University of California at San Diego – Department of Physics – Prof. John McGreevy

Physics 220 Symmetries Fall 2024
Assignment 5 – Solutions

Comment: this problem set contains many problems, but most of them are simple and many of them are bonus problems.

Due 3:30pm Thursday, October 31, 2024

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1. **Brain-warmer.** Given a unitary representation D of a finite group G consider the objects

$$P_a \equiv \frac{d_a}{|G|} \sum_{g\in\mathsf{G}} \chi_a(g)^\star D(g) \tag{1}$$

for each irrep a of G.

(a) Show that $P^\dagger = P$.

$$P_a^\dagger = \frac{d_a}{|G|} \sum_{g\in\mathsf{G}} \chi_a(g) D(g)^\dagger = \frac{d_a}{|G|} \sum_{g\in\mathsf{G}} \chi_a(g) D(g^{-1}) = \frac{d_a}{|G|} \sum_{h=g^{-1}\in\mathsf{G}} \chi_a(h)^\star D(h) = P_a. \tag{2}$$

(b) Show that $P_a P_b = \delta_{ab} P_a$.

$$P_a P_b = \frac{d_a d_b}{|G|^2} \sum_{g,h\in\mathsf{G}} \chi_a(g)^\star \chi_b(h)^\star D(g) D(h) \tag{3}$$

$$= \frac{d_a d_b}{|G|^2} \sum_{g,h\in\mathsf{G}} \chi_a(g)^\star \chi_b(h)^\star D(gh) \tag{4}$$

$$= \frac{d_a d_b}{|G|^2} \sum_{k=gh\in\mathsf{G}} D(k) \sum_{g\in\mathsf{G}} \chi_a(g)^\star \chi_b(g^{-1}k)^\star \ . \tag{5}$$

We can't directly decompose $\chi_b(g^{-1}k)$, but we can use the GOT:

$$P_a P_b = \frac{d_a d_b}{|G|^2} \sum_{k\in\mathsf{G}} D(k) \sum_{i=j,k,l} \sum_{g\in\mathsf{G}} \left(D_a(g)\right)^\star_{ij} \left(D_b(g^{-1})\right)^\star_{kl} \left(D_b(k)\right)^\star_{lk} \ . \tag{6}$$

and the GOT says

$$\frac{1}{|G|} \sum_{g} \left(D_a(g^{-1})\right)_{ji} \left(D_b(g)\right)_{lk} = \frac{1}{d_a} \delta_{jk} \delta_{il} \delta^{ab} \tag{7}$$

which gives

$$P_aP_b = \frac{d_a}{|G|}\sum_{k\in\mathsf{G}} D(k)\sum_{j,k,l}\delta_{jk}\delta_{jl}\left(D_a(k)\right)^\star_{lk}\delta^{ab} \tag{8}$$

$$= \delta^{ab}\frac{d_a}{|G|}\sum_k D(k)\chi^\star_a(k) = \delta^{ab}P_a. \tag{9}$$

2. **All functions on a group.** Show that the matrix elements of the unitary matrices representing the irreps of a finite group provide an orthonormal basis for all functions on the group (with respect to the inner product $\langle f_1|f_2\rangle = \frac{1}{|\mathsf{G}|}\sum_{g\in\mathsf{G}} f_1^\star(g)f_2(g)$).

   A nice way to think about the matrix elements of the irreps is as related to the unitary matrices:

$$\langle g|aij\rangle = \left(D^a(g)\right)_{ij}\sqrt{\frac{d_a}{|G|}} \tag{10}$$

   relating the basis of the regular rep labelled by group elements g and the basis labelled by irreps a and matrix elements ij.

   With this in mind, the statement we want to show is essentially the statement of the Grand Orthogonality Theorem:

$$\langle ajk|blm| = \sum_{g\in G}\langle ajk|g\rangle\langle g|blm\rangle \tag{11}$$

$$= \sum_{g\in G}\frac{d_a}{|G|}\left(D_a(g)\right)^\star_{jk}\left(D_b(g)\right)_{lm} = \delta_{ab}\delta_{jl}\delta_{km}. \tag{12}$$

   This says that these states are orthonormal. To see that they form a complete basis, we have to count how many such functions there are. But we know that

$$|\mathsf{G}| = \sum_a d_a^2. \tag{13}$$

   (For example, because this is the ee component of row orthogonality.)

3. **Diffusion on the vertices of the cube.**

   (a) The group $\mathsf{O} = S_4$ of rotational symmetries of the cube acts on functions on vertices of the cube. Decompose this representation into irreps of S_4.

   To compute the character, look at the list of generators of the S_4 action on the cube, and look at the fixed points. We see that the identity fixes everyone, and the $2\pi/3$ rotations fix the pair of opposite vertices passing through the rotation axis. Therefore

$$\chi_{\mathbf{8}} = \begin{pmatrix} 8 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \chi \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

where χ is the S_4 character table:

S_4		n_C	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{3}'$
$(1) =$	$\cdot$	1	1	1	2	3	3
$(12) =$	$\mathbb{Z}_2$	6	1	-1	0	1	-1
$(12)(34) =$	$\mathbb{Z}_2$	3	1	1	2	-1	-1
$(123) =$	$\mathbb{Z}_3$	8	1	1	-1	0	0
$(1234) =$	$\mathbb{Z}_4$	6	1	-1	0	-1	1

Therefore $\mathbf{8} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{3} \oplus \mathbf{3}'$. So in a sense this example is easier than the action on the faces, since every irrep appears only once. There are no matrices to diagonalize at all!

(b) Assign a temperature to each vertex of the cube T_i. Suppose the temperature evolves according to

$$T_i(t+1) = (1-\lambda)T_i(t) + \frac{\lambda}{3} \sum_{\text{nbrs} j \text{ of } i} T_j(t)$$

(here $\lambda \in (0,1)$ measures the rate of diffusion). Find the rate at which the temperature function approaches its final distribution. (Do it without explicitly diagonalizing any matrices.)

We are trying to find the eigenvalues of the adjacency matrix of the cube (divided by the number of neighbors, so that its action is to replace each value with the the average of its neighbors). First, the singlet $\mathbf{1}$ is just the uniform state where all sites have the same temperature – this is clearly mapped to itself by rotations. It has eigenvalue 1. All the other modes must be orthogonal to this. To figure out which of these modes is which, we can again notice that vertices separated by a large diagonal remain that way after rotation (just like opposite faces) – let's call these 'opposite vertices'. So again, like in the case of faces, we can consider states where opposite vertices are the same and opposite (related by a sign). In the former case, we must also have $w+x+y+z=0$ in order to be orthogonal to the uniform state. This produces a 3-dimensional invariant subspace (which is actually the **3**). A site with value w gets mapped to $(x+y+z)/3 = -w/3$, so the eigenvalue is $-1/3$. If opposite vertices are related by a minus sign, this produces a 4-dimensional subspace, which must be further reducible, since there is no irrep with dimension 4. The action a time step replaces a site labelled w with $-(x+y+z)/3$. If $x+y+z+w=0$, this is $+w/3$ and hence an eigenvector with eigenvalue $+1/3$. This is another 3-dimensional invariant subspace (which is in fact the $\mathbf{3'}$). What's the remaining $\mathbf{1'}$? Well, if in addition $x=y=z=w$, then $-(x+y+z)/3 = -w$ so this is also an eigenvector.

So the spectrum of the adjacency matrix is:

$$\{1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -1\}.$$

If I set $\lambda = 1$, the state after time t looks like

$$1^t c_1 |1\rangle + \left(\frac{1}{3}\right)^t c_3 |3\rangle + \left(\frac{-1}{3}\right)^t c_{3'} |3'\rangle + (-1)^t c_{1'} |1'\rangle .$$

So in fact at late times, there is a remaining oscillation (of the two sublattices) of magnitude $|c_{1'}|$.

With more general values of λ, then the group theory and eigenvectors would be all the same, but the eigenvalues would be replaced by

$$\{1, 1-\frac{2}{3}\lambda, 1-\frac{2}{3}\lambda, 1-\frac{2}{3}\lambda, 1-\frac{4}{3}\lambda, 1-\frac{4}{3}\lambda, 1-\frac{4}{3}\lambda, 1-2\lambda\}.$$

Now indeed the final state uniform, and the rate of approach is determined by the eigenvalue of largest magnitude less than 1. For $\lambda < 3/4$, this is which is $(1-\frac{2}{3}\lambda)$, but for $\lambda > 3/4$ $1-2\lambda$ has larger absolute value. So the behavior is $|\,(1-x\lambda)^t\,| = e^{\log|1-x\lambda| t}$, where $x = 2/3$ or $x = 2$ depending on λ, and so the rate is $\log|1-x\lambda|$.

(c) [bonus problem] Construct explicit projectors onto the eigenmodes using the character table of S_4.

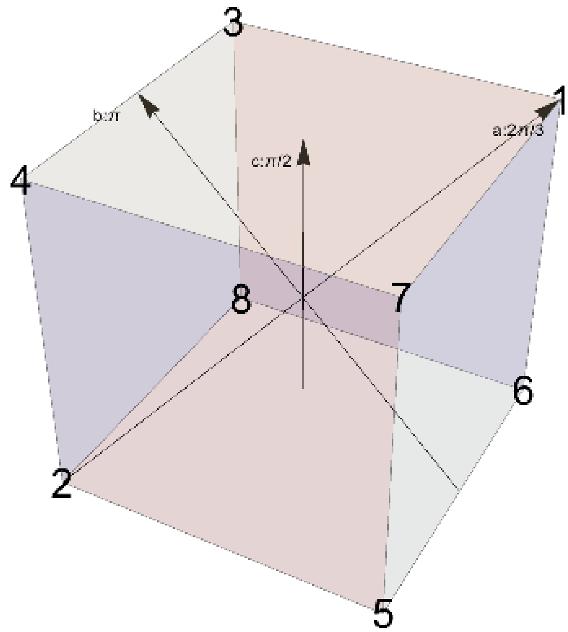
We'll use the presentation $S_4 = \langle a, b, c | a^3 = b^2 = c^4 = e, bca = c^2 \rangle$. so that we only need to enter 3 8×8 matrices, and then can generate the rest using the group law. We organize the elements into cosets by $\mathbb{Z}_3 = \langle a | a^3 = e \rangle$ because we suspect that the representation of S_4 on the vertices of the cube is the representation induced by the trivial representation of $\mathbb{Z}_3 \subset S_4$ (as on the previous homework). This gives an easy way to make the 8×8 matrices

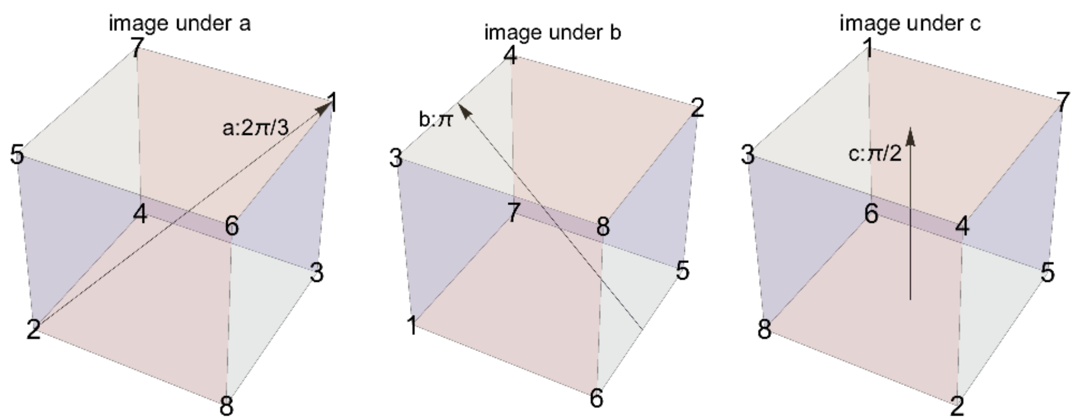
– we just have to know how the elements of S_4 act on the cosets.

element	cycle rep	conjugacy class	sign
e	()	1	1
a	(123)	4	1
a^2	(321)	4	1
b	(12)	2	-1
ba	(23)	2	-1
ba^2	(13)	2	-1
c	(1234)	5	-1
ca	(4132)	5	-1
ca^2	(41)	2	-1
bc	(234)	4	1
bca	(42)(31)	3	1
bca^2	(421)	4	1
cb	(341)	4	1
cba	(412)	4	1
cba^2	(41)(23)	3	1
c^2b	(3142)	5	-1
c^2ba	(1342)	5	-1
c^2ba^2	(42)	2	-1
c^3	(4321)	5	-1
c^3a	(43)	2	-1
c^3a^2	(4312)	5	-1
bc^3	(431)	4	1
bc^3a	(12)(34)	3	1
bc^3a^2	(243)	4	1

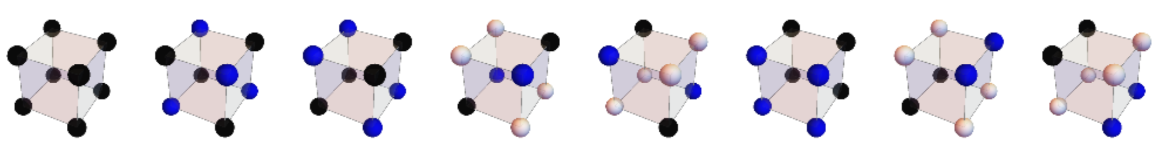
On the cosets, a acts (from the left) as (367)(485), b acts as (12)(34)(56)(78) and c acts as (1347)(2568).

Alternatively, we could have just labelled the vertices of the cube and noticed that these are the permutations by which the three generators of S_4 act. In retrospect, this might have been easier, but I found it very satisfying when the outcome of the calculation of the action on cosets perfectly matched the pictures of cubes.





We can visualize the normal modes as follows (black and blue are opposite signs; every component of every eigenvector has the same magnitude):



These pictures (made automatically with this mathematica file) confirm my claims above that the case with opposite vertices equal is the **3** and with opposite vertices opposite and summing to zero is the **3′**, and that the **1′** is the state where opposite vertices are opposite and $x = y = z = w$.

4. **Diffusion on the edges of the tetrahedron.** [bonus problem]

(a) Show that the group of rotations which are symmetries of a regular tetrahedron is isomorphic to A_4, the subgroup of S_4 consisting of just the even permutations.

The rotation symmetries are the $2\pi/3$ rotations whose axis goes through a vertex and the center of the opposite triangle (there are 4 choices of axis, and 2 rotations for each axis), and the π rotations through the centers of two edges which do not share a vertex (there are 3 such pairs of edges). This is 12 elements altogether. These correspond respectively to the elements such as (123) and (12)(34) of A_4. The four vertices transform in the fundamental representation.

(b) Construct the character table for A_4.

I will just write down the answer:

A_4		n_C	$\mathbf{1}$	$\mathbf{1}'$	$\mathbf{1}''$	$\mathbf{3}$
(1)	$\cdot$	1	1	1	1	3
(12)(34)	$\mathbb{Z}_2$	3	1	1	1	-1
(123)	$\mathbb{Z}_3$	4	1	ω	ω^2	0
(321)	$\mathbb{Z}_3$	4	1	ω^2	ω	0

Notice that the element (12) that conjugates (123) to (321) is not in A_4, so they are in different classes in A_4. Note that $1^2 + 1^2 + 1^2 + 3^2 = 12$. One way to find the answer is to decompose the irreps of S^4. Another way is to use the fact that the characters of the 1d reps for the order-3 elements must be cube roots of unity and demand column and row orthogonality.

(c) A_4 acts on the *edges* of the tetrahedron in a 6-dimensional representation. Decompose this into irreps.

The character is $\chi_{\mathbf{6}} = (6, 2, 0, 0) = \chi.(1, 1, 1, 1)$. So this rep is $\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{3}$.

The subgroup of S_4 which fixes an edge (say 12) is the $\mathbb{Z}_2$ generated by (12)(34). So the **6** that acts on the edges is the induced rep from the trival rep of this $\mathbb{Z}_2$ subgroup.

(d) Suppose we have a tight-binding model on the edges of the tetrahedron, $H = -t \sum_{\langle e_1 e_2 \rangle} |e_1\rangle\langle e_2| + h.c.$ where two edges are regarded as neighbors if they both lie in the boundary of the same face. Find the spectrum.

The uniform state $\sum_i |i\rangle$ has eigenvalue $-4t$. Opposite edges of opposite sign gives the **3** and has eigenvalue zero. Opposite edges same requires the sum to be zero to be orthogonal to the uniform state. This 2d rep decomposes into $\mathbf{1}' \oplus \mathbf{1}''$, but they have the same eigenvalue, $2t$. Altogether, the spectrum is

$$(-4t, 0, 0, 0, 2t, 2t).$$

Incidentally, the eigenvectors for the $\mathbf{1}'$ and $\mathbf{1}''$ are not real. They involve loop currents around the faces of the tetrahedron. By this I mean that the wavefunction for each link acquires a factor of $\omega = e^{2\pi \mathbf{i}/3}$ as we go around the face. The two reps are related by a parity transformation (switching the orientation of the currents), or by time reversal (i.e. complex conjugation).

5. **Projection operators.** [bonus problem] Write a program to make the pictures of the normal modes of the 'triatomic molecule'. Write it in such a way that it is easy to redo it for the generalization to n particles arranged in a regular n-sided polygon.

   Here's mine.

6. **Using the algebra of classes to construct the character table for S_3.** [bonus problem]

   (a) Write explicit expressions for the $\mathbf{C}_\alpha$ operators for the case $G = S_3$, and find the structure constants $c_{\alpha\beta}^{\gamma}$ of the algebra of classes.

   $$\mathbf{C}_1 = \mathbf{e}, \mathbf{C}_2 = \frac{1}{3}\left((12) + (13) + (23)\right), \mathbf{C}_3 = \frac{1}{2}\left((123) + (321)\right).$$

   $$\mathbf{C}_1\mathbf{C}_\beta = \mathbf{C}_\beta, \mathbf{C}_2\mathbf{C}_2 = \frac{1}{3}\mathbf{C}_1 + \frac{2}{3}\mathbf{C}_3, \mathbf{C}_3\mathbf{C}_3 = \frac{1}{2}\mathbf{C}_1 + \frac{1}{2}\mathbf{C}_3, \mathbf{C}_2\mathbf{C}_3 = \mathbf{C}_2.$$

   Notice that these rules are consistent with the fact that $\mathbf{C}_2$ contains the odd permutations, and odd with odd is even, odd with even is odd.

   (b) Construct the matrices $(\mathbf{C}_\alpha)^\gamma{}_\beta$, check that they commute, and find their eigenvalues, λ_α^a

   $$(\mathbf{C}_1)^\gamma{}_\beta = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, (\mathbf{C}_2)^\gamma{}_\beta = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{2}{3} & 0 \end{pmatrix}, (\mathbf{C}_3)^\gamma{}_\beta = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2} \end{pmatrix}.$$

   The eigenvalues are:

   $$\lambda_1^a = (1,1,1)^a, \lambda_2^a = (-1,1,0)^a, \lambda_3^a = (1,1,-\frac{1}{2})^a.$$

That is:

$$\lambda^a_\alpha = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & -\frac{1}{2} \end{pmatrix}$$

where a is the column index and α is the row index. We should switch the first two columns to put the trivial rep first.

(c) Check that you reproduce the character table for S_3 using the fact that the eigenvalues should be $\lambda^a_\alpha = \chi^a_\alpha / \chi^a_e$.

Since $\sum_\alpha n_\alpha \lambda^1_\alpha = 1 \cdot 1^2 + 3 \cdot (-1)^2 + 21^2 = 6$ we have $\chi^1_e = 1$. Since $\sum_\alpha n_\alpha \lambda^2_\alpha = 1 \cdot 1^2 + 3 \cdot 1^2 + 21^2 = 6$ we have $\chi^2_e = 1$. Since $\sum_\alpha n_\alpha \lambda^2_\alpha = 1 \cdot 1^2 + 3 \cdot 0^2 + 2\left(-\frac{1}{2}\right)^2 = \frac{3}{2}$ we have $\chi^3_e = 2$. Therefore, the character table is

$$\begin{matrix} 1 & 1 & 2 \\ -1 & 1 & 0 \\ 1 & 1 & -1 \end{matrix}$$

which is correct.