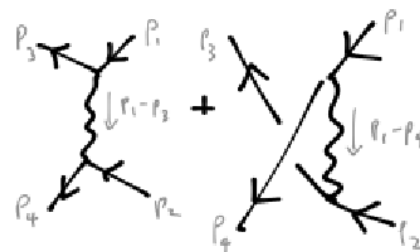
$$\Phi(x, P) = e^{i \int_P A} \Phi(x),$$

where P is a path which ends at x and infinity. This *Wilson line* $e^{i \int_P A}$ carries away the gauge transformation, so that $\Phi(x, P)$ is actually invariant under gauge transformations that fall off at infinity.

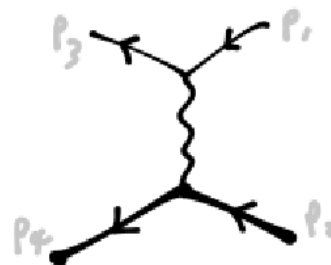
⁶⁶See figure 3 [here](#). Now remember that a person doesn't have much control over their name. By the way, I totally believe the bit about *non-perturbative strings = lint*.

interference terms. Interference terms mean that you have to be careful about the overall sign or phase of the amplitudes.

$e^-e^- \leftarrow e^-e^-$. What happens if instead we scatter two electrons (Möller scattering)? In that case, the leading order diagrams are the ones at right. Now the intermediate photons have $k_t = (p_1 - p_3)$ and $k_u = (p_1 - p_4)$ respectively, so that the denominator of the propagator is t and u in the two diagrams. The evaluation of these diagrams has a lot in common with the ones for $e^+e^- \rightarrow e^+e^-$, namely you just switch some of the legs between initial and final state.



The relation between such amplitudes is called *crossing symmetry*. Let's illustrate it instead for $e^-\mu^- \leftarrow e^-\mu^-$, where again there is only one diagram, related by crossing to (6.19). The diagram is the one at right. (The muon is the thicker fermion line.)



$$i\mathcal{M} = \begin{array}{c} p_3 \\ \swarrow \\ \text{---} \\ \searrow \\ p_4 \end{array} \begin{array}{c} p_1 \\ \swarrow \\ \text{---} \\ \searrow \\ p_2 \end{array} = (-ie\bar{u}_3\gamma^\mu u_1)_{\text{electrons}} \frac{-i\left(\eta_{\mu\nu} - \frac{(1-\xi)k_\mu k_\nu}{k^2}\right)}{k^2} (-ie\bar{u}_2\gamma^\nu u_4)_{\text{muons}} \quad (6.18)$$

with $k \equiv p_1 - p_3 = p_2 - p_4$. It differs from (6.19) by replacing the relevant vs with us for the initial/final antiparticles that were moved into final/initial particles, and relabelling the momenta. After the spin sum,

$$\frac{1}{4} \sum_{s_{1,2,3,4}} |\mathcal{M}|^2 = \frac{e^4}{4t^2} \text{tr} \left(\gamma^\mu (\not{p}_4 + m_\mu) \gamma^\nu (\not{p}_2 + m_\mu) \right) \text{tr} \left(\gamma_\nu (\not{p}_3 + m_e) \gamma_\mu (\not{p}_1 + m_e) \right)$$

this amounts to the replacement $(p_1, p_2, p_3, p_4) \rightarrow (p_1, -p_3, p_4, -p_2)$.

Crossing symmetry more generally. If you look at a Feynman diagram on its side (for example because someone else fails to use the convention that time goes to the left) it is still a valid amplitude for some process. Similarly, dragging particles between the initial and final state also produces a valid amplitude. Making this relation precise can save us some work. The precise relation for dragging an incoming particle into the

final state, so that it is an outgoing antiparticle, is:

$$i\mathcal{M}_{f \leftarrow iA}(p_f; p_i, p_A) = \text{diagram} = i\mathcal{M}_{\bar{A} f \leftarrow i}(p_f, k = -p_A; p_i) = \text{diagram}.$$

(If you must, note that this is another sense in which an antiparticle is a particle going backwards in time.) If A is a spinor particle, the sum relations for particles and antiparticles are different:

$$\sum_r u^r(p) \bar{u}^r(p) = \not{p} + m, \quad \sum_r v^r(k) \bar{v}^r(k) = \not{k} - m = -(\not{p} + m)$$

– after accounting for $k = -p_A$, they differ by an overall sign. Hence we must also append a fermion sign factor $(-1)^{\text{number of fermions shuffled between in and out}}$ in the unpolarized scattering probability. We'll study a well-motivated example in more detail next.

Mott formula. By studying scattering of an electron from a heavy charged fermion (a muon is convenient) we can reconstruct the cross section for scattering off a Coulomb potential (named after Mott). This example will be important next quarter, where you'll learn how it is corrected by other QED processes.

$\boxed{\mu^+ \mu^- \leftarrow e^+ e^-}$. Consider again the process $\mu^+ \mu^- \leftarrow e^+ e^-$. To try to keep things straight, I'll call the electron momenta p, p' and the muon momenta k, k' , since that won't change under crossing. We found the amplitude

$$i\mathcal{M}_{\mu^+ \mu^- \leftarrow e^+ e^-} = \text{diagram} = \left(-ie \bar{u}^s(k) \gamma^\mu v^{s'}(k') \right)_{\text{muons}} \frac{-i \left(\eta_{\mu\nu} - \frac{(1-\xi) q_\mu q_\nu}{q^2} \right)}{q^2} \left(-ie \bar{v}^{r'}(p') \gamma^\nu u^r(p) \right)_{\text{electrons}} \quad (6.19)$$

(with $q \equiv p + p' = k + k'$)⁶⁷ and the (unpolarized) scattering probability density

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \stackrel{\text{spinor traces}}{=} \frac{1}{4} \frac{e^4}{s^2} E^{\mu\nu} M_{\mu\nu},$$

where the tensor objects $E^{\mu\nu}, M^{\mu\nu}$ come respectively from the electron and muon lines,

$$\frac{1}{4} E_{\mu\nu} = p_\mu p'_\nu + p'_\mu p_\nu - \eta_{\mu\nu} (p \cdot p' + m_e^2)$$

⁶⁷Relative to the notation I used earlier, $p_1 = p, p_2 = p', p_3 = k, p_4 = k'$.

$$\frac{1}{4}M_{\mu\nu} = k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu}(k \cdot k' + m_\mu^2).$$

and they are contracted by the photon line, with $s = q^2 = (p + p')^2$.

$\boxed{e^- \mu^- \leftarrow e^- \mu^-}$. To get from this the amplitude (tree level, so far) for the process $e^- \mu^- \leftarrow e^- \mu^-$, we must move the incoming positron line to an outgoing electron line, and move the outgoing antimuon line to an incoming muon line (hence the sign in σ will be $(-1)^{\text{number of fermions shuffled between in and out}} = (-1)^2 = 1$). Relative to the amplitude for $\mu^+ \mu^- \leftarrow e^+ e^-$ (6.19), we must replace the relevant vs with us for the initial/final antiparticles that were moved into final/initial particles, and we must replace $p' \rightarrow -p', k' \rightarrow -k'$:

$$i\mathcal{M} = \text{[diagram]} = (-ie\bar{u}(p')\gamma^\mu u(p))_{\text{electrons}} \frac{-i\left(\eta_{\mu\nu} - \frac{(1-\xi)q_\mu q_\nu}{q_t^2}\right)}{q_t^2} (-ie\bar{u}(k)\gamma^\nu u(k'))_{\text{muons}} \quad (6.20)$$

with $q_t \equiv p - p' = k - k'$. After the spin sum,

$$\frac{1}{4} \sum_{s,s',r,r'} |\mathcal{M}|^2 = 4 \frac{e^4}{t^2} \left(-p_\mu p'_\nu - p'_\mu p_\nu - \eta_{\mu\nu}(-p \cdot p' + m_e^2) \right) \cdot \left(-k_\mu k'_\nu - k'_\mu k_\nu - \eta_{\mu\nu}(-k \cdot k' + m_\mu^2) \right) \quad (6.21)$$

On the Mandelstam variables, this is just the permutation $(s, t, u) \rightarrow (t, u, s)$.

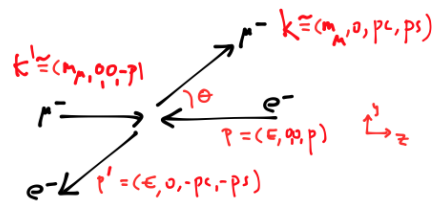
Payoff: the Mott formula. Recall other ways of figuring out the scattering cross

section from a Coulomb potential from a point charge of charge ze .



We think about scattering from a fixed electrostatic potential $A_0 = \frac{ze}{r}$ and do classical mechanics. I can never remember how this goes. Instead, let's just scatter an electron off a heavy charge, such as a muon. If the charge of the heavy object were z times that of the electron, we would multiply the amplitude by z and the cross section by z^2 .

'Heavy' here means that we can approximate the CoM frame by its rest frame, and its initial and final energy as $k'_0 = m_\mu, k_0 = \sqrt{m_\mu^2 + \vec{k}^2} = m_\mu + \frac{1}{2}\vec{k}^2/m_\mu + \dots \simeq m_\mu$. Also, this means the collision is approximately elastic, $E' \simeq E$. In the diagram of the kinematics at right, annoyingly, $s \equiv \cos\theta, c \equiv \sin\theta$. (Sorry.)



This means that the muon-line tensor factor $M_{\mu\nu}$ in (6.21) simplifies dramatically:

$$-\frac{1}{4}M_{\mu\nu} \simeq k_\mu k'_\nu + k'_\mu k_\nu - \eta_{\mu\nu} \left(\underbrace{k \cdot k' - m_\mu^2}_{=m_\mu^2 - m_\mu^2 = 0} \right) \simeq \delta_{\mu 0} \delta_{\nu 0} 2m_\mu^2.$$

In the electron line, we'll need the ingredient

$$-p \cdot p' + m_e^2 = -E^2 + \vec{p}^2 \cos \theta + m_e^2 = -\vec{p}^2(1 - \cos \theta). \quad (6.22)$$

So

$$\begin{aligned} E^{\mu\nu} M_{\mu\nu} = -8m_\mu^2 E^{00} &= 32m_\mu^2(2E^2 + \eta^{00}(-p \cdot p' + m_e^2)) \\ &\stackrel{(6.22)}{=} 32m_\mu^2(2E^2 - \vec{p}^2(1 - \cos \theta)) \\ &\stackrel{\text{trig}}{=} 32m_\mu^2 2(E^2 - \vec{p}^2 \sin^2 \theta/2) \stackrel{\beta^2 \equiv \vec{p}^2/E^2}{=} 64m_\mu^2 E^2(1 - \beta^2 \sin^2 \theta/2). \end{aligned}$$

From the two-body phase space, the cross section is

$$\begin{aligned} d\sigma &= \underbrace{\frac{1}{v_{\text{rel}}}}_{=\beta} \frac{1}{2E} \frac{1}{2m_\mu} \frac{z^2 e^4}{t^2} \frac{64}{4} m_\mu^2 E^2 (1 - \beta^2 \sin^2 \theta/2) \frac{d\Omega}{16\pi^2} \frac{p}{E_{\text{total}}} \\ &\stackrel{E_{\text{total}} \sim m_\mu}{=} \frac{4Ep}{\beta} \frac{z^2 \alpha^4 (1 - \beta^2 \sin^2 \theta/2)}{t^2} d\Omega \end{aligned}$$

Noting that $t = (p - p')^2 = -2\vec{p}^2(1 - \cos \theta)$, we get

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{Mott}} = z^2 \frac{\alpha^4 (1 - \beta^2 \sin^2 \theta/2)}{4\beta^2 \vec{p}^2 \sin^4 \theta/2}.$$

If we take $\beta \ll 1$ in this formula we get the Rutherford formula. Notice that it blows up at $\theta \rightarrow 0$. This is a symptom of the long-range nature of the Coulomb potential, *i.e.* the masslessness of the photon.

Electron-proton scattering. The answer is basically the same if we think of the heavy particle in (6.20) as a proton (we have to flip the sign of the charge but this gets squared away since there is no interference in this case). $ep \rightarrow ep$ is called Rutherford scattering, for good reason⁶⁸.

Electron-photon scattering. In the case of the process $e^- \gamma \leftarrow e^- \gamma$,⁶⁹ we meet

⁶⁸If you don't know why, you should go read *Inward Bound*, by Abraham Pais, as soon as possible.

⁶⁹which at high energy is called Compton scattering and at low energies is called Thomson scattering. Despite my previous curmudgeonly footnote chastising the innocent reader for an imagined incomplete knowledge of the history of science, I do have a hard time remembering which name goes where. Moreover, as much as I revere the contributions of many of these folks, I find that using their names makes me think about the people instead of the physics. No one owns the physics! It's the same physics for lots of space aliens, too.

a new ingredient, namely external photons:

$$\begin{aligned}
 i\mathcal{M} &= \text{diagram 1} + \text{diagram 2} \equiv i\mathcal{M}_s + i\mathcal{M}_t \\
 &= (-ie)^2 \epsilon_1^\mu \epsilon_4^{\nu*} \bar{u}_3 \left(\gamma_\nu \frac{\mathbf{i}k_s + m}{s - m^2} \gamma_\mu + \gamma_\mu \frac{\mathbf{i}k_t + m}{t - m^2} \gamma_\nu \right) u_2 .
 \end{aligned} \tag{6.23}$$

The two amplitudes have a relative plus since we only mucked with the photon contractions, they just differ by how the gamma matrices are attached. If you don't believe me, draw the contractions on this:

$$\langle \gamma e | (\bar{\Psi} A \Psi)_1 (\bar{\Psi} A \Psi)_2 | \gamma e \rangle$$

(I'm not going to TeX it, thank you).

Now, if we don't measure the photon polarizations, we need

$$P = \frac{1}{4} \sum_{\text{polarizations, spins}} |\mathcal{M}|^2.$$

The key ingredient is the completeness relation

$$\sum_{i=1,2} \epsilon_\mu^{i*}(k) \epsilon_\nu^i(k) = -\eta_{\mu\nu} + \text{something proportional to } k^\mu k^\nu.$$

We can do various incantations to find a definite coefficient, but *it will not matter* because of the Ward identity: anytime there is an external photon $\epsilon(k)_\mu$, the amplitude is $\mathcal{M} = \mathcal{M}_\mu \epsilon^\mu(k)$ and satisfies $k^\mu \mathcal{M}_\mu = 0$. Therefore, we can ignore the term about which I was vague and we have

$$\begin{aligned}
 \sum_{\text{polarizations}} |\mathcal{M}|^2 &= \sum_i \epsilon_\mu^{i*} \mathcal{M}^{\mu*} \mathcal{M}^\nu \epsilon_\nu^i = -\eta_{\mu\nu} \mathcal{M}^{\mu*} \mathcal{M}^\nu + (\text{terms with } \mathcal{M}_\mu k^\mu) \\
 &= -\mathcal{M}_\mu^* \mathcal{M}^\mu.
 \end{aligned}$$

Don't be scared of the minus sign, it's because of the mostly minus signature, and makes the thing positive. But notice the opportunity to get negative probabilities if the gauge bosons don't behave!

A dramatic process related by crossing to Compton scattering is pair annihilation, $\mathcal{M}_{\gamma\gamma \leftarrow e^+e^-}$. See the end of Peskin §5, where he has a nice plot comparing to experimental data the result for $\frac{d\sigma}{d\Omega}$ as a function of scattering angle.

To be continued ... *e.g.* [here](#).

A More about the non-perturbative proof of the Ward identity

[Schwartz §14.8] First, consider a Green's function from which we might make an S-matrix element by LSZ,

$$G \equiv \langle \Omega | \mathcal{T} \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | \Omega \rangle = \int D\Psi e^{iS} \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)$$

where the operators $\mathcal{O}_i(x_i) \mapsto e^{-iQ_i\alpha} \mathcal{O}_i(x_i)$ have charge Q_i under a global U(1) symmetry. For example the $\mathcal{O}(x)$ could be just the elementary field $\Psi(x)$ ⁷⁰.

Now change variables in the path integral so that $\mathcal{O}_i(x_i) \mapsto e^{-iQ_i\alpha(x_i)} \mathcal{O}_i(x_i)$; the action will shift by $S \mapsto S - \int \alpha \partial_\mu j^\mu$ where j^μ is the Noether current. The path integral doesn't change at all, so its infinitesimal variation is

$$0 = \delta G = \int D\Psi \left(- \int \mathbf{i} \alpha \partial^\mu j_\mu e^{iS} \mathcal{O}_1 \cdots \mathcal{O}_n - \mathbf{i} \sum_i Q_i \alpha(x_i) e^{iS} \mathcal{O}_1 \cdots \mathcal{O}_n \right) \quad (\text{A.1})$$

$$= \int d^d x \alpha(x) \left[\mathbf{i} \partial_\mu \langle j^\mu(x) \mathcal{O}_1 \cdots \mathcal{O}_n \rangle - \sum_i Q_i \delta(x - x_i) G \right]. \quad (\text{A.2})$$

Since this is true for any $\alpha(x)$, we learn that the thing in square brackets is zero: $\partial_\mu j^\mu = 0$ up to contact terms. This is called the Ward-Takahashi identity.

Now suppose we do this same manipulation in a gauge theory, say QED for definiteness (the story is slightly modified in scalar QED). The additional terms in S are $F_{\mu\nu} F^{\mu\nu} + \mathbf{i} A_\mu \bar{\Psi} \gamma^\mu \Psi$, which are invariant under the transformation, so don't change these statements. Notice that the transformation we're doing here is *not* the gauge transformation, since A_μ doesn't transform – we're only doing the gauge transformation on the matter fields here, so their kinetic terms actually shift and produce the $\alpha \partial^\mu j_\mu$ term above. Photon field insertions in G don't contribute, since they have charge zero here.

Next, think about the LSZ formula for an S-matrix element with (say) two external photons:

$$\begin{aligned} \mathcal{M} &= \langle \epsilon, \dots \epsilon_k \dots | S | \dots \rangle \stackrel{LSZ}{=} \epsilon^\mu \epsilon_k^\rho \mathbf{i}^n \int d^4 x e^{ipx} \square_{\mu\nu} \int d^4 x_1 e^{ip_k x_k} \square_{\rho\sigma}^k \int \dots \langle A^\nu(x) \dots A^\sigma(x_k) \dots \rangle \\ &\equiv \epsilon^\mu \mathcal{M}_\mu \end{aligned} \quad (\text{A.3})$$

where $\square_{\mu\nu}$ is shorthand for the photon kinetic operator $\square_{\mu\nu} = \square \eta_{\mu\nu} - \partial_\mu \partial_\nu / \mu^2$. Hidden in the \dots is all the factors of *e.g.* $\int dy e^{iq_1 y} (\mathbf{i} \not{\partial}_y + m_1)$ associated with external spinor

⁷⁰You'll have to trust me for now that the path integral for fermionic fields exists. That's the only information about it we'll need here.

particles, as well as the information that we have to take the limit where the particles go on-shell.

The Schwinger-Dyson equation for A_μ then implies that

$$\square_{\rho\sigma}^k \square_{\mu\nu} \langle A^\nu(x) \dots A^\sigma(x_k) \dots \rangle = \square_{\rho\sigma}^k (\langle j_\mu(x) \dots A_\sigma(x_k) \dots \rangle - \mathbf{i} \delta^4(x - x_k) \eta_{\mu\sigma} \langle \dots \rangle) \quad (\text{A.4})$$

$$= \langle j_\mu(x) \dots j_\sigma(x_k) \rangle - \mathbf{i} \square \delta(x - x_k) \eta_{\mu\sigma} \langle \dots \rangle \quad (\text{A.5})$$

First of all, this is why I said we could get the S-matrix elements with photons from correlators with currents. Notice that this is only true up to the contact terms. But those are disconnected amplitudes that we can ignore.

Finally, set the polarization of one of the photons equal to its momentum $\epsilon = p$. Then

$$\begin{aligned} p^\mu \mathcal{M}_\mu &= \lim_{q^2 \rightarrow m^2} \epsilon_k^\rho \mathbf{i}^n \int d^4x e^{ipx} \int d^4x_1 e^{ip_k x_k} \int dy e^{iq_1 y} (\mathbf{i} \not{\phi}_y + m_1) \dots \langle -\mathbf{i} \partial_\mu j^\mu \dots j_\rho(x_1) \dots \Psi(y) \dots \rangle \\ &\stackrel{(\text{A.2})}{=} \lim_{q^2 \rightarrow m^2} (\not{q}_1 - m_1) (\not{q}_2 - m_2) \dots \sum_j Q_j \tilde{G}(\dots, q_j \pm p, \dots) \end{aligned} \quad (\text{A.6})$$

where the \pm depends on whether particle j is incoming or outgoing. At the last step we used the Fourier transform of (A.2).

Now here's the punchline: The \tilde{G} on the RHS of (A.6) has poles at $(q_j \pm p)^2 = m_j^2$, and *not* at $q_j^2 = m_j^2$. So when it's multiplied by $\not{q}_j - m_j = \frac{q_j^2 - m_j^2}{\not{q}_j + m_j}$ and we put the external particles on shell, $q_j^2 \rightarrow m_j^2$, it will vanish. End of story. Notice that no use of perturbation theory was made here.

[End of Lecture 21]