University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 2 – Solutions

Comment: this problem set contains many problems, but most of them are simple and many of them are bonus problems.

Due 3:30pm Thursday, October 10, 2024

1. **Brain-warmer.** What is the cycle structure of the permutation (235)(245)?

   Repeatedly using $(abc) = (ab)(bc)$ and the fact that non-overlapping cycles commute we find

   $$(235)(245) = (23)(35)(24)(45) = (23)(24)(35)(45) = (234)(345) = (23)(34)(34)(45) = (23)(45).$$

   The last expression is two non-overlapping 2-cycles, so this is the cycle structure.

2. **Conjugacy classes of S_n.**

   (a) Write the following elements of S_n in cycle notation:

   $$\pi = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ 2\ 3\ 1\ 4\ 5 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ 2\ 1\ 3\ 4\ 5 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1\ 2\ 3\ 4\ 5 \\ 2\ 3\ 4\ 1\ 5 \end{pmatrix}.$$

   (b) Check that the cycle structure is preserved by conjugation, *e.g.* for $\pi^{-1}\sigma\pi$, $\pi^{-1}\rho\pi$.

   (c) Bonus problem: give a proof that this always works.

   Suppose σ has the cycle decomposition $\sigma = \sigma_1\sigma_2....$ where σ_i are non-overlapping cycles of length s_i.

   $$\pi^{-1}\sigma\pi = (\pi^{-1}\sigma_1\pi)(\pi^{-1}\sigma_2\pi)(\pi^{-1}\cdots)$$

   so we wish to show that $\pi^{-1}\sigma_i\pi$ is a cycle of length s_i, too. (Note that since π is a bijection on the n objects, disjoint cycles must remain disjoint under conjugation by π.) But consider $(\pi^{-1}\sigma_i\pi)^{s_i} = \pi^{-1}(\sigma_i)^{s_i}\pi = \pi^{-1}e\pi = e$. If $\pi^{-1}\sigma_i\pi$ were a cycle of any other length, this would not be the case.

   More constructively, the action by conjugation of π is

   $$\pi(ijk...)\pi^{-1} = (\pi_i\pi_j\pi_j...). \tag{1}$$

   To see this, let $\sigma \equiv (ijk...)$ and $\rho \equiv \pi\sigma\pi^{-1}$, so $\rho\pi = \pi\sigma$. $(\rho\pi)_i = (\pi\sigma)_i = \pi_j$, $(\rho\pi)_j = (\pi\sigma)_j = \pi_k$ and so on, which can be rewritten as $\rho_{\pi_i} = \pi_j, \rho_{\pi_j} = \pi_k$.

In contrast, when acting on an element not in the cycle $\sigma = (ijk..)$, ρ does nothing, since the action of π commutes with ρ on such an element.. This means ρ can be written as in (1).

Alternatively, we can use representation theory. Think about the action of S_n in the defining representation. The representative of π, $D(\pi)$ is an orthogonal matrix (in fact one with a single one in each row and column). But this means that conjugation by π is represented as a simple basis transformation. The cycle structure of σ is determined by the dimensions of the invariant subspaces of $D(\sigma)$. This data does not change under a basis transformation.

3. **Brain-warmer.**

   (a) What group is this: $G = \langle a, b | aba^{-1}b^{-1} = e\rangle$?

   The relation says that a and b commute, so this group is $\mathbb{Z} \times \mathbb{Z} = \{(n, m)\}$, with the identification $a^n b^m \to (n, m)$.

   (b) [Bonus problem] Find a space X so that $\pi_1(X) = G$ above.

   Take the plane and divide it up into unit squares. Now identify two points if they are related by a translation by $(n, m), \in \mathbb{Z} \times \mathbb{Z}$. This is a description of the torus T^2, which makes clear that its fundamental group is $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$. To see this, we can appeal to the fact that $\pi_1(X/G) = G$ if G acts freely and $\pi_1(X) = \{e\}$. Alternatively, we can draw a small loop in the middle of one of the squares (this clearly represents the identity in $\pi_1(T^2)$) and then homotope it to the boundary of the square, where it becomes $aba^{-1}b^{-1} = e$, where a and b are the loops crossing the square horizontally and vertically.

   You could arrive at this answer by following the construction of X_G that I described in lecture: make the Cayley diagram for the group. This is just the edges of the square lattice. There is a relation for each square. If we fill in these squares as instructed, we get the whole plane $\mathbb{R}^2$. Now we quotient by the action of $\mathbb{Z}^2$, which is generated by unit translations in x and y, to get $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, whose fundamental group is therefore $\mathbb{Z}^2$.

   But I have to admit that there is an even easier way. Given that $\pi_1(S^1) = \mathbb{Z}$, we can use the fact that $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$[1] to conclude that the space we want is $S^1 \times S^1 = T^2$.

[1] The proof of this fact follows from the definition without too much trouble: assume we have representatives of $\pi_1(X)$ and $\pi_1(Y)$; these can be embedded in $X \times Y$ by mapping *e.g.* a curve in X to (the curve in X, the base point in Y). Then the generators of $\pi_1(X \times Y)$ are just (γ_x, γ_y).

4. **Quaternions.**

   Decompose the quaternion group Q_8 into conjugacy classes.

   Conjugating $\mathbf{i}$ by $\mathbf{j}$ gives

$$\mathbf{j}\mathbf{i}(-\mathbf{j}) = -\mathbf{k}(-\mathbf{j}) = -\mathbf{i}.$$

   Similarly, conjugating $\pm\mathbf{i}$ or $\pm\mathbf{j}$ or $\pm\mathbf{k}$ by any of the other two reverses the sign. All other conjugations leave the elements invariant. Therefore each of $\pm\mathbf{i}$, $\pm\mathbf{j}$ and $\pm\mathbf{k}$ constitute a conjugacy class of two elements. ± 1 are by themselves.

5. **Brain-warmer.** Check the relation $|G| = |Z(g)||C(g)|$ for $g = (12)$ in $G = S_n$. Here $Z(g)$ is the centralizer of g (the set of elements of G that commute with g) and $C(g)$ is the conjugacy class of g (by $|C(g)|$ I mean the number of elements in the conjugacy class).

   $Z(12) = \mathbb{Z}_2 \times S_{n-2}$ where the $\mathbb{Z}_2 = \{e, (12)\}$. This has $2(n-2)!$ elements. The conjugacy class contains all interchanges on two of n objects, of which there are $n(n-1)/2$. So the product is indeed $n!$.

6. **A presentation of A_4.** Prove that the group $\langle a, b | a^2 = e, b^3 = e, (ab)^3 = e \rangle$ is isomorphic to A_4, the group of even permutations of 4 objects,

$$A_4 = \{e, (12)(34), (14)(23), (13)(24), (123), (132), (243), (234), (341), (314), (421), (412)\}.$$

   The isomorphism takes a to an order-two element and b to an order-three element. The order-three element could be (123) (or one of its conjugates). (12) is not in A_4, so the order-two element must be $a = (12)(34)$ (or one of its conjugates). Now we check that their product ab is indeed order three:

$$ab = (12)(34)(123) = (12)(43)(312) = (12)(4312) = (12)(1243) = (12)(12)(243) = (243).$$

   The inverse of $(ab) = (234)$ is then just $(ab)^2$. To get the other order-two elements $(13)(24)$ and $(14)(23)$ we can just conjugate a by b and b^2: bab^2, b^2ab. There are 4 other order-3 elements $(143), (134), (142), (124)$ which are obtained from ab^2 and its square and aba and its square. To be completely explicit, then the 12 elements of A_4 are

$$e, (12)(34), (14)(23), (13)(24), (123), (132), (243), (234), (341), (314), (421), (412)$$

$$e, a, bab^2, b^2ab, b, b^2, ab, (ab)^2 = b^2a, ba, (ba)^2 = ab^2, aba, (aba)^2 = ab^2a$$

   To make sure that the map is onto we should make sure that there are no other independent words. An argument is something like this: an arbitrary word is a product of factors ab^m, with $m = 1, 2$. But using the relations above we can reduce any word with three or more as to one of the above elements.

A nice way to check that there are 12 elements of the group with this presentation is to make the Cayley diagram. Here are two ways to draw it (from Jiashu Han and Yuan Zhang respectively):

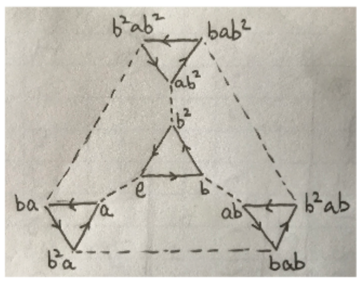
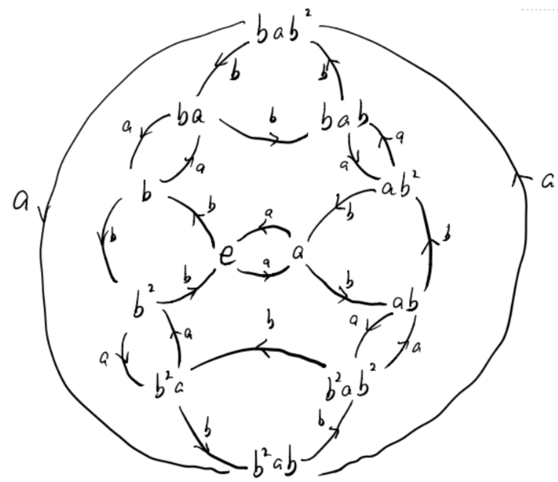


Figure 1: Cayley diagram of the group $\langle a, b | a^2 = e, b^3 = e, (ab)^3 = e \rangle$.



I like them both.

7. **Free groups are weird.** [Bonus problem] Show that the free group on two elements $\langle a, b | \rangle$ contains subgroups isomorphic to the free group on any number of elements.

   We can make a generator of the subgroup out of a string of letters, like $abab$, and another generator out of another, different string, like $a^2b^2a^2b^2$, and so on. The group generated by such elements is a subgroup, and there are no relations between them. I think the simplest basis of generators for F_n is $a^kb^k, k = 1..n$.

8. **Counting elements of conjugacy classes of S_k.** Here is a cool trick, related to Polya enumeration, for counting the number of elements in the conjugacy class of S_k associated to a given Young diagram (cycle structure), λ.

   (a) [bonus problem]. Fill in the missing details of the following argument.

   First, recall the object $z(\sigma) \equiv z_1^{c_1(\sigma)} z_2^{c_2(\sigma)} \cdots$, where $c_i(\sigma)$ is the number of cycles of length i in the permutation σ. This is a conjugation-invariant weight over which we can sum:

$$Z_G(z_1, z_2, \cdots) \equiv \sum_{\sigma \in S_k} z(\sigma).$$

   This is (proportional to) the object we called the cycle index in our discussion of Polya enumeration (for the case with $G = S_k$).

Now consider the case where $|X| = k$ and $G = S_k$, the whole permutation group on the k objects, and we'll take n colors (*i.e.* an n-state Potts model on X). Weight a coloring with l_i objects of color i with a factor of $W = u_1^{l_1} u_2^{l_2} \cdots$. Polya's enumeration theorem says that the partition sum is then

$$\sum_{\text{orbits } O} W(O) = Z_{S_k}\left(z_1 = u_1 + u_2 + \cdots, z_2 = u_1^2 + u_2^2 + \cdots, \cdots\right) \tag{2}$$

$$= \sum_{l_1, l_2, \cdots l_n} (\#\text{ of orbits with } l_1 \text{ 1s, } l_2 \text{ 2s...})\, u_1^{l_1} u_2^{l_2} \cdots \tag{3}$$

What is this number of orbits? Because we are modding out by the whole permutation group, an orbit is entirely determined by specifying the number of each color. So this number is only ever 1 or 0.

To avoid the cases where it's zero, here's the final trick, familiar from statistical mechanics as the grand canonical ensemble: sum over k (!). Let

$$P(t, u_1, u_2, \cdots) \equiv \sum_{k=1}^{\infty} t^k \sum_{\text{orbits } O \text{ of } S_k} W(O).$$

On the one hand, this is

$$P(t, u_1, u_2, \cdots) = \sum_k t^k Z_{S_k}\left(z_1 = u_1 + u_2 + \cdots, z_2 = u_1^2 + u_2^2 + \cdots, \cdots\right).$$

On the other hand, this is

$$P(t, u_1, u_2, \cdots) = \sum_k t^k \sum_{l_1, l_2, \cdots l_n} (\#\text{ of orbits with } l_1 \text{ 1s, } l_2 \text{ 2s...})\, u_1^{l_1} u_2^{l_2} \cdots \tag{4}$$

$$= \sum_{l_1=0}^{\infty} (tu_1)^{l_1} \sum_{l_2=0}^{\infty} (tu_2)^{l_2} \cdots \tag{5}$$

$$= \exp\left(z_1 t + z_2 t^2/2 + z_3 t^3/3 + \cdots\right) \tag{6}$$

where $z_i \equiv u_1^i + u_2^i + \cdots$.

The only step I really left out is summing the geometric series

$$\sum_{l_i=0}^{\infty} (tu_i)^{l_i} = \frac{1}{1 - u_i t} = e^{-\log(1-u_i t)} = \exp\left(u_i t + (u_i t)^2/2 + (u_i t)^3/3 + \cdots\right).$$

Then

$$P(t, u_1, u_2, \cdots) = \prod_i \exp\left(u_i t + (u_i t)^2/2 + (u_i t)^3/3 + \cdots\right) \tag{7}$$

$$= \exp\left((\sum_i u_i) t + (\sum_i u_i^2) t^2/2 + (\sum_i u_i^3) t^3/3 + \cdots \right) \tag{8}$$

$$= \exp\left(z_1 t + z_2 t^2/2 + z_3 t^3/3 + \cdots\right). \tag{9}$$

I learned this from this website.

So for example, to compute the sizes of the conjugacy classes of S_7, let $T = z_1 t + z_2 t^2/2 + z_3 t^3/3 + \cdots$ (you can stop at some number bigger than 7), and just find the coefficient of t^7 in e^T. The result is a polynomial in the z_i where the sum of the subscripts of each term adds up to 7. Each term is then associated with a Young diagram λ and hence a conjugacy class.

(b) What should you get if you set $z_i = 1$ for all i and why?

If we set $z_i = 1, \forall i$, then we are just counting orbits of S_n on n elements, of which there is exactly 1. Also, the sum over conjugacy classes of the number of elements in each class is the order of the group.

(c) Find the size of each conjugacy class of S_4 and S_5 using the result above. (I recommend Mathematica's Series and Coefficient commands and the method described in the previous part of the problem.) Check that your polynomial satisfies the check of the previous part.

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In[95]:= n = 12; ss = Series[Exp[Sum[p[l] t^l/l, {l, 1, n}]], {t, 0, n}];

In[96]:= k = 4; cs3 = Coefficient[ss, t^k] * Factorial[k] // Expand
Out[96]= p[1]^4 + 6 p[1]^2 p[2] + 3 p[2]^2 + 8 p[1] p[3] + 6 p[4]

In[88]:= k = 5; cs3 = Coefficient[ss, t^k] * Factorial[k] // Expand
Out[88]= p[1]^5 + 10 p[1]^3 p[2] + 15 p[1] p[2]^2 + 20 p[1]^2 p[3] + 20 p[2] p[3] + 30 p[1] p[4] + 24 p[5]

In[89]:= k = 6; cs3 = Coefficient[ss, t^k] * Factorial[k] // Expand
Out[89]= p[1]^6 + 15 p[1]^4 p[2] + 45 p[1]^2 p[2]^2 + 15 p[2]^3 + 40 p[1]^3 p[3] +
          120 p[1] p[2] p[3] + 40 p[3]^2 + 90 p[1]^2 p[4] + 90 p[2] p[4] + 144 p[1] p[5] + 120 p[6]

         (* check that we got all the group elements: *)

In[97]:= cs3/Factorial[k] /. p[i_] :> 1
Out[97]= 1
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(d) Actually, there is a better way to learn the sizes of conjugacy classes of S_n. The order of the centralizer of g, $|Z_g|$, depends only on its conjugacy class. The size of the conjugacy class is then $|C_g| = |G|/|Z_g|$ (by Lagrange's theorem). Show that the centralizer of an element g of S_n with c_j cycles of length j is

$$|Z_g| = \prod_j (c_j)! j^{c_j}. \tag{10}$$

[Hint: Think about what elements that commute with a permutation of a given cycle structure can do, then count them.] Write a formula for the number of elements of the conjugacy class C_g and compare with your results from the previous part.

An element of the centralizer can only permute cycles of equal length or cyclically permute within a cycle. The number of such elements is (10).

$$|C_g| = \frac{n!}{\prod_j (c_j)! j^{c_j}}. \tag{11}$$

Some useful special cases:

$$|C_{\begin{smallmatrix}\square\\\square\\\square\\\square\\\square\end{smallmatrix}}| = (n-1)!, \qquad |C_{\begin{smallmatrix}\square&\square&\square&\square\\\square&&&\end{smallmatrix}}| = \frac{n(n-1)}{2}, \qquad |C_{\begin{smallmatrix}\square&\square&\square&\square&\square\end{smallmatrix}}| = 1.$$

9. **Counting non-isomorphic graphs.** A graph with k vertices can be regarded as a choice of $\{0,1\}$ for each of the $\binom{k}{2} = \frac{k(k-1)}{2}$ pairs of vertices ('1' means no edge and '1' means yes edge). Two graphs are isomorphic if they are related by a relabelling of the vertices. How many non-isomorphic graphs on 4 vertices are there? (The result of the previous problem will be useful.)

   Construct the partition function which weights a graph by the number of edges, $Z(t) = \sum_{\text{graphs, }\Gamma} t^{\#\text{ of edges of }\Gamma}$ for $k = 4$.

   Bonus problem: answer the above questions for 5 vertices.

   We take the set on which the group acts to be $X = \{\text{edges}\} = \{12, 13, 14, 23, 24, 34\}$ (for $k = 4$), and we color the edges with one of two colors. Let X_l be the set of graphs with with l edges, and X_l^g the set of such graphs fixed by the action of the group element g. According to not-Burnside,

$$Z(t) = \sum_l |X_l/G| t^l = \sum_l \frac{1}{|G|} \sum_{g \in G} |X_l^g| t^l = \frac{1}{|G|} \sum_g \left(\sum_l |X_l^g| t^l \right).$$

   Since again we are modding out by the whole permutation group, we only care about conjugacy classes – that is, every element of a conjugacy class C contributes the same amount to Z:

$$Z(t) = \frac{1}{|G|} \sum_C n_C \left(\sum_l |X_l^g| t^l \right).$$

   Therefore

conjugacy class	# of elements	orbits	# of orbits	weighted contribution
e	1	12, 13, 14, 23, 24, 34	$\binom{4}{2} = 6$	$(1+t)^6$
(12)	6	{13, 23}, {14, 24}, 12, 34	4	$(1+t^2)^2(1+t)^2$
(12)(34)	3	{13, 24}, {14, 23}, 12, 34	4	$(1+t^2)^2(1+t)^2$
(123)	8	{12, 13, 23}, {14, 24, 34}	2	$(1+t^3)^2$
(1234)	6	{12, 23, 34, 41}, {13, 24}	4	$(1+t^4)(1+t^2)$

$$\begin{aligned} Z(t) &= \frac{1}{4!}\left((1+t)^6 + (1+t^2)^2(1+t)^2 + (1+t^2)^2(1+t)^2 + (1+t^3)^2 + (1+t^4)(1+t^2)\right) \\ &= 1 + t + 2t^2 + 3t^3 + 2t^4 + t^5 + t^6. \end{aligned}$$

If we set $t = 1$ we get the total number, which is 11.

Alternatively, we can use Polya enumeration. The cycle index for the action of S_4 on the 6 edges is

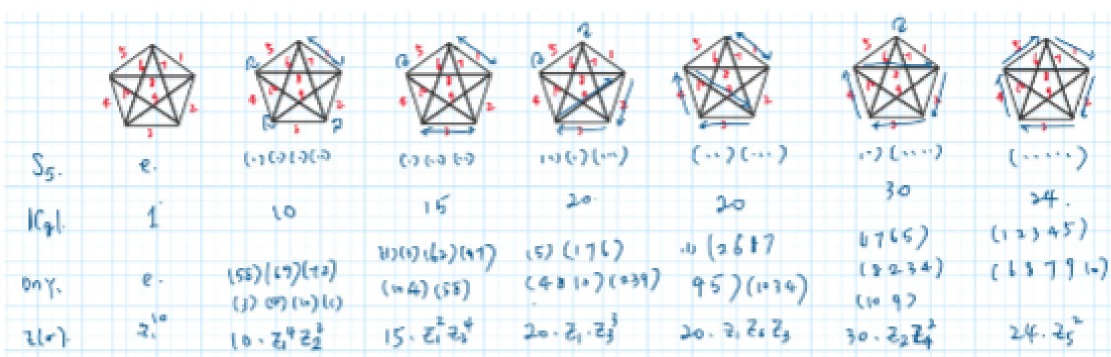
$$Z = \frac{1}{24}\left(z_1^6 + 9z_1^2z_2^2 + 8z_3^3 + 6z_2z_4\right).$$

(check: $Z(z_i = 1) = 1$). Making this polynomial requires the same work as the table above. Setting $z_i = 1 + t^i$ gives the polynomial above.

For 5 vertices, the cycle index of S_5 on the 10 edges is

$$Z = \frac{1}{120}\left(z_1^{10} + 10z_1^4z_2^2 + 15z_1^2z_2^4 + 20z_1z_3^3 + 20z_1z_6z_3 + 30z_2z_4^2 + 24z_5^2\right).$$

For the explanation, see the following nice figure from Xiang Li:



Now plug in $z_i = 1 + t^i$ and expand. Setting $t = 1$ gives 34 graphs on 5 vertices.

10. **Quotients of the spherical model.** I'll post my solution of this problem with the next problem set.