University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 3

Comment: this problem set contains many problems, but most of them are simple and many of them are bonus problems.

Due 3:30pm Thursday, October 17, 2024

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1. **Brain-warmer.**

   (a) Consider the action of a group G on itself by left multiplication. What kinds of terms are allowed in the cycle indicator for this group action?

   Compute the cycle indicators for the action by left multiplication of Q_8, of D_4 and of $\mathbb{Z}_n$ (do the case of n prime first, and do general n as a bonus problem). What do you get if you set all the $z_i = 1$?

   (b) Find the cycle index for the action of Q_8 on itself by conjugation.

2. **Brain-warmer.** Given a group homomorphism $\phi : G \to K$, show that its kernel $H \equiv \ker \phi \equiv \{g \in G | \phi(g) = e \in K\} \subset G$ is always a subgroup. Further, show that it is a *normal* subgroup.

3. **Representation on cosets.** Given a group G and a subgroup H, we can construct a representation of G by its action on the cosets $x \in G/H$. The action is

$$\begin{array}{ccc} G/H & \to & G/H \\ x = \{g_1, g_2 \cdots\} & \mapsto & \{gg_1, gg_2 \cdots\} \end{array}.$$

   To see more explicitly what this does, choose a representative x_i of each distinct coset. Then $G = \cup_i x_i H$, and for each $g \in G$, gx_iH is again a coset and therefore equal to one of the x_iH. So left multiplication by g permutes the cosets. [1]

   Consider the case where $H = \langle(123)\rangle$ is the subgroup of S_3 generated by the order-3 element (123). First decompose S_3 into cosets by H. Write out the matrices in a basis. What representation of S_3 do you get by the construction above? (If it is reducible, decompose it into irreps, for example by computing its character.)

   What about the case where $H' = \langle(12)\rangle$? In each case, decompose it into irreps.

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[1] A word of clarification here: an action of a group G on a set (recall that for a set $X = \{x_1, ..x_k\}$ of order k, this is a homomorphism from $G \to S_k$) is also a representation in the following sense: define the carrier space $V = \text{span}\{|x_i\rangle, i = 1..k\}$, and define the linear operators to act on the basis vectors by $D(g)|x\rangle = |gx\rangle$. This is a special kind of representation called a permutation representation, since the linear operators take basis vectors to basis vectors, rather than making linear combinations – the matrix elements in this basis are a single 1 in each row and column and zeros in all the other entries.

This construction plays an important role in the idea of induced representations. We will learn to call it the representation of G induced by the trivial representation of H.

4. **The class of inverses.** Given a conjugacy class c, define the class $\bar{c}$ to consist of the inverses of each of the elements in c. Convince yourself that this is well-defined. Show that for a unitary representation R,

$$\chi_R(\bar{c}) = \chi_R(c)^\star.$$

5. **Character exercise.** Recall the definition of the regular representation of a finite group G:

$$\mathcal{H}_\mathsf{G} \equiv \text{span}\{|g\rangle\,,\ g \in \mathsf{G}\}.$$

This space also carries a representation of $\mathsf{G} \times \mathsf{G}$ by

$$\Gamma(m,n)\,|h\rangle = \left|mhn^{-1}\right\rangle\,,\ (m,n) \in \mathsf{G} \times \mathsf{G}.$$

$\mathcal{H}_\mathsf{G}$ is also reducible as a representation of $\mathsf{G} \times \mathsf{G}$. Show that

$$\mathcal{H}_\mathsf{G} = \oplus_{R,\text{irreps of }\mathsf{G}} R \otimes \bar{R}$$

where $\bar{R}$ is the conjugate representation to R, with $\Gamma_{\bar{R}}(g) = \Gamma_R(g)^\star$.

Hint: Find the characters of $\mathcal{H}_\mathsf{G}$ as a representation of $\mathsf{G} \times \mathsf{G}$, and compute $\left\langle \chi_{\mathcal{H}_\mathsf{G}} | \chi_{R_i \otimes R_j} \right\rangle$, where R_i are all the irreps of G.

6. **Reps from characters.** Using the character table, make unitary representation matrices for the 2-dimensional representation of $D_3 = S_3$ in which $D(123)$ and $D(132)$ are diagonal. Note that $1 + \omega + \omega^2 = 0$ where $\omega \equiv e^{2\pi \mathbf{i}/3}$.