University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 4 – Solutions

Comment: this problem set contains many problems, but most of them are simple and many of them are bonus problems.

Due 3:30pm Thursday, October 24, 2024

---

1. **Brain-warmer.** If I tell you that $\langle \chi_R, \chi_R \rangle \equiv \frac{1}{|G|} \sum_{g \in G} \chi_R^\star(g)\chi_R(g) = 2$ for some representation R of a finite group G, do you know whether R is irreducible? Explain.

   It is not. As a special case of character orthonormality, an irrep has $\langle \chi_a, \chi_a \rangle = 1$. Moreover, we know that it decomposes into two irreps, since if its character is $\chi = \chi_1 + \chi_2$ then

$$\left(\langle \chi_1| + \langle \chi_2|\right)\left(|\chi_1\rangle + |\chi_2\rangle\right) = \langle \chi_1|\chi_1\rangle + \langle \chi_2|\chi_2\rangle = 2.$$

2. **Brain-warmer.** Given two representations A, B of a group G (with carrier spaces U and V respectively) and an intertwiner between them

$$\Lambda : U \to V, \quad \Lambda A(g) = B(g)\Lambda \forall g \in G$$

   show that $\ker \Lambda \subset U$ and $\text{Im}\Lambda \subset V$ are invariant subspaces. (Recall that this was the crucial ingredient in Schur's Lemma.)

   Recall that an invariant subpace of a representation D is $W \subset V$ such that $D(g)\,|w\rangle \in W$ for all $w \in W, g \in G$.

   If $|a\rangle \in \ker \Lambda$, $\Lambda\,|a\rangle = 0$. Then

$$\Lambda A_\alpha\,|a\rangle = B_\alpha \Lambda\,|a\rangle = 0,$$

   so $A_\alpha\,|a\rangle \in \ker \Lambda$, too.

   Similarly, if $|b\rangle \in \text{Im}\,\Lambda$, then $|b\rangle = \Lambda\,|a\rangle$ for some $|a\rangle \in U$, so

$$\Lambda A(g)\,|a\rangle = B(g)\Lambda\,|a\rangle = B(g)\,|b\rangle \in \text{Im}\,\Lambda, \forall g \in G.$$

3. **Character table for the quaternions.** Figure out the character table for the quaternion group Q_8 (on page 11 of the lecture notes) by whatever means necessary (don't look it up).

One way to do it is to find all the irreps. We know the trivial rep. Since there are 5 conjugacy classes, the dimensions (1,b,c,d,e) have to satisfy $8 = 1+b^2+c^2+d^2+e^2$ which Diophantine equation is only solved by $(1, 1, 1, 1, 2)$. The 2d rep is

$$D(\pm 1) = \pm \mathbb{I}, D(\pm \mathbf{i}) = \pm \mathbf{i} X, D(\pm \mathbf{j}) = \pm \mathbf{i} Y, D(\pm \mathbf{k}) = \mp \mathbf{i} Z$$

where X, Y, Z are the Pauli matrices. These are traceless. The other three one-dimensional reps are $D(\mathbf{i}) = -1, D(\mathbf{j}) = D(\mathbf{k}) = +1$ and its cyclic permutations. This gives:

	Q_8	n_C	1	1_x	1_y	1_z	2
	e	1	1	1	1	1	2
$\mathbb{Z}_2$	$-e$	1	1	1	1	1	-2
$\mathbb{Z}_4$	$\pm\mathbf{i}$	2	1	1	-1	-1	0
$\mathbb{Z}_4$	$\pm\mathbf{j}$	2	1	-1	1	-1	0
$\mathbb{Z}_4$	$\pm\mathbf{k}$	2	1	-1	-1	1	0

But we actually don't need to find all the reps. We know the first column of the character table. Once we know the dimensions of the irreps, we know the first row. Consider the second row. Since $-e$ generates a $\mathbb{Z}_2$ subgroup, the 1d characters can only be ± 1. The condition $\sum_r \chi_r^\star(c)\chi_r(c') = \frac{|G|}{n_c}\delta^{cc'}$ with $c = c' = -e$ requires that $\chi_r(-e)$ satisfy the same equation as the dimensions of the irreps and therefore requires $\chi_2(-e) = \pm 2$ which then gives the second row by orthonormality with the first row.

	Q_8	n_C	1	1_x	1_y	1_z	2
	e	1	1	1	1	1	2
$\mathbb{Z}_2$	$-e$	1	1	1	1	1	-2
$\mathbb{Z}_4$	$\pm\mathbf{i}$	2	1	x	y	z	a
$\mathbb{Z}_4$	$\pm\mathbf{j}$	2	1	x'	y'	z'	b
$\mathbb{Z}_4$	$\pm\mathbf{k}$	2	1	x''	y''	z''	c

Then column orthonormality of the last column with itself requires

$$0 = 2^2 + (-2)^2 + |a|^2 + |b|^2 + |c|^2$$

so $a = b = c = 0$.

Since the elements of the remaining three conjugacy classes each generate a $\mathbb{Z}_4$ subgroup, the 1d characters must satisfy $x^4 = y^4 = z^4 = 1$. Row orthonormality of the 3rd row with itself requires

$$1^2 + |x|^2 + |y|^2 + |z|^2 + 0^2 = 8/2 \quad (1)$$

so each of $|x| = |y| = |z| = 1$. You might have worried that we could have $(1, x, y, z) = (1, -1, \mathbf{i}, -\mathbf{i})$ which satisfies orthogonality with the first two rows,

$1+x+y+z=0$ as well as (1). But then there is no choice of $(1,x',y',z')$ which satisfies both $1+x'+y'+z'$ and $(1,x',y',z')\cdot(1,x,y,z)=0$. Therefore, each of (x,y,z) and the primed and double primed things must have two -1 and one +1.

Note that Q_8 and D_4 have the same character table. But they are actually different groups (Q_8 has six elements of order four while D_4 has only two).

4. **Irreps and conjugacy classes.**

Consider the object

$$S^a_\alpha \equiv \frac{1}{n_\alpha}\sum_{g\in C_\alpha} D^a(g),$$

a linear operator on V_a, the carrier space for an irrep of G. C_α is a conjugacy class of G.

(a) Show that S^a_α commutes with all the $D^a(g)$.

This is the same manipulation as the one we did to show that the character table is square. For any class function $f(g)$,

$$D^a(h)S = \sum_g f(g)D^a(hg) = \sum_{g'=h^{-1}gh} f(hg'h^{-1})D^a(g'h) = \sum_{g'} f(g)D^a(g')D^a(h) = SD^a(h).$$

(To remove clutter, I dropped the indices on S in this equation.) Now take $f(g) = \begin{cases} 1, & g\in C_\alpha \\ 0, & \text{else}\end{cases}.$

(b) Use Schur's lemma to conclude that $S^a_\alpha = \lambda^a_\alpha \mathbb{I}_a$ and find λ^a_α in terms of familiar objects.

Well, this is just exactly the conclusion of Schur's lemma. Taking trace of both sides, we get

$$\frac{1}{n_\alpha}\sum_{g\in C_\alpha}\chi^a(\alpha) = \lambda^a_\alpha d_a$$

which says

$$\lambda^a_\alpha = \chi^a_\alpha/\chi^a_e$$

since $\chi^a_e = d_a$.

(c) [Bonus problem] Conclude that $\mathbf{C}_\alpha\mathbf{P}^a = \lambda^a_\alpha\mathbf{P}^a$ where $\mathbf{P}^a = \frac{d_a}{|G|}\sum_{g\in G}(\chi^a(g))^\star\,\mathbf{g}$ is the projector in the group algebra associated with the irrep a and $\mathbf{C}_\alpha \equiv \frac{1}{n_\alpha}\sum_{g\in C_\alpha}\mathbf{g}$ are elements of the center of the group algebra.

The idea is that $\mathbf{C}_\alpha$ commutes with everyone in the group algebra. This means, by Schur's lemma, that acting on an irrep, it must be proportional to the identity. $\mathbf{P}^a$ restricts its action to the irrep a.

To be more precise and more explicit, we have

$$\mathbf{C}_\alpha \mathbf{P}^a = \frac{d_a}{|G| n_\alpha} \sum_{g\in G, h\in C_\alpha} \chi^a(g^{-1}) \mathbf{hg} \tag{2}$$

$$= \frac{d_a}{|G| n_\alpha} \sum_{g\in G, h\in C_\alpha} \chi^a(g^{-1}h) \mathbf{g} \tag{3}$$

$$= \frac{d_a}{|G| n_\alpha} \sum_{g\in G, h\in C_\alpha} \mathrm{tr}_a \left(D^a(g^{-1}) D^a(h) \right) \mathbf{g} \tag{4}$$

$$= \frac{d_a}{|G|} \sum_{g\in G} \mathrm{tr}_a \left(D^a(g^{-1}) \underbrace{\frac{1}{n_\alpha} \sum_{h\in C_\alpha} D^a(h)}_{=S^a_\alpha} \right) \mathbf{g} \tag{5}$$

$$\stackrel{4b}{=} \lambda^\alpha_a \frac{d_a}{|G|} \sum_{g\in G} \mathrm{tr}_a D^a(g^{-1}) \mathbf{g} = \lambda^\alpha_a \mathbf{P}^a. \tag{6}$$

5. **Statistics of cycle lengths of permutations.** [This is a bonus problem]

(a) What fraction of permutations in S_n have just one big cycle?

The size of the conjugacy class with one column of n boxes is $\frac{n!}{\prod_j j^{k_j} k_j!} = (n-1)!$. So the fraction is $\frac{(n-1)!}{n!} = \frac{1}{n}$.

(b) What fraction of permutations in S_{2n} have no cycle of length greater than n? What is the large-n limit of this number?

[Hint: Count the number of ways to assign elements to such a cycle.]

An element of S_{2n} can have at most one cycle of length greater than n. The number of elements with a cycle of length $l > n$ is

$$N_l = \binom{2n}{l} (l-1)!(2n-l)!$$

The first factor is the number of ways to choose which elements are in the l-cycle, the second factor is the number of ways to choose the order of those elements (up to cyclic permutations which don't change the cycle), and the last factor counts the permutations of the remaining elements. This is

$$N_l = (2n)!/l.$$

Therefore the probability that a uniformly random permutation has a cycle of length $> n$ is

$$\frac{1}{(2n)!}\sum_{l=n+1}^{2n}\frac{(2n)!}{l}=\sum_{l=n+1}^{2n}\frac{1}{l}=H_{2n}-H_n$$

where $H_n \equiv \sum_{k=0}^{n} 1/k$ is the nth harmonic number. These numbers approach

$$H_n \overset{n\to\infty}{\to} \ln n+\gamma$$

where γ is the Euler-Mascheroni constant. So the probability that a permutation of $2n$ elements has no cycle of length greater than n is

$$1-(H_{2n}-H_n) \overset{n\to\infty}{\to} 1-\ln 2n+\ln n = 1-\ln 2 \approx 0.30685. \qquad (7)$$

(c) What fraction of permutations in S_n have no fixed points? What is the large-n limit of this number?

[Hint: One way to do it is to use inclusion-exclusion. Another way is to find a recursion relation for $D(n)$, the number of elements of S_n with no fixed point. A third, trickier, way is to use our formula for cycle-lengths; to use this method, I had to pass through the grand canonical ensemble, *i.e.* sum over n.]

Method 1: There are $\binom{n}{1}=n$ ways to choose an element to be in a one-cycle, and then $(n-1)!$ permutations of the remaining elements. However, this number

$$\binom{n}{1}(n-1)!$$

overcounts permutations with two one-cycles (which could be either the one we chose or part of the remaining permutation). So let's subtract

$$\binom{n}{2}(n-2)!\frac{1}{2!}=\frac{n!}{2!}\ .$$

The $1/2!$ permutes the two one-cycles. But now we have undercounted permutations with at least three one-cycles, so we add back

$$\binom{n}{3}(n-3)!\frac{1}{3!}=\frac{n!}{3!}\ .$$

Continuing in this way until we get to n one-cycles, we have the fraction of permutations with at least one one-cycle is

$$1 - \frac{1}{2!} + \frac{1}{3!} \ldots \pm \frac{1}{n!}. \tag{8}$$

Perhaps more clearly, let S^k be the subset of S_n that fixes the kth object. Then

$$|S^1 \cup \cdots \cup S^n| = \sum_i |S^i| - \sum_{i<j} |S^i \cap S^j| + \sum_{i<j<k} |S^i \cap S^j \cap S^k| - \cdots \tag{9}$$

$$= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \binom{n}{3}(n-3)! - \cdots \tag{10}$$

Thus, the fraction without any one-cycle is

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \ldots \mp \frac{1}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \stackrel{n\to\infty}{\to} \frac{1}{e} \approx 0.36787944. \tag{11}$$

Method 2: Another solution is to find a recursion relation for $D(n)$, the number of fixed-point-free permutations (which are called *derangements*). Let me describe it in terms of the riddle about exams in the next problem. In a derangement, there are $n-1$ possibilities for whose exam person n gets, let's call it i. Let's think about whose exam person i gets (for $n \geq 2$). There are two cases:

i. Person i gets person n's exam, so there is a cycle of length two. Fixing i and n, the number of ways this can happen is $D(n-2)$, since we just remove i and n from the discussion.

ii. Person i gets some other person's exam. In this case, the number of ways this can happen is just $D(n-1)$, since there are $n-1$ people involved, and each one has exactly one exam they can't get (in the case of person i it is person n's exam).

Therefore

$$D(n) = (n-1)(D(n-2) + D(n-1)), \; n \geq 2 \tag{12}$$

with the initial condition $D(0) = 1, D(1) = 0$. A nice way of rewriting this equation (I learned from Xiang Li) is

$$D(n) - nD(n-1) = -(D(n-1) - (n-1)D(n-2)) \tag{13}$$

so that $G(n) \equiv D(n) - nD(n-1)$ satisfies $G(n) = -G(n-1)$ with $G(2) = 1$, which is solved by $G(n) = (-1)^n$.

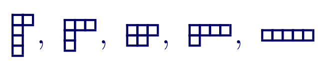
Mathematica can solve the recursion relation (12) by the command `RSolve`, with the result

$$D(n) = \frac{\Gamma(1+n,-1)}{e}. \tag{14}$$

In fact, Mathematica has a command called `Subfactorial` which is the answer to this question. Alternatively, you can check that the solution above (11) (times $n!$) solves this linear second order recursion relation and is therefore the unique solution.

Method 3: Let's count how many elements of S_n *do* have at least one fixed point, this is $F(n) \equiv n! - D(n)$. We know that the cycle structure is a property of a conjugacy class of S_n, which corresponds to a Young tableau with n boxes. Demanding that the permutations in a conjugacy class do have a fixed point means that they have at least one cycle of length one, and hence that the corresponding tableau has at least one column of a single box.

But every such tableau of n boxes can be made by attaching an extra column of one row to all the tableau for S_{n-1}. For example, for $n = 5$:



(15)

So

$$F(n) = \sum_{\{k_j | \sum_j jk_j = n-1\}} \prod_j \frac{n!}{j^{\tilde{k}_j} \tilde{k}_j!} \tag{16}$$

where $\tilde{k}_j \equiv (k_1 + 1, k_2 \cdots)_j$. This sum doesn't seem so easy because of the constraint on the ks.

As a warmup, let's think about how to do the sum over conjugacy classes of the number of elements in each conjugacy class:

$$\sum_{\{k_j | \sum_j jk_j = n\}} \prod_j \frac{n!}{j^{k_j} k_j!} = |S_n| = n! \ . \tag{17}$$

When we have a sum over a constrained variable a very useful trick is to sum over the constraint, with some fugacity variable as a bookkeeping device. So consider $\frac{1}{n!}\sum_{n=0}^{\infty}(BHS)x^n$ with some fugacity x. The LHS gives

$$
\begin{aligned}
S(x) \equiv \sum_{n=0}^{\infty} x^n \sum_{\{k_j | \sum_j jk_j = n\}} \prod_j \frac{1}{j^{k_j} k_j!} \\
= \sum_{\{k_j\}} x^{\sum_j jk_j} \prod_{j=1} \frac{1}{j^{k_j} k_j!} \\
= \sum_{\{k_j\}} \prod_{j=1} \frac{x^{jk_j}}{j^{k_j} k_j!} \\
= \prod_j \sum_{k_j=0}^{\infty} \frac{(x^j/j)_j^k}{k_j!} \\
= \prod_j e^{x^j/j} = e^{\sum_{j=1}^{\infty} x^j/j} \\
= e^{-\log(1-x)} = \frac{1}{1-x} \ .
\end{aligned}
\tag{18}
$$

Which is just what we get from the RHS: $S(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.
So let's use the same trick for the counting permutations with fixed points:

$$
\begin{aligned}
F(x) \equiv \sum_{n=0}^{\infty} x^n \sum_{\{k_j | \sum_j jk_j = n-1\}} \prod_j \frac{1}{j^{\tilde{k}_j} \tilde{k}_j!} \\
= \sum_{\{k_j\}} x^{\sum_j jk_j + 1} \prod_{j=1} \frac{1}{j^{\tilde{k}_j} \tilde{k}_j!} \\
= \sum_{\{k_j\}} \frac{x^{k_1+1}}{(k_1+1)!} \prod_{j>1} \frac{x^{jk_j}}{j^{k_j} k_j!} \\
= \sum_{\tilde{k}_1=1}^{\infty} \frac{x^{\tilde{k}_1}}{\tilde{k}_1!} \prod_{j>1} \sum_{k_j=0}^{\infty} \frac{x^{jk_j}}{j^{k_j} k_j!} \\
= (e^x - 1) \prod_{j=2}^{\infty} e^{x^j/j} \\
= \frac{1}{1-x}(1 - e^{-x}).
\end{aligned}
\tag{19}
$$

To extract the coefficient of x^n in $F(x) = \sum_{n=0}^{\infty} q_n x^n$, we can take

$$
\frac{1}{n!}\partial_x^n F(x)|_{x=0} = q_n. \tag{20}
$$

The first few entries are

$$
\{0, 1, 1, 4, 15, 76, 455, 3186, 25487, 229384, 2293839...\}
$$

which agrees with the values of $F(n)$. To see directly that

$$(1-e^{-x})\frac{1}{1-x} \stackrel{?}{=} \sum_{n=1}^{\infty} x^n F(n) \tag{21}$$

we can Taylor expand each factor on the LHS as

$$\begin{aligned}(1-e^{-x})\frac{1}{1-x} &= -\sum_{j=1}^{\infty}\frac{(-x)^j}{j!}\sum_{l=0}^{\infty}x^l \\ &= \sum_{j=1,l=0}^{\infty}\frac{(-1)^{j+1}}{j!}x^{j+l} \\ &= \sum_{n=j+l=1}^{\infty}x^n\sum_{j=1}^{n}\frac{(-1)^{j+1}}{j!}.\end{aligned} \tag{22}$$

At the last step, the summation variable j must be $\leq n$ since $n = j + l$.

Method 4: (I learned from Deepak Aryal and Haoran Sun)

$$n! = \sum_{k=0}^{n}\binom{n}{k} D(k)$$

where the kth term counts the permutations with exactly $n-k$ one-cycles. Therefore

$$D(n) = n! - \sum_{k=0}^{n-1}\binom{n}{k} D(k) \tag{23}$$

is a recursion relation for $D(k)$ (which apparently Mathematica can solve). But supposed we are just interested in the large-n limit of $r_n \equiv D(n)/n!$. The recursion says

$$r_n = 1 - \sum_{k=0}^{n-1}\frac{1}{k!(n-k)!}D(k) = 1 - \sum_{k=0}^{\infty}\frac{r_k}{(n-k)!}. \tag{24}$$

Assuming the limit r_∞ exists, this says something like

$$r_\infty = 1 - r_\infty \sum_{l=1}^{n}\frac{1}{l!} \tag{25}$$

or $1 = r_\infty \sum_{l=0}^{\infty}\frac{1}{l!} = r_\infty e$.

6. **Completely unrelated riddles.**

(a) There are 100 students in the group theory class. The professor is tired of grading and decides to let the students grade each others' exams. He arranges 100 boxes and uniformly randomly puts one exam in each box. The students must each open one box and grade the exam they find. What's the probability that no student grades their own exam?

(b) [I recommend this puzzle very highly.] The next year, there are again 100 students in the group theory class. This time, for the final exam, the evil professor decides to assign a group project. He arranges 100 numbered boxes and uniformly randomly puts the name of one student in each box. Each student is allowed to open half the boxes. If every student finds their own name then all the students pass, otherwise they all fail. The students are not allowed to communicate with each other after the box-opening begins, but they are allowed to develop a strategy together beforehand. What is a strategy that allows them all to pass with probability $> .3$?

   I first heard this problem from Brian Swingle.

(c) A group of n comic artists is playing a game of exquisite corpse. This means that each person draws one frame of the comic, and then passes it on to the next. If the order of artists is determined by a uniformly random element of S_n, what is the probability that the result is a comic with n frames, each by a different artist?

   I guess I could have made this problem clearer. The idea is we pick a permutation $\pi \in S_n$, and after each round (everyone draws their frame), person i passes their comic to person $\pi(i)$.